Spectral shift function for the Stark operator in the large coupling constant limit

A. Pushnitski*† and V. Sloushch‡

Abstract

Let H be the Stark operator in $\mathbb{R}^3$ and let $V = V(x)$ be a non-negative potential which decays at infinity as $|x|^{-l}$ for a sufficiently large $l > 0$. The spectral shift function for the pair of operators $H + tV$, H is studied in the asymptotic regime $t \to \infty$. The leading term of the asymptotics is obtained; the result is in agreement with the phase space volume considerations.

Keywords: Stark operator, spectral shift function, large coupling constant.

1 Introduction and Main Result

1. In $L^2(\mathbb{R}^3, dx)$, $x = (x_1, x_2, x_3)$, consider the Stark operator

$$H = -\Delta + x_1, \quad \operatorname{Dom} H = \{u \in L^2(\mathbb{R}^3) : -\Delta u + x_1 u \in L_2(\mathbb{R}^3)\}.$$

The operator H is self-adjoint; its spectrum $\sigma(H)$ is purely absolutely continuous and coincides with the real axis; see [3]. Consider also an operator of multiplication by a function (potential in physical terminology) $V : \mathbb{R}^3 \to [0, \infty)$, which is assumed to satisfy the condition

$$0 \leqslant V(x) \leqslant C(1 + |x|^2)^{-l/2}, \quad l > \frac{7}{2}, \quad C > 0. \tag{1.1}$$

We shall consider the operator $H + tV$, where $t > 0$ is a parameter (coupling constant) which will be taken large in the sequel.

2. The following preliminary result asserts the existence of the spectral shift function (SSF) for the pair of operators $H + tV$, H:

*Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, U.K. email: a.b.pushnitski@lboro.ac.uk telephone: +44(1509)222867, fax: +44(01509)223969

†The corresponding author

‡Department of Mathematical Physics, Faculty of Physics, St.Petersburg State University, 198504, St.Petersburg, Russia. email: sloushch@mph.phys.spbu.ru

Proposition 1.1. *Let V satisfy* (1.1) *and let $t > 0$. Then for all $\varphi \in C_0^\infty(\mathbb{R})$, the operator $\varphi(H+tV) - \varphi(H)$ belongs to the trace class. There exists a function $\xi = \xi(\lambda, t)$, $0 \leqslant \xi(\cdot, t) \in L^\infty_{\text{loc}}(\mathbb{R})$ such that*

$$\operatorname{Tr}(\varphi(H+tV) - \varphi(H)) = \int_{-\infty}^{\infty} \xi(\lambda, t)\varphi'(\lambda)d\lambda, \quad \forall \varphi \in C_0^\infty(\mathbb{R}) \tag{1.2}$$

and

$$\operatorname*{ess\,sup}_{\mu \leq \lambda} \xi(\mu, t) \to 0 \text{ as } \lambda \to -\infty. \tag{1.3}$$

The trace formula (1.2) determines ξ up to an additive constant, and the normalisation condition (1.3) fixes this constant.

The spectral shift function for a pair of self-adjoint operators was introduced by M. G. Krein in [10]. Background information on the SSF theory can be found in [5, 16, 13]. The SSF for the Stark operator has been studied by a number of authors; here we note the papers [15, 6]. The paper [6] contains an extensive list of literature on the subject.

The above Proposition 1.1 is essentially well known. In particular, similar statements (for somewhat different classes of V) are contained in Lemma 2.3 of [15] and Lemma 2 of [6]. Proposition 1.1 can also be easily proven by using the recent result [17]. As a by-product of our construction in Section 3, we obtain another proof of Proposition 1.1. We note that the assumption $V \geq 0$ is not essential in Proposition 1.1 but is essential in Theorem 1.2 below.

3. In addition to (1.1), assume that $V(x)$ has the following asymptotics:

$$\lim_{|x|\to\infty} |x|^l |V(x) - V_\infty(x)| = 0, \tag{1.4}$$

where $V_\infty : \mathbb{R}^3 \to [0, \infty)$ is a function which is continuous on $\mathbb{R}^3 \setminus \{0\}$ and homogeneous of the degree $-l$: $V_\infty(tx) = t^{-l}V_\infty(x)$, $t > 0$, $x \in \mathbb{R}^3$. Our main result is

Theorem 1.2. *Assume* (1.1), (1.4). *Then for any compact interval $\delta \subset \mathbb{R}$, one has*

$$\lim_{t\to+\infty} \operatorname*{ess\,sup}_{\lambda\in\delta} \left| t^{-\varkappa}\xi(\lambda, t) - \Delta(l)\right| = 0, \tag{1.5}$$

where $\varkappa = \frac{9}{2(l+1)}$, and

$$\Delta(l) = \frac{1}{6\pi^2} \int_{\mathbb{R}^3} [(-x_1)_+^{3/2} - (-x_1 - V_\infty(x))_+^{3/2}]dx.$$

4. Remarks: 1. An elementary calculation shows that the integral in the definition of $\Delta(l)$ converges as long as $l > \frac{7}{2}$ (see (1.1)).

2. Note that the asymptotic coefficient $\Delta(l)$ does not depend on $\lambda \in \mathbb{R}$.

3. The asymptotics (1.5) is in agreement with the phase space volume considerations. Indeed, let us denote (cf. [13])

$$\xi^{\text{cl}}(\lambda, t) = (2\pi)^{-3} \operatorname{vol}\{(x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 : |p|^2 + x_1 + tV(x) > \lambda > |p|^2 + x_1\}.$$

It is easy to check that $\xi^{\rm cl}$ has the asymptotic behaviour given by (1.5).

5. Let us describe the technique and the structure of the paper. Broadly speaking, our approach is a combination of ideas from [2] and [12]. In [2], the following problem was considered. Let $H = -\Delta + p(x)$ be the Schrödinger operator in $\mathbb{R}^d$ such that the integrated density of states for H exists, let $V \geq 0$ be of the same type as in this paper and let λ belong to a gap in the spectrum of H. The object of study in [2] was the number of eigenvalues of $H+\tau V$ which cross λ as τ grows monotonically from 0 to t. The behaviour of this eigenvalue counting function was studied for $t \to \infty$. Essentially, the approach of [2] was to 'truncate' the operator $H + tV$, replacing it by an operator with the same coefficients on a large cube in $\mathbb{R}^d$. The spectrum of the truncated operator is discrete and therefore can be studied by variational methods. If the size of the cube grows together with t, the truncated problem approximates the initial problem. Note that it is not difficult to see (cf. [5]) that if the spectral shift function for the pair of operators $H + tV$, H exists, then it coincides with the eigenvalue counting function of [2].

Next, in [12], the spectral shift function for the pair of operators $(-\Delta) + tV$, $(-\Delta)$ in $\mathbb{R}^d$ was considered for $\lambda > 0$. The approach of [12] was different from [2] and heavily based on the explicit form of spectral representation for $-\Delta$. In this paper, we use a mixed approach, employing the technique of both [2] and [12].

In Section 2, we recall the spectral representation formulae for the Stark operator, introduce the sandwiched resolvent (2.10) and discuss the properties of its limiting values on the real axis. In Section 3, we prove Proposition 1.1 and recall the representation (3.5) for the SSF from [11]. As well as in [12], the representation (3.5) is our main tool. With the help of this representation, in Section 4 we show how to reduce the problem to computing spectral asymptotics of the sandwiched resolvent. In computing this spectral asymptotics, we follow the method of [2]; the main steps of this method are repeated in Sections 5–9. The additional complications as compared to [2] are due to two circumstances: (i) in our case, the spectral parameter λ belongs to the spectrum of H; (ii) the operator H is not semi-bounded from below.

6. Notation For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote $x_\perp = (x_2, x_3) \in \mathbb{R}^2$. We will use the anisotropic weighted space

$$L^2_\rho(\mathbb{R}^3) \equiv L^2(\mathbb{R}^3, \langle x_1\rangle^{2\rho} dx), \qquad \rho > 0, \qquad \langle x_1\rangle \equiv (1 + x_1^2)^{1/2},$$

and denote by $\|\cdot\|_{2,\rho}$ the norm in this space. We use the notation

$$W = \sqrt{V}.$$

We denote by S_∞ the class of compact operators and by S_q, $q > 0$, the standard Schatten classes of compact operators (see e.g. [4]). We do not distinguish between a function on $\mathbb{R}^3$ and an operator of multiplication by this function in $L^2(\mathbb{R}^3)$. We use the notation $x_+ = \max\{x, 0\}$, $x \in \mathbb{R}$.

Some notation concerning the eigenvalue counting functions of compact operators will be introduced in the beginning of Section 3.

2 Preliminary resolvent estimates

First we recall the formulae for the spectral decomposition of H due to [3]. Let $f : \mathbb{R}^3 \to \mathbb{C}$ be of the Schwartz class $\mathcal{S}(\mathbb{R}^3)$. Define

$$\widetilde{f}(p) = (2\pi)^{-1} \int_{\mathbb{R}^3} e^{-i\langle p_\perp, x_\perp\rangle} \operatorname{Ai}(x_1 - p_1) f(x) dx, \tag{2.1}$$

where Ai is the Airy function. The mapping $f \mapsto \widetilde{f}$ is a unitary transformation in $L^2(\mathbb{R}^3)$, and thus can be extended onto the whole of $L^2(\mathbb{R}^3)$. This unitary transformation diagonalises H:

$$\widetilde{g}(p) = (p_1 + |p_\perp|^2)\widetilde{f}(p), \text{ if } f \in \operatorname{Dom}(H), \quad g = Hf. \tag{2.2}$$

Let $E(\lambda)$ be the spectral projection of H associated with the interval $(-\infty, \lambda)$. By (2.2), one has

$$(E(\lambda)f, f) = \int_{p_1 + |p_\perp|^2 < \lambda} |\widetilde{f}(p)|^2 dp = \int_{\mathbb{R}^2} dp_\perp \int_{-\infty}^{\lambda - |p_\perp|^2} |\widetilde{f}(p)|^2 dp_1.$$

Differentiation with respect to λ gives

$$\frac{d}{d\lambda}(E(\lambda)f, f) = (J(\lambda)f, J(\lambda)f), \quad f \in \mathcal{S}(\mathbb{R}^3),$$

where $J(\lambda) : \mathcal{S}(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ is given by

$$(J(\lambda)f)(p_\perp) = (2\pi)^{-1} \int_{\mathbb{R}^3} e^{-i\langle p_\perp, x_\perp\rangle} \operatorname{Ai}(x_1 + |p_\perp|^2 - \lambda) f(x) dx.$$

Lemma 2.1. *Let $F \in L^2_{1/4}(\mathbb{R}^3)$. Then, for any $\lambda \in \mathbb{R}$, the operator*

$$J(\lambda)F : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$$

belongs to the Hilbert-Schmidt class S_2 and obeys the estimates

$$\|J(\lambda)F\|^2_{S_2} \le C \int_{\mathbb{R}^3} |F(x)|^2 \langle x_1 - \lambda\rangle^{1/2} dx, \tag{2.3}$$

$$\|J(\lambda)F\|_{S_2} \to 0 \quad \text{as } \lambda \to -\infty. \tag{2.4}$$

If, in addition, $F \in L^2_{\frac{1}{4}+\frac{\theta}{2}}(\mathbb{R}^3)$, $0 < \theta < 1$, then one has the estimate

$$\|(J(\lambda) - J(\lambda'))F\|^2_{S_2} \le C|\lambda - \lambda'|^{2\theta} \int_{\mathbb{R}^3} |F(x)|^2 \langle x_1 - \lambda\rangle^{\frac{1}{2}+\theta} dx, \quad |\lambda - \lambda'| \leqslant 1. \tag{2.5}$$

Before proceeding to the proof, let us collect necessary estimates for the Airy function and its derivative. As it is well known [1], one has the estimates

$$|\operatorname{Ai}(x)| \leqslant C\langle x\rangle^{-1/4} \exp(-\tfrac{2}{3}x_+^{3/2}), \quad |\operatorname{Ai}'(x)| \leqslant C\langle x\rangle^{1/4} \exp(-\tfrac{2}{3}x_+^{3/2}), \quad x \in \mathbb{R}. \tag{2.6}$$

From here one easily obtains the following bounds for any $a \in \mathbb{R}$:

$$\int_a^\infty \mathrm{Ai}(t)^2 dt \leq C\langle a\rangle^{\frac{1}{2}} \exp(-\tfrac{4}{3}a_+^{3/2}), \quad a \in \mathbb{R}, \tag{2.7}$$

$$\int_a^\infty \mathrm{Ai}'(t)^2 dt \leq C\langle a\rangle^{\frac{3}{2}}, \quad a \in \mathbb{R}. \tag{2.8}$$

Proof. Using (2.7), we get

$$\|J(\lambda)F\|_{S_2}^2 = (2\pi)^{-2} \int_{\mathbb{R}^3} dx |F(x)|^2 \int_{\mathbb{R}^2} dp_\perp |\mathrm{Ai}(x_1 + |p_\perp|^2 - \lambda)|^2$$
$$\leq C \int_{\mathbb{R}^3} |F(x)|^2 \langle x_1 - \lambda\rangle^{1/2} \exp(-\tfrac{4}{3}(x_1 - \lambda)_+^{3/2}) dx.$$

This yields (2.3) and (2.4). Combining the two estimates (2.6), we obtain

$$|\mathrm{Ai}(x+y) - \mathrm{Ai}(x)| \leq C|y|^\theta \langle x\rangle^{-\frac{1}{4}+\frac{\theta}{2}} \exp(-\tfrac{2}{3}(x-1)_+^{3/2}), \quad |y| \leq 1. \tag{2.9}$$

As in the first part of the proof, the estimate (2.9) yields (2.5). ∎

Recall the notation $W = \sqrt{V}$. For $\mathrm{Im}\, z > 0$, denote

$$T(z) = W(H - z)^{-1} W. \tag{2.10}$$

Lemma 2.2. *Let V satisfy the bound* (1.1)*. Then for all $\lambda \in \mathbb{R}$, the limit $T(\lambda + i0)$ exists in the operator norm, is compact and continuous in $\lambda \in \mathbb{R}$ and satisfies the estimate*

$$\|T(\lambda + i0)\| \to 0, \quad \lambda \to -\infty. \tag{2.11}$$

Moreover, $\mathrm{Im}\, T(\lambda + i0)$ belongs to the trace class S_1, is continuous in the trace norm and obeys the estimate

$$\|\mathrm{Im}\, T(\lambda + i0)\|_{S_1} \to 0, \quad \lambda \to -\infty. \tag{2.12}$$

Proof. Let us prove the inclusion

$$W(H - z)^{-1} \in S_2, \quad \mathrm{Im}\, z \neq 0. \tag{2.13}$$

By the diagonalisation formulae, it suffices to check that

$$W(x) e^{-i\langle p_\perp, x_\perp\rangle} \mathrm{Ai}(x_1 - p_1)(p_1 + |p_\perp|^2 - z)^{-1} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3, dx\, dp).$$

Using (2.7) and the elementary estimate $\langle a - b\rangle \leq C\langle a\rangle\langle b\rangle$, we obtain

$$\int_{\mathbb{R}^3} dx\, |W(x)|^2 \int_{\mathbb{R}^3} dp\; \mathrm{Ai}(x_1 - p_1)^2 |p_1 + |p_\perp|^2 - z|^{-2}$$
$$= \pi \int_{\mathbb{R}^3} dx\, |W(x)|^2 \int_{\mathbb{R}} dt |t - z|^{-2} \int_{x_1 - t} ds\; \mathrm{Ai}(s)^2$$
$$\leq C \int_{\mathbb{R}^3} dx\, |W(x)|^2 \langle x_1\rangle^{1/2} \int_{\mathbb{R}} dt |t - z|^{-2} \langle t\rangle^{1/2} < \infty.$$

From (2.13) by a standard approximation argument it follows that $W|(H-z)^{-1}|^{1/2}$ is compact and therefore $T(z)$ is compact for any $z \in \mathbb{C} \setminus \mathbb{R}$.

2. Let $\lambda_0 \in \mathbb{R}$; denote $\delta = (\lambda_0 - 1, \lambda_0 + 1)$. Let us prove that for all $\lambda \in \delta$, the limit $T(\lambda + i0)$ exists in the operator norm, is continuous in $\lambda \in \delta$ and $\operatorname{Im} T(\lambda + i0)$ belongs to the trace class and is continuous in $\lambda \in \delta$ the trace norm.

3. Let $\omega_{2\delta}$ be the characteristic function of the interval $(\lambda_0 - 2, \lambda_0 + 2)$. For $\operatorname{Im} z > 0$, let

$$T_{2\delta}(z) = W(H-z)^{-1}\omega_{2\delta}(H)W, \qquad \widetilde{T}_{2\delta}(z) = W(H-z)^{-1}(I - \omega_{2\delta}(H))W.$$

Consider the operator $T_{2\delta}(z)$. By the spectral theorem,

$$T_{2\delta}(z) = \int_{\mathbb{R}} \frac{\omega_{2\delta}(t)}{t-z}(J(t)W)^*(J(t)W)dt.$$

From here and Lemma 2.1 we obtain that the limit $T_{2\delta}(\lambda + i0)$ exists for all $\lambda \in \delta$ and is continuous in $\lambda \in \delta$ in the trace norm. On the other hand, for all $\lambda \in \delta$, the limit $\widetilde{T}_{2\delta}(\lambda + i0)$ exists in the operator norm, is self-adjoint and continuous in $\lambda \in \delta$ in the operator norm.

4. It remains to prove the estimates (2.11), (2.12). The estimate (2.11) is borrowed from [9]. From the relation

$$\operatorname{Im} T(\lambda + i0) = \pi (J(\lambda)W)^*(J(\lambda)W)$$

we obtain $\|\operatorname{Im} T(\lambda + i0)\|_{S_1} = \pi \|J(\lambda)W\|_{S_2}^2$. Now (2.12) follows from (2.4). ∎

3 Representation for the spectral shift function

1. First we recall the standard notation for the eigenvalue counting functions of compact operators. For a compact self-adjoint operator A, we denote by $\lambda_n(A)$ its eigenvalues enumerated with multiplicities taken into account and for $s > 0$ define

$$n_+(s; A) = \sharp\{n : \lambda_n(A) > s\}, \quad n_-(s; A) = \sharp\{n : \lambda_n(A) < -s\}.$$

For a compact but not necessarily self-adjoint operator T, we use the notation

$$n(s; T) \equiv n_+(s^2; T^*T), \quad s > 0$$

for the counting function of its s-numbers. We will frequently use H. Weyl's inequalities for the sum and the product of compact operators

$$n(s_1 + s_2; T_1 + T_2) \leqslant n(s_1; T_1) + n(s_2; T_2), \quad s_1, s_2 > 0, \tag{3.1}$$

$$n(s_1 s_2; T_1 T_2) \leqslant n(s_1; T_1) + n(s_2; T_2), \quad s_1, s_2 > 0. \tag{3.2}$$

In terms of the counting function, the trace norm and the Hilbert-Schmidt norm of an operator T can be expressed as

$$\|T\|_{S_1} = \int_0^\infty n(s; T)ds, \qquad \|T\|_{S_2}^2 = 2\int_0^\infty s\,n(s; T)ds. \tag{3.3}$$

From here using monotonicity of $n(s,T)$, it is easy to obtain the estimates

$$n(s;T) \leqslant s^{-1}\|T\|_{S_1}, \qquad n(s;T) \leqslant s^{-2}\|T\|_{S_2}^2. \tag{3.4}$$

2. Consider the limiting values $T(\lambda+i0)$, $\lambda \in \mathbb{R}$, discussed in Lemma 2.2 and denote

$$A(\lambda) = \operatorname{Re} T(\lambda+i0), \quad B(\lambda) = \operatorname{Im} T(\lambda+i0).$$

Let us define the function

$$\xi(\lambda,t) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\tau}{1+\tau^2} n_-(1/t; A(\lambda)+\tau B(\lambda)). \tag{3.5}$$

The integral in (3.5) converges due to the inclusions $A(\lambda) \in S_\infty$, $B(\lambda) \in S_1$ (see (3.7) below). In this section, we prove

Theorem 3.1. *The function $\xi(\lambda,t)$, defined by* (3.5)*, satisfies the hypothesis of Proposition 1.1.*

Formula (3.5) is the representation for the SSF due to [11]. In [11], this formula has been proven in an abstract operator theoretic framework in the case when the perturbation V is a trace class operator. Some extensions of this result to the cases when V is not in the trace class but satisfies some 'relatively trace class' conditions are given in [11]. None of these extensions covers the case of the Stark operator, so below we prove the representation (3.5) in the required case. The last part of the following proof uses an argument similar to the one of [6].

Proof of Theorem 3.1. Without the loss of generality we can consider the case $t=1$.

Let P_n be a sequence of orthogonal projections in $L^2(\mathbb{R}^3)$ of a finite rank such that $P_n \xrightarrow{s} I$, $n \to \infty$. Denote $V^{(n)} = WP_nW$; this operator is of the trace class (even of a finite rank) and therefore we can invoke the general operator theoretic construction of [10]. This construction asserts the existence of the SSF $0 \leqslant \xi_n \in L^1(\mathbb{R})$ such that for all $\varphi \in C_0^\infty(\mathbb{R})$, the trace formula

$$\operatorname{Tr}(\varphi(H+V^{(n)}) - \varphi(H)) = \int_{-\infty}^{\infty} \varphi'(\lambda)\xi_n(\lambda)d\lambda \tag{3.6}$$

holds true. As $V^{(n)}$ is of the trace class, we can also apply the result of [11] which gives a formula representation for the SSF:

$$\xi_n(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\tau}{1+\tau^2} n_-(1; P_nA(\lambda)P_n + \tau P_nB(\lambda)P_n), \quad \text{a.e. } \lambda \in \mathbb{R}.$$

Below we prove that:

(i) For a.e. $\lambda \in \mathbb{R}$, $\xi_n(\lambda) \to \xi(\lambda,1)$ as $n \to \infty$, where $\xi(\lambda,t)$ is defined by (3.5).

(ii) One has $0 \leqslant \xi_n(\lambda) \leqslant \xi(\lambda,1)$ and $\xi(\cdot,1) \in L^\infty_{\text{loc}}(\mathbb{R})$.

(iii) For all $\varphi \in C_0^\infty(\mathbb{R})$, one has $\varphi(H+V) - \varphi(H) \in S_1$ and

$$\|\varphi(H+V^{(n)}) - \varphi(H+V)\|_{S_1} \to 0 \text{ as } n \to \infty.$$

(iv) The function $\xi(\lambda, 1)$ satisfies the normalization condition (1.3).

Using (i)–(iii) and applying the dominated convergence theorem, we can pass to the limit as $n \to \infty$ in (3.6). Together with (iv), this shows that the function $\xi(\lambda, 1)$ satisfies the hypothesis of Proposition 1.1.

Let us prove (i). By Lemma 2.2, the operator $A(\lambda)$ is compact and $B(\lambda)$ is of the trace class. It follows (see [16, Lemma 6.1.3]) that

$$\|A(\lambda)(I - P_n)\| = \|(I - P_n)A(\lambda)\| \to 0, \qquad \text{as } n \to \infty,$$
$$\|B(\lambda)(I - P_n)\|_{S_1} = \|(I - P_n)B(\lambda)\|_{S_1} \to 0, \qquad \text{as } n \to \infty.$$

From here we obtain

$$\|P_n A(\lambda) P_n - A(\lambda)\| \to 0, \quad n \to \infty,$$
$$\|P_n B(\lambda) P_n - B(\lambda)\|_{S_1} \to 0, \quad n \to \infty.$$

Now we can apply the stability result [11, Lemma 2.5], which asserts that the r.h.s. of (3.5) is operator norm continuous in $A(\lambda)$ and trace norm continuous in $B(\lambda)$ as long as -1 is not an eigenvalue of $A(\lambda) + iB(\lambda)$ (here $t = 1$). It is easy to see that the last condition holds true for a.e. $\lambda \in \mathbb{R}$. Indeed, if $T(\lambda + i0)\psi + \psi = 0$ for some $\psi \in L^2(\mathbb{R}^3)$, then, by [8, Proposition 4.2], λ is an eigenvalue of $H + V$, but $H + V$ can have at most countably many eigenvalues.

Let us prove (ii) and (iv). The inequality $\xi_n(\lambda) \leqslant \xi(\lambda, 1)$ follows from variational considerations. Fix $\varepsilon \in (0, 1)$; using Weyl's inequality (3.1) for the sum of operators and formula (3.3), we obtain

$$\begin{aligned} \xi(\lambda; 1) \leqslant n_-(1 - \varepsilon; A(\lambda)) + \frac{1}{\pi}\int_{-\infty}^{0} \frac{d\tau}{1+\tau^2} n_-(\varepsilon; \tau B(\lambda)) = n_-(1-\varepsilon; A(\lambda)) \\ + \frac{1}{\pi}\int_0^\infty \frac{ds}{1+s^2} n_+(s; \varepsilon^{-1} B(\lambda)) \leqslant n_-(1-\varepsilon; A(\lambda)) + \frac{1}{\pi\varepsilon}\|B(\lambda)\|_{S_1}. \end{aligned} \quad (3.7)$$

The estimate (3.7), together with Lemma 2.2, yields the inclusion $\xi(\cdot, 1) \in L^\infty_{\mathrm{loc}}$ and the normalisation condition (1.3).

Let us prove (iii). Let us first consider the case $\varphi(\lambda) = (\lambda - z)^{-1}$, $\operatorname{Im} z \neq 0$. From the inclusion (2.13) by using the resolvent identity, we get

$$W(H + V - z)^{-1} \in S_2. \quad (3.8)$$

Further, by a repeated application of the resolvent identity, one has

$$\begin{aligned} &(H + V^{(n)} - z)^{-1} - (H + V - z)^{-1} \\ &\quad = (I + (H + V^{(n)} - z)^{-1} W(I - P_n)W)(H + V - z)^{-1} W(I - P_n) W (H + V - z)^{-1}, \end{aligned}$$

and so

$$\begin{aligned}
&\|(H+V^{(n)}-z)^{-1}-(H+V-z)^{-1}\|_{S_1}\\
&\qquad\leq (1+C|\mathrm{Im}\, z|^{-1})\left\|\frac{H+V-i}{H+V-z}\right\|^2\|(I-P_n)W(H+V-i)^{-1}\|_{S_2}^2\\
&\qquad\qquad\leq \frac{C}{|\mathrm{Im}\, z|^2}(1+\tfrac{C}{|\mathrm{Im}\, z|})\|(I-P_n)W(H+V-i)^{-1}\|_{S_2}^2. \quad (3.9)
\end{aligned}$$

Note that by (3.8), we have (see [16, Lemma 6.1.3]):

$$\|(I-P_n)W(H+V-i)^{-1}\|_{S_2}^2\to 0 \text{ as } n\to\infty.$$

Next, let $\varphi\in C_0^\infty(\mathbb{R})$, and let $\widetilde{\varphi}\in C_0^\infty(\mathbb{C})$ be an almost analytic extension of φ, i.e. a function with the properties $\widetilde{\varphi}\,|_{\mathbb{R}}=\varphi$, $\frac{\partial\widetilde{\varphi}(z)}{\partial\overline{z}}=O(|\mathrm{Im}\, z|^\infty)$ as $\mathrm{Im}\, z\to 0$ (see e.g. [7] for the discussion of almost analytic extension and references). Then

$$\varphi(\lambda)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\widetilde{\varphi}(z)}{\partial\overline{z}}\frac{1}{\lambda-z}dx\,dy,\quad z=x+iy,$$

and therefore, by (3.9),

$$\begin{aligned}
\|\varphi(H+V^{(n)})-\varphi(H+V)\|_{S_1}&\leq\frac{1}{\pi}\int_{\mathbb{C}}\left|\frac{\partial\widetilde{\varphi}(z)}{\partial\overline{z}}\right|\|(H+V^{(n)}-z)^{-1}-(H+V-z)^{-1}\|_{S_1}dx\,dy\\
&\leq C\|(I-P_n)W(H+V-i)^{-1}\|_{S_2}^2\int_{\mathbb{C}}\left|\frac{\partial\widetilde{\varphi}(z)}{\partial\overline{z}}\right||\mathrm{Im}\, z|^{-2}(1+\tfrac{C}{|\mathrm{Im}\, z|})dx\,dy\to 0
\end{aligned}$$

as $n\to\infty$. ∎

4 Proof of Theorem 1.2

Our main operator theoretic tool is formula (3.5). The analytic ingredient of the proof is the following

Proposition 4.1. *Assume* (1.1) *and* (1.4) *and let* $r>0$. *Then*

$$\lim_{s\to+0}\sup_{|\lambda|\leq r}|s^{\varkappa}n_-(s,A(\lambda))-\Delta(l)|=0 \tag{4.1}$$

and for any $\theta>0$

$$\sup_{|\lambda|\leq r}n(s,B(\lambda))=O(s^{-\frac{6}{2l-1}-\theta}),\quad s\to+0. \tag{4.2}$$

Note that $\frac{6}{2l-1}<\varkappa$ if $l>7/2$.

Proposition 4.1 is proved in Section 9 by using the technique of [2]. In Sections 5–8, the necessary statements for the proof of Proposition 4.1 are prepared. Now, using Proposition 4.1 and formula (3.5), we shall prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to prove two estimates:

$$\limsup_{t\to+\infty} \operatorname*{ess\,sup}_{|\lambda|\le r} (t^{-\varkappa}\xi(\lambda,t) - \Delta(l)) \leqslant 0, \tag{4.3}$$

$$\liminf_{t\to+\infty} \operatorname*{ess\,inf}_{|\lambda|\le r} (t^{-\varkappa}\xi(\lambda,t) - \Delta(l)) \geqslant 0. \tag{4.4}$$

Let us prove the upper bound (4.3); the lower bound (4.4) can be proven similarly. Fix $\varepsilon \in (0,1)$. Using the representation (3.5), arguing similarly to (3.7), and using the bound (4.2), we obtain

$$\operatorname*{ess\,sup}_{|\lambda|\le r}\{t^{-\varkappa}\xi(\lambda,t) - \Delta(l)\} \leqslant \sup_{|\lambda|\le r}\{t^{-\varkappa}n_-(1-\varepsilon; tA(\lambda)) - (1-\varepsilon)^{-\varkappa}\Delta(l)\}$$
$$+ t^{\frac{6}{2l-1}+\theta-\varkappa}C(\theta,r)\frac{1}{\pi}\int_0^\infty \frac{d\tau}{1+\tau^2}(\varepsilon\tau)^{-\frac{6}{2l-1}-\theta} + ((1-\varepsilon)^{-\varkappa}-1)\Delta(l) \tag{4.5}$$

for any $\theta > 0$. Choose $\theta > 0$ sufficiently small so that $\frac{6}{2l-1} + \theta - \varkappa < 0$. Then, by (4.1) and (4.5), we get

$$\limsup_{t\to+\infty} \operatorname*{ess\,sup}_{|\lambda|\le r} (t^{-\varkappa}\xi(\lambda,t) - \Delta(l)) \leqslant ((1-\varepsilon)^{-\varkappa}-1)\Delta(l).$$

As $\varepsilon \in (0,1)$ can be taken arbitrary small, we obtain (4.3). ∎

5 Potential cutoff

Let $B_m := [-m,m)^3 \subset \mathbb{R}^3$, let θ_m be the characteristic function of B_m in $\mathbb{R}^3$, and let $\widetilde{\theta}_m(x) = 1 - \theta_m(x)$, $x \in \mathbb{R}^3$. Our first step in the proof of Proposition 4.1 is to replace V by the product $V\theta_{2m}$. We will choose m depending on the parameter s in (4.1), (4.2). The choice will be different for the proof of (4.1) and (4.2). For the proof of (4.2), we shall take $m = s^{-2/(2l-1)}$, and for the proof of (4.1) we shall take $m = s^{-q}$, where q is any exponent in the range

$$\tfrac{1}{l} < q < \tfrac{13}{10}\tfrac{1}{l+1}. \tag{5.1}$$

For the rest of the paper, let us fix the parameter $r > 0$ from the hypothesis of Proposition 4.1.

Proposition 5.1. *Assume* (1.1) *and let* $r > 0$. *Then:*

(i) For all $\theta > 0$ *and* $\varepsilon > 0$, *one has*

$$\sup_{|\lambda|\le r}\big(n(s,B(\lambda)) - n(s(1-\varepsilon),\theta_{2m}B(\lambda)\theta_{2m})\big)_+ = O(s^{-\frac{6}{2l-1}-\theta}), \quad s\to+0, \tag{5.2}$$

where $m = s^{-\frac{2}{2l-1}}$.

(ii) For all q *in the range* (5.1) *and all* $\varepsilon \in (0,1)$, *the following estimates hold true*

$$n_-(s,\theta_{2m}A(\lambda)\theta_{2m}) \le n_-(s,A(\lambda)), \quad \lambda\in\mathbb{R}, \tag{5.3}$$

$$\sup_{|\lambda|\le r}\{n_-(s,A(\lambda)) - n_-(s(1-\varepsilon),\theta_{2m}A(\lambda)\theta_{2m})\} = o(s^{-\varkappa}), \quad s\to+0, \tag{5.4}$$

where $m = s^{-q}(1-\varepsilon)^{-q}$.

In order to prove Proposition 5.1, we need to prepare several auxiliary statements. First, let $\chi_1 \in C_0^\infty(\mathbb{R})$, $0 \leq \chi_1 \leq 1$, $\chi_1(t) = 1$ for $|t| \leqslant \frac{3}{2}r$, $\chi_1(t) = 0$ for $|t| \geqslant 2r$, and let $\chi_2(t) = 1 - \chi_1(t)$. Let

$$R_1(z) = R(z)\chi_1(H), \quad R_2(z) = R(z)\chi_2(H). \tag{5.5}$$

Note that the operator $R_2(\lambda)$ is well defined and bounded for $\lambda \in (-\frac{3}{2}r, \frac{3}{2}r)$.

Proposition 5.2. *Let $F, G \in L^2_{1/4}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and let R_2 be as in (5.5). Then for all $s > 0$, one has*

$$\sup_{|\lambda|\leq r} n(s, FR_2(\lambda)G) \leqslant C(r)s^{-3/2}\|F\|_{2,1/4}\|G\|_{2,1/4}\|F\|_\infty^{1/2}\|G\|_\infty^{1/2}.$$

Proof. 1. For $a > 0$, write

$$FR_2(\lambda)G = T_a^{(1)}(\lambda) + T_a^{(2)}(\lambda), \qquad T_a^{(1)}(\lambda) = F\chi_{(\lambda-a,\lambda+a)}(H)R_2(\lambda)G,$$

where $\chi_{(\lambda-a,\lambda+a)}$ is the characteristic function of the interval $(\lambda-a, \lambda+a)$. Let us estimate the trace norm of $T_a^{(1)}(\lambda)$ and the operator norm of $T_a^{(2)}(\lambda)$. For $\|T_a^{(2)}(\lambda)\|$, one has a trivial estimate

$$\|T_a^{(2)}(\lambda)\| \leqslant \|F\|_\infty\|G\|_\infty a^{-1}. \tag{5.6}$$

Next,

$$\|T_a^{(1)}(\lambda)\|_{S_1} \leqslant \int_{\lambda-a}^{\lambda+a} \chi_2(t)|\lambda - t|^{-1}\|J(t)F\|_{S_2}\|J(t)G\|_{S_2}dt.$$

From here, using (2.3) and the estimate

$$\int_{\mathbb{R}^3} |F(x)|^2\langle x_1 - t\rangle^{1/2}dx \leqslant 2\langle t\rangle^{1/2} \int_{\mathbb{R}^3} |F(x)|^2\langle x_1\rangle^{1/2}dx,$$

we obtain

$$\|T_a^{(1)}(\lambda)\|_{S_1} \leqslant C(r)a^{1/2}\|F\|_{2,1/4}\|G\|_{2,1/4}. \tag{5.7}$$

2. For a given $s > 0$, choose $a = s^{-1}2\|F\|_\infty\|G\|_\infty$. Using (5.6), (5.7), and (3.4), we obtain

$$\begin{aligned} n(s, T_a^{(1)}(\lambda)+T_a^{(2)}(\lambda)) \leqslant n(s/2, T_a^{(1)}(\lambda))+n(s/2, T_a^{(2)}(\lambda)) \leqslant 2s^{-1}C(r)a^{1/2}\|F\|_{2,1/4}\|G\|_{2,1/4} \\ = C'(r)s^{-3/2}\|F\|_{2,1/4}\|G\|_{2,1/4}\|F\|_\infty^{1/2}\|G\|_\infty^{1/2}, \end{aligned}$$

which completes the proof. ∎

Lemma 5.3. *Let F, G belong to $L^2_\rho(\mathbb{R}^3)$ for some $\rho > 1/4$ and let R_1 be as in (5.5). Then for all $\lambda \in \mathbb{R}$, the limit $FR_1(\lambda + i0)G$ exists in the trace class and one has*

$$\sup_{|\lambda|\leq r} \sup_{0<\mu<1} \|FR_1(\lambda + i\mu)G\|_{S_1}^2 \leq C(\rho, r)\|F\|_{2,\rho}^2\|G\|_{2,\rho}^2.$$

Proof. immediately follows from Lemma 2.1 ∎

Lemma 5.4. *Assume* (1.1). *Then for any $\alpha > 0$ and $\theta > 0$, there exists a constant $C = C(\alpha, \theta, r) > 0$ such that*

$$\sup_{|\lambda|\le r} n(s, \theta_m T(\lambda)\widetilde{\theta}_{2m}) \leqslant C s^{-\alpha} m^{3+\theta}, \qquad s > 0, \quad m \geqslant 1.$$

The proof of Lemma 5.4 is given in Section 6.

Proof of the Proposition 5.1. First note that (5.3) follows from variational considerations. Let us prove (5.2) and (5.4). Denote

$$M(\lambda) := T(\lambda) - \theta_{2m} T(\lambda)\theta_{2m}, \quad T_j(\lambda) := \lim_{\mu\to+0} W R_j(\lambda + i\mu) W, \quad j = 1, 2,$$

where R_1, R_2 are as defined by (5.5). Write $M(\lambda) = M_1(\lambda) + M_2(\lambda) + M_3(\lambda)$, where

$$\begin{aligned}
M_1(\lambda) &= \widetilde{\theta}_{2m} T_1(\lambda)\widetilde{\theta}_{2m} + (\theta_{2m} - \theta_m)T_1(\lambda)\widetilde{\theta}_{2m} + \widetilde{\theta}_{2m} T_1(\lambda)(\theta_{2m} - \theta_m)\\
M_2(\lambda) &= \widetilde{\theta}_{2m} T_2(\lambda)\widetilde{\theta}_{2m} + (\theta_{2m} - \theta_m)T_2(\lambda)\widetilde{\theta}_{2m} + \widetilde{\theta}_{2m} T_2(\lambda)(\theta_{2m} - \theta_m)\\
M_3(\lambda) &= \theta_m T(\lambda)\widetilde{\theta}_{2m} + \widetilde{\theta}_{2m} T(\lambda)\theta_m.
\end{aligned}$$

Let us apply Lemma 5.3 to M_1, Proposition 5.2 to M_2, and Lemma 5.4 to M_3. We obtain:

$$\begin{aligned}
&\sup_{|\lambda|\le r} n(s, M_1(\lambda)) \leqslant C(\theta, r)s^{-1} m^{\frac{7}{2} - l + \theta},\\
&\sup_{|\lambda|\le r} n(s, M_2(\lambda)) \leqslant C(r)s^{-3/2} m^{\frac{7}{2} - \frac{3l}{2}},\\
&\sup_{|\lambda|\le r} n(s, M_3(\lambda)) \leqslant C(\alpha, \theta, r)s^{-\alpha} m^{3+\theta}.
\end{aligned} \tag{5.8}$$

Taking $m = s^{-\frac{2}{2l-1}}$, combining the above three inequalities and using Weyl's inequality for the sum of operators, we obtain

$$\sup_{|\lambda|\le r} n(s, M(\lambda)) = O(s^{-\frac{6}{2l-1} - \theta}), \quad s \to +0, \tag{5.9}$$

for any $\theta > 0$. Combining (5.9) with the estimate

$$n(s, B(\lambda)) \leqslant n(s(1-\varepsilon), \theta_{2m} B(\lambda)\theta_{2m}) + 2n(s\varepsilon; M(\lambda)), \quad s > 0, \quad 0 < \varepsilon < 1,$$

yields (5.2).

Finally, taking $m = (1-\varepsilon)^{-q} s^{-q}$ and combining the inequalities (5.8), we obtain

$$\sup_{|\lambda|\le r} n(s, M(\lambda)) = o(s^{-\varkappa}), \quad s \to +0.$$

Together with the estimate

$$n_-(s, A(\lambda)) \leqslant n_-(s(1-\varepsilon), \theta_{2m} A(\lambda)\theta_{2m}) + 2n(s\varepsilon; M(\lambda)), \quad s > 0,$$

this yields (5.4). ∎

6 Proof of Lemma 5.4

First, we need a statement similar to Lemma 5.3. Denote $\nabla_j = \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$ and let R_1 be as in (5.5).

Lemma 6.1. *Let $F \in L^2_{\rho+\frac{1}{2}}(\mathbb{R}^3)$, $G \in L^2_\rho(\mathbb{R}^3)$ for some $\rho > 1/4$ and let $r > 0$. Then for all $\lambda \in \mathbb{R}$ and for $j = 1, 2, 3$, the limit $\lim_{\mu\to+0} F\nabla_j R_1(\lambda + i\mu)G$ exists in the trace class and*

$$\sup_{|\lambda|\le r}\sup_{0<\mu<1} \|F\nabla_j R_1(\lambda + i\mu)G\|^2_{S_1} \le C(\rho, r)\|F\|^2_{2,\rho+\frac{1}{2}}\|G\|^2_{2,\rho}$$

Proof. Similarly to Lemma 2.1, one obtains for $j = 1, 2, 3$:

$$\sup_{|\lambda|\le r}\|F\nabla_j J(\lambda)^*\|^2_{S_2} \leqslant C(r)\int_{\mathbb{R}^3}|F(x)|^2\langle x_1\rangle^{3/2}dx,$$

$$\sup_{|\lambda|\le r}\sup_{|\lambda-\lambda'|\le 1}\|F\nabla_j(J(\lambda) - J(\lambda'))^*\|^2_{S_2} \leqslant C(r)|\lambda-\lambda'|^{2\theta}\int_{\mathbb{R}^3}|F(x)|^2\langle x_1\rangle^{\frac{3}{2}+\theta}dx,$$

for $\theta > 0$ and $|\lambda - \lambda'| \le 1$. From here, as in Lemma 5.3, one obtains the desired estimates. ∎

Let the operator $R_2(z) = (H - z)^{-1}\chi_2(H)$ be as defined by (5.5) and let $\mathcal{R}_2(x, y, z)$, $x, y \in \mathbb{R}^3$, be the integral kernel of $R_2(z)$.

Proposition 6.2. *For any $M = 2, 3, 4, \dots$ and $\mu \in (0, 1)$, one has the estimates*

$$|\mathcal{R}_2(x, y, i\mu)| \leqslant C_M|x - y|^{-2M}\langle x_1 + y_1\rangle^M\langle x_1\rangle^{\frac{1}{4}}\langle y_1\rangle^{\frac{1}{4}}, \quad x \ne y, \tag{6.1}$$

$$\left|\frac{\partial}{\partial x_j}\mathcal{R}_2(x, y, i\mu)\right| \leqslant C_M|x - y|^{-2M}(1 + |x - y|^{-1})\langle x_1 + y_1\rangle^M\langle x_1\rangle^{\frac{3}{4}}\langle y_1\rangle^{\frac{1}{4}}, \quad x \ne y. \tag{6.2}$$

Proof. 1. We start with a simple estimate. Suppose that a function $\phi : \mathbb{R} \to \mathbb{C}$ obeys the estimate $|\phi(t)| \leqslant C\langle t\rangle^m$, $m < -3/2$. Then, by (2.7),

$$\int_{\mathbb{R}^3}|\mathrm{Ai}(x_1 - p_1)|^2|\phi(p_1 + |p_\perp|^2)|dp = \pi\int_{-\infty}^{\infty}dt\,|\phi(t)|\int_{x_1-t}^{\infty}ds\ \mathrm{Ai}(s)^2$$
$$\le C\int_{-\infty}^{\infty}dt\,|\phi(t)|\langle x_1 - t\rangle \le C\langle x_1\rangle^{1/2}.$$

Next, for $x, y, p \in \mathbb{R}^3$, let

$$T(x, y; p) = (2\pi)^{-2}\,\mathrm{Ai}(x_1 - p_1)\,\mathrm{Ai}(y_1 - p_1)e^{ip_\perp(x_\perp - y_\perp)};$$

then by Cauchy-Schwartz and the previous estimate,

$$\int_{\mathbb{R}^3}|T(x, y; p)\phi(p_1 + p_\perp^2)|dp \leqslant C\langle x_1\rangle^{1/4}\langle y_1\rangle^{1/4}. \tag{6.3}$$

2. Let $R_2'(z) = \frac{d}{dz}R_2(z) = (H-z)^{-2}\chi_2(H)$, and let $\mathcal{R}_2'(x,y;z)$, $x,y \in \mathbb{R}^3$, be the corresponding integral kernel. By the diagonalization formulae (2.1), (2.2), we have

$$\mathcal{R}_2'(x,y;z) = \int_{\mathbb{R}^3} T(x,y;p)\phi_z(p_1+p_\perp^2)dp, \qquad \phi_z(t) = \chi_2(t)(t-z)^{-2}, \tag{6.4}$$

and by (6.3), the above integral converges absolutely.

Consider the differential expression

$$B(p,s) = -\Delta_{p_\perp} - \frac{\partial^4}{\partial p_1^4} - 4p_1\frac{\partial^2}{\partial p_1^2} - 6\frac{\partial}{\partial p_1} + 2s\frac{\partial^2}{\partial p_1^2}, \quad s \in \mathbb{R}.$$

By a straightforward computation, one checks the identity

$$B(p, x_1+y_1)T(x,y;p) = |x-y|^2 T(x,y;p). \tag{6.5}$$

Using this identity and integrating by parts in (6.4), we obtain

$$\mathcal{R}_2'(x,y;z) = |x-y|^{-2}\int_{\mathbb{R}^3} T(x,y;p)B^\dagger(p,x_1+y_1)\phi_z(p_1+p_\perp^2)dp,$$

where

$$B^\dagger(p,s) = -\Delta_{p_\perp} - \frac{\partial^4}{\partial p_1^4} - 4p_1\frac{\partial^2}{\partial p_1^2} - 2\frac{\partial}{\partial p_1} + 2s\frac{\partial^2}{\partial p_1^2}, \quad s \in \mathbb{R}$$

is the formal conjugate of $B(p,s)$. Next, one has

$$\mathcal{R}_2(x,y,z) = \int_{+i\infty}^{z} \mathcal{R}_2'(x,y;\zeta)d\zeta = |x-y|^{-2}\int_{\mathbb{R}^3} T(x,y;p)B^\dagger(p,x_1+y_1)\psi_z(p_1+p_\perp^2)dp, \tag{6.6}$$

where

$$\psi_z(t) = \int_{+i\infty}^{z} \phi_\zeta(t)d\zeta = \chi_2(t)(t-z)^{-1}.$$

Let us discuss the convergence of the integral in the r.h.s. of (6.6). By the explicit formula

$$\begin{aligned} B^\dagger(p,s)\psi_z(p_1+p_\perp^2) = -4\psi_z'(p_1+p_\perp^2) - 4(p_1+p_\perp^2)\psi_z''(p_1+p_\perp^2) - \psi_z^{(4)}(p_1+p_\perp^2) \\ - 2\psi_z'(p_1+p_\perp^2) + 2s\psi_z''(p_1+p_\perp^2), \end{aligned} \tag{6.7}$$

we conclude that $B^\dagger(p,s)\psi_z(p_1+p_\perp^2) = \tau_z(p_1+p_\perp^2)$, where $|\tau_z(t)| \leqslant C\langle t\rangle^{-2}$. Thus, by the estimate (6.3), the integral in the r.h.s. of (6.6) is absolutely convergent.

3. Repeatedly integrating by parts in (6.6) and using (6.5), we obtain for any $M \in \mathbb{N}$:

$$\mathcal{R}_2(x,y,z) = |x-y|^{-2M}\int_{\mathbb{R}^3} T(x,y;p)(B^\dagger(p,x_1+y_1))^M\psi_z(p_1+p_\perp^2)dp \tag{6.8}$$

Note that the function ψ_z satisfies the estimates

$$\left|\left(\frac{d}{dt}\right)^n \psi_{i\mu}(t)\right| \leqslant C_n\langle t\rangle^{-1-n}, \quad t\in\mathbb{R}, \quad \mu\in(0,1).$$

Using this fact and iterating (6.7), we obtain

$$|(B^\dagger(p,x_1+y_1))^M\psi_{i\mu}(p_1+p_\perp^2)| \leqslant C_M\langle x_1+y_1\rangle^M\langle p_1+p_\perp^2\rangle^{-1-M}, \quad \mu\in(0,1).$$

Combining the last estimate, (6.8), and (6.3), we obtain (6.1).

4. It remains to prove the estimate (6.2). Similarly to (6.3), using (2.7) and (2.8), one obtains

$$\int_{\mathbb{R}^3}\left|\frac{\partial}{\partial x_j}T(x,y;p)\phi(p_1+p_\perp^2)\right| dp \leqslant C\langle x_1\rangle^{3/4}\langle y_1\rangle^{1/4}, \quad j=1,2,3,$$

if $|\phi(t)|\leqslant\langle t\rangle^m$, $m<-5/2$. Differentiating (6.8) and applying the last estimate, we obtain (6.2). ∎

Proof of Lemma 5.4. 1. Let $e_1=(1,0,0)\in\mathbb{R}^3$ and $X_\lambda(x)=W(x+\lambda e_1)\langle x\rangle^{l/2}$. The shift of variable $x\mapsto x-\lambda e_1$ generates a unitary transformation in $L^2(\mathbb{R}^3)$ which transforms $T(\lambda)$ into $X_\lambda T_0 X_\lambda$, where $T_0=\lim_{\mu\to+0}W_0R(i\mu)W_0$, $W_0(x)=\langle x\rangle^{-l/2}$. As $\sup_{|\lambda|\le r}\|X_\lambda\|\le C(r)$, it suffices to prove that

$$n(s,\theta_m T_0\widetilde{\theta}_{2m}) \le C(\alpha,\theta)s^{-\alpha}m^{3+\theta}, \quad s>0, \quad m\ge 1. \tag{6.9}$$

In fact, we shall prove that

$$\sup_{0<\mu<1} n(s,\theta_m W_0R(i\mu)W_0\widetilde{\theta}_{2m}) \le C(\alpha,\theta)s^{-\alpha}m^{3+\theta}, \quad s>0, \quad m\ge 1, \tag{6.10}$$

from which (6.9) clearly follows.

2. Fix an integer $k\geqslant 1$, $k+1\geqslant\frac{2}{\alpha}$. Let $\varphi_0,\dots,\varphi_{k-1}\in C_0^\infty(\mathbb{R}^3)$ be the functions with the following properties:

$$\varphi_j(x)=\begin{cases} 1, \text{ if } x\in B_{(1+\frac{j}{k}+\frac{1}{4k})}, \\ 0, \text{ if } x\in\mathbb{R}^3\setminus B_{(1+\frac{j}{k}+\frac{3}{4k})}, \end{cases}$$

$$0\leqslant\varphi_j(x)\leqslant 1, \quad x\in\mathbb{R}^3, \text{ and } \sup_{x\in\mathbb{R}^3}|\partial^\gamma\varphi_j(x)|\leqslant C(\gamma), \quad \gamma\in\mathbb{Z}_+^3.$$

Next, for any $m\geqslant 1$, let $\varphi_j^m(x)=\varphi_j(x/m)$.

For any $\varphi\in C_0^\infty(\mathbb{R}^3)$, let $[\varphi]$ be the first order differential operator $[\varphi]=-(\Delta\varphi)-2\langle\nabla\varphi,\nabla\rangle$; it is easy to see that

$$\varphi R(i\mu)-R(i\mu)\varphi = R(i\mu)[\varphi]R(i\mu), \quad \mu\neq 0.$$

From here, using the obvious identities $\theta_m\varphi_0^m=\theta_m$, $\widetilde{\theta}_{2m}\varphi_0^m=0$, we obtain

$$\theta_m R(i\mu)\widetilde{\theta}_{2m} = \theta_m(\varphi_0^m R(i\mu)-R(i\mu)\varphi_0^m)\widetilde{\theta}_{2m} = \theta_m R(i\mu)[\varphi_0^m]R(i\mu)\widetilde{\theta}_{2m}.$$

In the same way, using the identities $[\varphi_0^m]\varphi_1^m = [\varphi_0^m]$, $\varphi_1^m\widetilde{\theta}_{2m} = 0$, we get

$$\theta_m R(i\mu)[\varphi_0^m]R(i\mu)\widetilde{\theta}_{2m} = \theta_m R(i\mu)[\varphi_0^m]R(i\mu)[\varphi_1^m]R(i\mu)\widetilde{\theta}_{2m}.$$

Proceeding this way, we eventually obtain

$$\theta_m R(i\mu)\widetilde{\theta}_{2m} = \theta_m R(i\mu)[\varphi_0^m]R(i\mu)[\varphi_1^m]\cdots R(i\mu)[\varphi_{k-1}^m]R(i\mu)\widetilde{\theta}_{2m}. \tag{6.11}$$

Let ζ_j^m be the characteristic function of the set $B_{m(1+\frac{j}{k}+\frac{3}{4k})} \setminus B_{m(1+\frac{j}{k}+\frac{1}{4k})}$ in $\mathbb{R}^3$. Clearly, $[\varphi_j^m] = \zeta_j^m[\varphi_j^m]$, $j = 0, \ldots, k-1$, so (6.11) implies

$$\theta_m W_0 R(i\mu)W_0\widetilde{\theta}_{2m} = (\theta_m W_0 R(i\mu)\zeta_0^m)([\varphi_0^m]R(i\mu)\zeta_1^m)\cdots([\varphi_{k-1}^m]R(i\mu)W_0\widetilde{\theta}_{2m}). \tag{6.12}$$

3. Let us estimate the eigenvalue counting function for each term in the product in the r.h.s. of (6.12). In each term, let us split the resolvent as $R(i\mu) = R_1(i\mu) + R_2(i\mu)$ according to (5.5). First let us consider the terms with $R_1(i\mu)$. By Lemma 5.3 and Lemma 6.1, we obtain for any $\theta > 0$:

$$\sup_{0<\mu<1} \|\theta_m W_0 R_1(i\mu)\zeta_0^m\|_{S_1} \leqslant Cm^{\frac{7}{4}+\theta}, \tag{6.13}$$

$$\sup_{0<\mu<1} \|[\varphi_j^m]R_1(i\mu)\zeta_{j+1}^m\|_{S_1} \leqslant Cm^{3+\theta}, \quad j = 0, \ldots, k-2, \tag{6.14}$$

$$\|[\varphi_{k-1}^m]R_1(i\mu)W_0\widetilde{\theta}_{2m}\|_{S_1} \leqslant Cm^{\frac{5}{4}+\theta}. \tag{6.15}$$

Here the constant C may depend on θ and k but not on $m \geqslant 1$.

Next, consider the terms with $R_2(i\mu)$. Using Proposition 6.2 with a sufficiently large M, we obtain:

$$\sup_{0<\mu<1} \|\theta_m W_0 R_2(i\mu)\zeta_0^m\|_{S_2} \leqslant C, \tag{6.16}$$

$$\sup_{0<\mu<1} \|[\varphi_j^m]R_2(i\mu)\zeta_{j+1}^m\|_{S_2} \leqslant C, \quad j = 0, \ldots, k-2, \tag{6.17}$$

$$\sup_{0<\mu<1} \|[\varphi_{k-1}^m]R_2(i\mu)W_0\widetilde{\theta}_{2m}\|_{S_2} \leqslant C, \tag{6.18}$$

where the constant C may depend on k, but not on $m \geqslant 1$.

4. Combining the estimates (6.13)-(6.18), using Weyl's inequality (3.1) for the sum of operators and the estimate (3.4), we obtain for all $m \geqslant 1$,

$$\sup_{0<\mu<1} n(s; \theta_m W_0 R(i\mu)\zeta_0^m) \leqslant Cs^{-2}m^{\frac{7}{4}+\theta},$$

$$\sup_{0<\mu<1} n(s; [\varphi_j^m]R(i\mu)\zeta_{j+1}^m)) \leqslant Cs^{-2}m^{3+\theta}, \quad j = 0, \ldots, k-2,$$

$$\sup_{0<\mu<1} n(s; [\varphi_{k-1}^m]R(i\mu)W_0\widetilde{\theta}_{2m}) \leqslant Cs^{-2}m^{\frac{5}{4}+\theta}.$$

Substituting these estimates into (6.12) and using Weyl's inequality (3.2) for the product, we obtain

$$\sup_{0<\mu<1} n(s; \theta_m W_0 R(i\mu)W_0\widetilde{\theta}_{2m}) \leqslant Cs^{-\frac{2}{k+1}}m^{3+\theta}, \quad s > 0, \quad m \geq 1.$$

By the choice of k, the last estimate yields (6.10). ∎

7 Reduction to a boundary value problem

For $m \geq 1$ and $t \geq 1$, consider the operators $H^D(B_{2m}) = -\Delta + x_1$ and $H^D(B_{2m}) + tV\theta_m$ in $L^2(B_{2m})$ with the Dirichlet boundary conditions. The spectra of these operators are discrete and bounded from below. One has

$$\begin{aligned}\xi(\lambda; H^D(B_{2m}) + tV\theta_m, H^D(B_{2m}))\\ = N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m), \quad \text{a.e. } \lambda \in \mathbb{R},\end{aligned}$$

where $N(\lambda; A)$ stands for the total number of eigenvalues (counting multiplicities) of an operator A in the interval $(-\infty, \lambda]$.

Proposition 7.1. *Assume* (1.1). *Then:*

(i) For all $\theta > 0$ *one has*

$$\sup_{|\lambda| \leq r} n(s, \theta_m B(\lambda)\theta_m) = O(s^{-\frac{6}{2l-1}-\theta}), \quad s \to +0, \tag{7.1}$$

where $m = s^{-\frac{2}{2l-1}}$.

(ii) For all q *in the range* (5.1) *and all* $\varepsilon \in (0,1)$, *one has*

$$\sup_{|\lambda| \leq r} \big(N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + (1-\varepsilon)s^{-1}V\theta_m) - n_-(s, \theta_m A(\lambda)\theta_m)\big)_+ = o(s^{-\varkappa}) \tag{7.2}$$

$$\sup_{|\lambda| \leq r} \big(n_-(s, \theta_m A(\lambda)\theta_m) - N(\lambda; H^D(B_{2m})) + N(\lambda; H^D(B_{2m}) + (1+\varepsilon)s^{-1}V\theta_m)\big)_+ = o(s^{-\varkappa}), \tag{7.3}$$

as $s \to +0$, *where* $m = 2s^{-q}$.

For any parallelepiped $Q = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$, let us define the operators $H^D(Q)$, $H^N(Q)$ acting in $L_2(Q)$ and generated by differential operation $-\Delta + x_1$ and Dirichlet and Neumann boundary conditions respectively. We need precise spectral asymptotics for $H^D(Q)$ and $H^N(Q)$:

Proposition 7.2. *There exists a constant* $C > 0$ *independent of* a_i, b_i, $i = 1, 2, 3$, *such that for any* $\lambda \in \mathbb{R}$ *and* $i = D, N$, *one has*

$$\begin{aligned}\left| N(\lambda; H^i(Q)) - \frac{1}{15\pi^2}(b_2 - a_2)(b_3 - a_3)[(\lambda - a_1)_+^{5/2} - (\lambda - b_1)_+^{5/2}] \right|\\ \leqslant C(\lambda - a_1)_+ \max\{(\lambda - a_1)_+, (b_2 - a_2), (b_3 - a_3)\}^2 + C.\end{aligned} \tag{7.4}$$

Proof. 1. First consider the eigenvalues of the operator

$$-\frac{d^2}{dx^2} + x \text{ on } (\gamma, \infty), \text{ Dirichlet boundary condition at } x = \gamma. \tag{7.5}$$

The eigenvalues λ of the operator (7.5) coincide with the solutions to the equation $\mathrm{Ai}(\gamma-\lambda)=0$. Thus, denoting by $N_\gamma(0)$ the number of eigenvalues of (7.5) in the interval $(-\infty,0]$, we see that $N_\gamma(0)$ coincides with the number of zeros of the Airy function $\mathrm{Ai}(t)$ for $t\geq\gamma$. From the asymptotics of zeros of the Airy function (see [1, (10.4.94)]) we obtain

$$|N_\gamma(0)-\frac{2}{3\pi}(-\gamma)_+^{3/2}|\leq C,\quad \gamma\in\mathbb{R} \tag{7.6}$$

for some constant C.

2. Next, let $N(\lambda;(a_1,b_1))$ be the number of eigenvalues in $(-\infty,\lambda]$ of the operator

$$-\frac{d^2}{dx^2}+x \text{ on } (a_1,b_1) \tag{7.7}$$

with Dirichlet or Neumann boundary conditions at $x=a_1,b_1$. By a shift of variable x, the number $N(\lambda;(a_1,b_1))$ equals the number of eigenvalues in $(-\infty,0]$ of the operator $-\frac{d^2}{dx^2}+x$ on $(a_1-\lambda,b_1-\lambda)$ with the corresponding boundary conditions at $x=a_1-\lambda,b_1-\lambda$.

Changing a boundary condition or introducing a new boundary condition for a one-dimensional Schrödinger operator is a rank one perturbation and so it can change the number of eigenvalues in any interval by at most one. Thus, we obtain

$$|N(\lambda;(a_1,b_1))-(N_{a_1-\lambda}(0)-N_{b_1-\lambda}(0))|\leq 2. \tag{7.8}$$

Combining (7.6) and (7.8), we get

$$|N(\lambda;(a_1,b_1))-\frac{2}{3\pi}((\lambda-a_1)_+^{3/2}-(\lambda-b_1)_+^{3/2})|\leq C \tag{7.9}$$

for some constant $C>0$ independent of a_i, b_i, λ. Note also that the l.h.s. of (7.9) vanishes for $\lambda<a_1$.

3. Consider the operator $H^D(Q)$. By separation of variables, the spectrum of $H^D(Q)$ can be explicitly described:

$$\sigma(H^D(Q))=\{\lambda_{j_1}+(j_2/l_2)^2+(j_3/l_3)^2 : j_1,j_2,j_3\in\mathbb{N}\},$$

where λ_{j_1} are the eigenvalues of the Dirichlet problem (7.7) and $l_2=(b_2-a_2)/\pi$, $l_3=(b_3-a_3)/\pi$. Thus, using (7.9), we obtain

$$\begin{aligned}N(\lambda;H^D(Q))&=\sharp\{j_1,j_2,j_3\in\mathbb{N}:\lambda_{j_1}+(j_2/l_2)^2+(j_3/l_3)^2\leq\lambda\}\\&=\sum_{j_1,j_2\in\mathbb{N}}N(\lambda-(j_2/l_2)^2+(j_3/l_3)^2;(a_1,b_1))\\&=\frac{2}{3\pi}\sum_{j_2,j_3\in\mathbb{N}}\Big(\big(\lambda-b_1-(j_2/l_2)^2-(j_3/l_3)^2\big)_+^{3/2}-\big(\lambda-a_1-(j_2/l_2)^2-(j_3/l_3)^2\big)_+^{3/2}\Big)+R,\end{aligned} \tag{7.10}$$

where the error term R can be estimated as

$$|R|\leq C\sharp\{j_2,j_3\in\mathbb{N}:\lambda-(j_2/l_2)^2-(j_3/l_3)^2\geq a_1\}\leq C(l_2l_3(\lambda-a_1)_++1)$$

with C independent of the parameters a_i, b_i, λ.

4. It remains to transform the r.h.s. of (7.10) into the required form appearing in (7.4). Denote

$$f(x_2, x_3) = (\lambda - a_1 - (x_2/l_2)^2 - (x_3/l_3)^2)_+^{3/2}, \quad x_2, x_3 \in \mathbb{R}.$$

Using the monotonicity of f and arguing as in the proof of the integral test of convergence of series, one obtains

$$\left| \sum_{j_2, j_3 \in \mathbb{N}} f(j_2, j_3) - \int_0^\infty \int_0^\infty f(x_2, x_3) dx_2 dx_3 \right|$$
$$\leq \int_0^\infty f(0, x_3) dx_3 + \int_0^\infty f(x_2, 0) dx_2 + f(0, 0). \quad (7.11)$$

Computing the integrals in (7.11), we obtain

$$\left| \sum_{j_2, j_3 \in \mathbb{N}} \left(\lambda - a_1 - (j_2/l_2)^2 - (j_3/l_3)^2\right)_+^{3/2} - \frac{(\lambda - a_1)_+^{5/2}}{10\pi^2}(b_2 - a_2)(b_3 - a_3) \right|$$
$$\leqslant C l_2 (\lambda - a_1)_+^2 + C l_3 (\lambda - a_1)_+^2 + (\lambda - a_1)_+^{3/2},$$

and the same estimate with a_1 replaced by b_1. This gives (7.4) for $i = D$. The case $i = N$ differs only by the fact that the indices j_2, j_3 run over $\mathbb{Z}_+$; all estimates remain valid. ■

Proof of Proposition 7.1.

1. For any $m \geqslant 1$ and $\lambda \in (-r, r)$, let $E_m(\lambda)$ be the spectral projection of the operator $H^D(B_{2m})$ associated with the interval $(\lambda - m^{-1/2}, \lambda + m^{-1/2})$. By Proposition 7.2, we have the estimate

$$\sup_{|\lambda| \leq r} \operatorname{rank} E_m(\lambda) \leqslant C m^3, \quad m \geqslant 1. \quad (7.12)$$

Denote $R^m(z) = (H^D(B_{2m}) - zI)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, and

$$R_1^m(\lambda + i\mu) = R^m(\lambda + i\mu) E_m(\lambda), \quad \mu > 0,$$
$$R_2^m(\lambda + i\mu) = R^m(\lambda + i\mu)(I - E_m(\lambda)), \quad \mu \geqslant 0.$$

Let us prove that for any $\beta > 0$ and $\alpha > 0$, there exists a constant $C = C(\beta, \alpha)$ such that

$$\sup_{|\lambda| \leq r} \sup_{0 < \mu < 1} n(s; \theta_m T(\lambda + i\mu)\theta_m - \theta_m W R_2^m(\lambda + i\mu) W \theta_m) \leqslant C s^{-\alpha} m^{3+\beta}, \quad s > 0, \quad m \geq 1. \quad (7.13)$$

2. Let us follow the construction used in the proof of Lemma 5.4. Choose an integer $k \in \mathbb{N}$ sufficiently large so that $k > 1/\alpha$ and $k > l/\beta$. Let $\varphi_0^m, \dots \varphi_{k-1}^m$ and $\zeta_0^m, \dots \zeta_{k-1}^m$ be the same functions as in the proof of Lemma 5.4. It is easy to see that for all $g \in L^2(B_{2m})$, $\varphi \in C_0^\infty(B_{2m})$, $f = \varphi(R(z) - R^m(z))g$, $\operatorname{Im} z > 0$, one has

$$f \in \operatorname{Dom} H, \quad (H - zI)f = [\varphi](R(z) - R^m(z))g, \quad (7.14)$$

where $[\varphi]$ is the first order differential operator $[\varphi] = -(\Delta\varphi) - 2\langle\nabla\varphi, \nabla\rangle$. The identity (7.14) can be written as

$$\varphi(R(z) - R^m(z))g = R(z)[\varphi](R(z) - R^m(z))g, \quad g \in L^2(B_{2m}). \tag{7.15}$$

From (7.15) and the identity $\varphi_0^m\theta_m = \theta_m$ we get

$$\begin{aligned}\theta_m(R(z) - R^m(z))\theta_m &= \theta_m\varphi_0^m(R(z) - R^m(z))\theta_m \\ &= \theta_m R(z)[\varphi_0^m](R(z) - R^m(z))\theta_m, \quad \operatorname{Im} z > 0, \quad m \geq 1.\end{aligned}$$

Proceeding as in the proof of Lemma 5.4, we eventually obtain

$$\theta_m(R(z) - R^m(z))\theta_m = \theta_m R(z)[\varphi_0^m]R(z)[\varphi_1^m]\dots R(z)[\varphi_{k-1}^m](R(z) - R^m(z))\theta_m.$$

From here we get

$$\begin{aligned}&\theta_m W R(z) W\theta_m - \theta_m W R_2^m(z) W\theta_m = M_1(z) + M_2(z), \\ &M_1(z) = \theta_m W R_1^m(z) W\theta_m - \theta_m W R(z)[\varphi_0^m]\cdots R(z)[\varphi_{k-1}^m]R_1^m(z)W\theta_m \\ &M_2(z) = (\theta_m W R(z)[\varphi_0^m])(\zeta_0^m R(z)[\varphi_1^m])\cdots(\zeta_{k-2}^m R(z)[\varphi_{k-1}^m])(\zeta_{k-1}^m(R(z) - R_2^m(z))W\theta_m).\end{aligned}$$

By (7.12), we obtain

$$n(s; M_1(\lambda + i\mu)) \leq Cm^3, \quad s > 0,$$

uniformly over $\mu > 0$ and $\lambda \in (-r, r)$. It remains to estimate the eigenvalue counting function for $M_2(\lambda + i\mu)$. As in the proof of Lemma 5.4, using Lemma 5.3, Lemma 6.1, and Proposition 6.2, we obtain for any $\gamma > 0$,

$$\sup_{|\lambda|\leq r}\sup_{0<\mu<1} n(s; \theta_m W R(\lambda + i\mu)[\varphi_0^m]) \leqslant C(\gamma, r)s^{-2}m^{\frac{5}{4}+\gamma}, \tag{7.16}$$

$$\sup_{|\lambda|\leq r}\sup_{0<\mu<1} n(s; \zeta_j^m R(\lambda + i\mu)[\varphi_{j+1}^m]) \leqslant C(\gamma, r)s^{-2}m^{3+\gamma}, \quad j = 0, \dots, k-2. \tag{7.17}$$

We also have trivial estimates of the operator norm (recall that $W_0(x) = \langle x\rangle^{-l/2}$):

$$\begin{aligned}&\sup_{|\lambda|\leq r}\sup_{0<\mu<1} \|\zeta_{k-1}^m(R(\lambda + i\mu) - R_2^m(\lambda + i\mu))W\theta_m\| \\ &\quad\leqslant \sup_{|\lambda|\leq r}\sup_{0<\mu<1}\left(\|\zeta_{k-1}^m W_0^{-1}\|\|W_0 R(\lambda + i\mu)W\| + C\|R_2^m(\lambda + i\mu)\|\right) \leqslant Cm^{l/2}, \quad m \geq 1.\end{aligned} \tag{7.18}$$

Combining the estimates (7.16)–(7.18) and using Weyl's inequality (3.2) for the product, we obtain

$$\begin{aligned}&\sup_{|\lambda|\leq r}\sup_{0<\mu<1} n(s; M_2(\lambda + i\mu)) \\ &\quad\leqslant \sup_{|\lambda|\leq r}\sup_{0<\mu<1} n(s; Cm^{l/2}(\theta_m W R(\lambda + i\mu)[\varphi_0^m])\dots(\zeta_{k-2}^m R(\lambda + i\mu)[\varphi_{k-1}^m])) \\ &\qquad\leqslant C(r, k, \gamma)(sm^{-l/2})^{-\frac{2}{k}}m^{3+\gamma} = C(r, k, \gamma)s^{-\frac{2}{k}}m^{3+\gamma+\frac{l}{k}}.\end{aligned}$$

By the choice of k and by the arbitrariness of $\gamma > 0$, we obtain (7.13).

3. As $\operatorname{Im} \theta_m W R_2^m(\lambda) W \theta_m = 0$, from (7.13) we immediately obtain (7.1). It remains to prove (7.2) and (7.3).

From (7.13), taking $m = 2s^{-q}$ and choosing $\alpha > 0$ and $\beta > 0$ sufficiently small, we obtain

$$\sup_{|\lambda|\le r} \{n_-(s; \theta_m A(\lambda)\theta_m) - n_-(s(1-\varepsilon); \theta_m W R_2^m(\lambda) W\theta_m)\} \leqslant o(s^{-\varkappa}), \quad s \to +0. \quad (7.19)$$

By the Birman-Schwinger principle and the estimate (7.12), we have

$$\begin{aligned} n_-(s(1-\varepsilon); \theta_m W R_2^m(\lambda) W\theta_m) &= N(\lambda; H^D(B_{2m})(I - E_m(\lambda))) \\ &- N(\lambda; H^D(B_{2m})(I - E_m(\lambda)) + \tfrac{1}{s(1-\varepsilon)}(I - E_m(\lambda))\theta_m V (I - E_m(\lambda))) \\ &= N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + \tfrac{1}{s(1-\varepsilon)} V\theta_m) + O(m^3) \quad (7.20) \end{aligned}$$

as $m \to \infty$, uniformly on $\lambda \in (-r, r)$. Combining (7.19) and (7.20), we obtain the upper bound (7.3). The lower bound (7.2) can be obtained in a similar fashion. ∎

8 Variational technique

Proposition 8.1. *Assume* (1.1) *and* (1.4). *Then for all* q *in the range* (5.1) *and all* $c > 0$*, one has*

$$\lim_{t\to+\infty} \sup_{|\lambda|\le r} |t^{-\varkappa}(N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m)) - \Delta(l)| = 0,$$

where $m = ct^q$.

Proof. 1. Let us prove the upper bound

$$\limsup_{t\to+\infty} \sup_{|\lambda|\le r} \{t^{-\varkappa}(N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m)) - \Delta(l)\} \le 0.$$

Using the same method, one easily proves the lower bound

$$\liminf_{t\to+\infty} \inf_{|\lambda|\le r} \{t^{-\varkappa}(N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m)) - \Delta(l)\} \ge 0.$$

2. Consider the partition

$$B_{2m} = B_m \cup \bigcup_{n=1}^{56} B_m^n,$$

where the cubes B_m^n, $j = 1, \dots, 56$ are disjoint, have the side length m, and form a tiling of the 'cubic layer' $B_{2m} \setminus B_m$. Recall that for a parallelepiped $Q \subset \mathbb{R}^3$, by $H^N(Q)$ we denote the differential operator $-\Delta + x_1$ in $L^2(Q)$ with the Neumann boundary conditions. By variational considerations (Dirichlet-Neumann bracketing), one has

$$N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m) \leqslant N(\lambda; H^N(B_m)) - N(\lambda; H^D(B_m) + tV) + \sum_{n=1}^{56} \{N(\lambda; H^N(B_m^n)) - N(\lambda; H^D(B_m^n))\}. \quad (8.1)$$

Let us estimate the sum in the r.h.s. of (8.1). Using Proposition 7.2, we get for all j,

$$\sup_{|\lambda|\le r} \{N(\lambda; H^N(B_m^n)) - N(\lambda; H^D(B_m^n)) \leqslant Ct^{3q} = o(t^{\varkappa}),$$

as $t \to \infty$. Thus,

$$\sup_{|\lambda|\le r} \{N(\lambda; H^D(B_{2m})) - N(\lambda; H^D(B_{2m}) + tV\theta_m)\} \leqslant \sup_{|\lambda|\le r} \{N(\lambda; H^N(B_m)) - N(\lambda; H^D(B_m) + tV)\} + o(t^{\varkappa}), \quad t \to \infty.$$

3. Fix $\delta > 0$ and consider the following partition of $\mathbb{R}^3$: $\mathbb{R}^3 = \cup_j Q_j$, $Q_j = \delta t^{\frac{1}{l+1}}([0,1)^3 + j)$, $j = (j_1, j_2, j_3) \in \mathbb{Z}^3$. For simplicity of further exposition, let us assume that t runs through a sequence such that $m/(\delta t^{\frac{1}{l+1}}) = ct^{q - \frac{1}{l+1}}/\delta$ are integers. By this assumption, each cube B_m contains an integer number of cubes Q_j. It is not difficult to lift this assumption, but this would make our exposition rather cumbersome.

By variational considerations,

$$N(\lambda; H^N(B_m)) \leqslant \sum_{Q_j \subset B_m} N(\lambda; H^N(Q_j)),$$

$$N(\lambda; H^D(B_m) + tV) \geqslant \sum_{Q_j \subset B_m} N(\lambda; H^D(Q_j) + tV) \geqslant \sum_{Q_j \subset B_m} N(-t \operatorname*{ess\,sup}_{x \in Q_j} V(x); H^D(Q_j)),$$

By Proposition 7.2, we obtain:

$$N(\lambda; H^N(Q_j)) \leqslant \frac{1}{15\pi^2} (\delta t^{\frac{1}{l+1}})^2 \{(\lambda - t^{\frac{1}{l+1}}\delta j_1)_+^{5/2} - (\lambda - t^{\frac{1}{l+1}}\delta(j_1+1))_+^{5/2}\} + Ct^{\frac{2}{l+1}+q}.$$

Similarly,

$$N(\lambda - t \operatorname*{ess\,sup}_{x \in Q_j} V(x); H^D(Q_j)) \geqslant \frac{1}{15\pi^2} (\delta t^{\frac{1}{l+1}})^2 \{(\lambda - t^{\frac{1}{l+1}}\delta j_1 - t \operatorname*{ess\,sup}_{x \in Q_j} V(x))_+^{5/2} - (\lambda - t^{\frac{1}{l+1}}\delta(j_1+1) - t \operatorname*{ess\,sup}_{x \in Q_j} V(x))_+^{5/2}\} - Ct^{\frac{2}{l+1}+q},$$

where C does not depend on $\lambda \in (-r, r)$. Thus, we obtain

$$N(\lambda; H^N(B_m)) - N(\lambda; H^D(B_m) + tV) \leqslant \frac{1}{6\pi^2} I(t, \delta, \lambda) + o(t^{\varkappa}), \quad t \to \infty,$$

uniformly in $\lambda \in (-r, r)$, where

$$I(t,\delta,\lambda) = \sum_{Q_j \subset B_m} (\delta t^{\frac{1}{l+1}})^3 \int_0^1 ds \{(\lambda - t^{\frac{1}{l+1}}\delta(j_1+s))_+^{3/2} - (\lambda - t^{\frac{1}{l+1}}\delta(j_1+s) - t \operatorname*{ess\,sup}_{x\in Q_j} V(x))_+^{3/2}\}. \tag{8.2}$$

4. It remains to use some elementary analysis to show that

$$\limsup_{\delta\to+0} \limsup_{t\to+\infty} \sup_{|\lambda|\le r} \{t^{-\varkappa} I(t,\delta,\lambda) - \Delta(l)\} = 0. \tag{8.3}$$

First we would like to replace V by V_∞ in the expression (8.2) for $I(t,\delta,\lambda)$. For any $\varepsilon > 0$, there exists sufficiently large $R > 0$ such that

$$(1-\varepsilon)V_\infty(x) \leqslant V(x) \leqslant (1+\varepsilon)V_\infty(x), \qquad |x| \geqslant R.$$

Let us show that the values of $V(x)$ for $|x| \le R$ do not affect the asymptotics (8.3). Indeed, for all sufficiently large t, the behaviour of $V(x)$ for $|x| \leqslant R$ affects only the eight terms in the sum (8.2) corresponding to the eight cubes Q_j that 'touch' the origin. These terms can be estimated above by $C\delta^{9/2}t^{\varkappa}$, and therefore they do not give contribution to the asymptotics (8.3). Thus, we obtain:

$$I(t,\delta,\lambda) \leqslant \sum_{Q_j \subset B_m} (\delta t^{\frac{1}{l+1}})^3 \int_0^1 ds \{(\lambda - t^{\frac{1}{l+1}}\delta(j_1+s))_+^{3/2} - (\lambda - t^{\frac{1}{l+1}}\delta(j_1+s) - t(1+\varepsilon)V_\infty(\hat{x}_j))_+^{3/2}\} + C\delta^{9/2}t^{\varkappa}, \tag{8.4}$$

where $\hat{x}_j \in \overline{Q_j}$ is the point where $\max_{x\in\overline{Q_j}} V_\infty(x)$ is attained. The bound (8.4) is uniform in $\lambda \in (-r, r)$. A similar lower bound holds true with $(1-\varepsilon)$ instead of $(1+\varepsilon)$.

5. Denote $\hat{y}_j = t^{-\frac{1}{l+1}}\hat{x}_j$. With this notation, the sum in r.h.s. of (8.4) becomes:

$$\sum_{Q_j \subset B_m} (\delta t^{\frac{1}{l+1}})^3 \int_0^1 ds \{(\lambda - t^{\frac{1}{l+1}}\delta(j_1+s))_+^{3/2} - (\lambda - t^{\frac{1}{l+1}}\delta(j_1+s) - t(1+\varepsilon)V_\infty(\hat{x}_j))_+^{3/2}\}$$
$$= t^{\varkappa} \sum_{Q_j \subset B_m} \delta^3 \int_0^1 ds \{(\lambda t^{-\frac{1}{l+1}} - \delta(j_1+s))_+^{3/2} - (\lambda t^{-\frac{1}{l+1}} - \delta(j_1+s) - (1+\varepsilon)V_\infty(\hat{y}_j))_+^{3/2}\}.$$

The r.h.s. of the last equality can be represented as an integral

$$t^{\varkappa} \int_0^1 ds \int_{\mathbb{R}^3} \phi_\delta(y,s,t,\lambda)dy,$$

where $\phi_\delta(y,s,t,\lambda)$ is a step function,

$$\phi_\delta(y,s,t,\lambda) = \begin{cases} (\lambda t^{-\frac{1}{l+1}} - \delta(j_1+s))_+^{3/2} - (\lambda t^{-\frac{1}{l+1}} - \delta(j_1+s) - (1+\varepsilon)V_\infty(\hat{y}_j))_+^{3/2}, \\ \qquad \text{if } t^{\frac{1}{l+1}}y \in Q_j \subset B_m, \\ 0, \ \text{if } t^{\frac{1}{l+1}}y \notin B_m \end{cases}$$

For a.e. $y \in \mathbb{R}^3$, we have

$$\phi_\delta(y,s,t,\lambda) \to (-y_1)_+^{3/2} - (-y_1 - (1+\varepsilon)V_\infty(y))_+^{3/2},$$

as $t \to \infty$, $\delta \to +0$, uniformly in $\lambda \in (-r,r)$. One also has an integrable bound for ϕ_δ:

$$\phi_\delta(y,s,t,\lambda) \leqslant (-y_1+1)_+^{3/2} - (-y_1+1-C|y|^{-l})_+^{3/2},$$

for all sufficiently large t, sufficiently small $\delta > 0$, and all $\lambda \in (-r,r)$. By dominated convergence, we get

$$\lim_{\delta\to+0}\lim_{t\to+\infty}\int_0^1 ds \int_{\mathbb{R}^3}\phi_\delta(y,s,t,\lambda)dy = \int_{\mathbb{R}^3}[(-x_1)_+^{3/2} - (-x_1-(1+\varepsilon)V_\infty(x))_+^{3/2}]dx.$$

As $\varepsilon > 0$ can be chosen arbitrary small, we get (8.3). ∎

9 Proof of Proposition 4.1: putting the estimates together

The estimate (4.2) follows directly from Proposition 5.1(i) and Proposition 7.1(i).

Let us prove the estimate (4.1). Fix a number $\varepsilon \in (0,1)$ and an exponent q in the range (5.1). Let $\sigma = (1-\varepsilon)s$, $m = 2\sigma^{-q}$, $t = (1+\varepsilon)\sigma^{-1}$. Using Proposition 5.1(ii), Proposition 7.1(ii), and Proposition 8.1 (with $c = \frac{1}{2}(1+\varepsilon)^{-q}$), we get

$$\begin{aligned}
&\sup_{|\lambda|\leq r}\{s^\varkappa n_-(s;A(\lambda)) - \Delta(l)\}\\
&\leqslant \sup_{|\lambda|\leq r}\{s^\varkappa n_-(s;A(\lambda)) - s^\varkappa n_-(\sigma;\theta_{2\sigma^{-q}}A(\lambda)\theta_{2\sigma^{-q}})\}\\
&\quad + (1-\varepsilon)^{-\varkappa}\sup_{|\lambda|\leq r}\{\sigma^\varkappa n_-(\sigma;\theta_{2\sigma^{-q}}A(\lambda)\theta_{2\sigma^{-q}})\\
&\qquad - \sigma^\varkappa(N(\lambda;H^D(B_{2m})) - N(\lambda;H^D(B_{2m})+(1+\varepsilon)\sigma^{-1}V\theta_m))\}\\
&\quad + (1-\varepsilon)^{-\varkappa}(1+\varepsilon)^\varkappa\sup_{|\lambda|\leq r}\{t^{-\varkappa}(N(\lambda;H^D(B_{2m})) - N(\lambda;H^D(B_{2m})+tV\theta_m))) - \Delta(l)\}\\
&\quad + (1-\varepsilon)^\varkappa(1+\varepsilon)^\varkappa\Delta(l) - \Delta(l)\\
&\leqslant (1-\varepsilon)^{-\varkappa}(1+\varepsilon)^\varkappa\Delta(l) - \Delta(l) + o(1), \quad s \to +0.
\end{aligned}$$

As $\varepsilon \in (0,1)$ can be taken arbitrary small, we obtain the upper bound

$$\limsup_{s\to+0}\sup_{|\lambda|\leq r}\{s^\varkappa n_-(s,A(\lambda)) - \Delta(l)\} \leqslant 0. \tag{9.1}$$

In the same way, one proves the lower bound

$$\liminf_{s\to+0}\inf_{|\lambda|\leq r}\{s^\varkappa n_-(s,A(\lambda)) - \Delta(l)\} \geqslant 0. \tag{9.2}$$

Combining (9.1) and (9.2) yields (4.1).

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