Spectral Shift Function in Strong Magnetic Fields

Vincent Bruneau, Alexander Pushnitski, Georgi Raikov

Dedicated to Professor Mikhail Birman
on the occasion of his 75th birthday

Abstract. We consider the three-dimensional Schrödinger operator H with constant magnetic field of strength $b > 0$ and continuous electric potential $V \in L^1(\mathbb{R}^3)$ which admits certain power-like estimates at infinity. We study the asymptotic behaviour as $b \to \infty$, of the spectral shift function $\xi(E; H, H_0)$ for the pair of operators (H, H_0) at energies $E = \mathcal{E}b + \lambda$, $\mathcal{E} > 0$ and $\lambda \in \mathbb{R}$ being fixed. We distinguish two asymptotic regimes. In the first one called *asymptotics far from the Landau levels* we pick $\mathcal{E}/2 \notin \mathbb{Z}_+$ and $\lambda \in \mathbb{R}$; then the main term is always of order $\sqrt{b}$, and is independent of λ. In the second asymptotic regime called *asymptotics near a Landau level* we choose $\mathcal{E} = 2q_0$, $q_0 \in \mathbb{Z}_+$, and $\lambda \neq 0$; in this case the leading term of the SSF could be of order b or $\sqrt{b}$ for different λ.

Contents

1 Introduction

The main object of investigation in the present paper is the spectral shift function (SSF) for the three-dimensional Schrödinger operator with constant magnetic field, perturbed by an electric potential which decays sufficiently fast at infinity. Let us recall the abstract setting in which the SSF for a pair of self-adjoint operators occurs. Let at first $\mathcal{T}_0$ and $\mathcal{T}$ be two self-adjoint operators acting in the same Hilbert space, and $\mathcal{T} - \mathcal{T}_0 \in S_1$ where S_1 denotes the trace class. Then there exists a unique function $\xi(.;\mathcal{T},\mathcal{T}_0) \in L^1(\mathbb{R})$ such that *the Lifshits-Krein trace formula*

$$\mathrm{Tr}(f(\mathcal{T}) - f(\mathcal{T}_0)) = \int_{\mathbb{R}} \xi(E;\mathcal{T},\mathcal{T}_0) f'(E)dE \tag{1.1}$$

holds for every $f \in C_0^\infty(\mathbb{R})$ (see [19] or e.g. [31, Theorem 8.3.3]). Let now $\mathcal{H}_0$ and $\mathcal{H}$ be two lower-bounded self-adjoint operators acting in the same Hilbert space. Assume that for some $\gamma > 0$, and $\lambda_0 \in \mathbb{R}$ lying strictly below the infima of the spectra of $\mathcal{H}_0$ and $\mathcal{H}$, we have

$$(\mathcal{H} - \lambda_0)^{-\gamma} - (\mathcal{H}_0 - \lambda_0)^{-\gamma} \in S_1. \tag{1.2}$$

Set

$$\xi(E;\mathcal{H},\mathcal{H}_0) := \begin{cases} -\xi((E-\lambda_0)^{-\gamma};(\mathcal{H}-\lambda_0)^{-\gamma},(\mathcal{H}_0-\lambda_0)^{-\gamma}) & \text{if} \quad E > \lambda_0, \\ 0 \quad \text{if} \quad E \le \lambda_0. \end{cases}$$

Then we have (see [31, Theorem 8.9.1])

$$\mathrm{Tr}(f(\mathcal{H}) - f(\mathcal{H}_0)) = \int_{\mathbb{R}} \xi(E;\mathcal{H},\mathcal{H}_0) f'(E)dE, \quad f \in C_0^\infty(\mathbb{R}).$$

The function $\xi(\cdot;\mathcal{H},\mathcal{H}_0)$ is called the SSF for the pair of the operators $(\mathcal{H},\mathcal{H}_0)$; it does not depend on the particular choice of γ and λ_0 in (1.2) and belongs to the class $L^1((\lambda_0,\infty),(E-\lambda_0)^{-\gamma-1}dE)$.
If E lies below the infimum of the spectrum of $\mathcal{H}_0$, then $\mathcal{H}$ can have only finitely many eigenvalues below E, and we have

$$\xi(E;\mathcal{H},\mathcal{H}_0) = -N(E;\mathcal{H}) \tag{1.3}$$

where $N(E;\mathcal{H})$ denotes the number of the eigenvalues of $\mathcal{H}$ lying on the interval $(-\infty, E]$, and counted with the multiplicities. On the other hand, for almost every E in the absolutely continuous spectrum of $\mathcal{H}_0$, the SSF $\xi(E;\mathcal{H},\mathcal{H}_0)$ is related to the scattering determinant $\det S(E;\mathcal{H},\mathcal{H}_0)$ for the pair $(\mathcal{H},\mathcal{H}_0)$ by *the Birman-Krein* formula

$$\det S(E;\mathcal{H},\mathcal{H}_0) = e^{-2\pi i \xi(E;\mathcal{H},\mathcal{H}_0)} \tag{1.4}$$

(see [9] or [31, Section 8.4]).
In the present paper the role of $\mathcal{H}_0$ is played by the operator $H_0 := (i\nabla + \mathbf{A})^2 - b$, essentially self-adjoint on the Schwartz class $\mathcal{S}(\mathbb{R}^3)$. Here the magnetic potential $\mathbf{A} = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ generates the constant magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A} = (0,0,b)$, $b > 0$. It is well-known that $\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = [0,\infty)$ (see [3]), where $\sigma(H_0)$ denotes the spectrum of H_0, and $\sigma_{\mathrm{ac}}(H_0)$ - its absolutely continuous spectrum. Moreover, the Landau levels $2bq$, $q \in \mathbb{Z}_+$ play the role of thresholds in the spectrum of H_0 (here and in what follows $\mathbb{Z}_+$ denotes the set of all non-negative integers).

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1, x_2)$ the variables in the plane perpendicular to the magnetic field. We assume that V satisfies

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad |V(\mathbf{x})| \leq C_0 \langle X_\perp \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3}, \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.5}$$

with $C_0 > 0$, $m_\perp > 2$, $m_3 > 1$, and $\langle x \rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $d \geq 1$. Set $H := H_0 + V$. Obviously, $\inf \sigma(H) \geq -C_0$. The role of the perturbed operator $\mathcal{H}$ is played in this paper by H. By (1.5) and the diamagnetic inequality (see e.g. [3]), $V^{1/2}(H_0 - \lambda_0)^{-1}$ with $\lambda_0 < 0$ is a Hilbert-Schmidt operator. Therefore, the resolvent identity implies $(H - \lambda_0)^{-1} - (H_0 - \lambda_0)^{-1} \in S_1$ for $\lambda_0 < \inf \sigma(H) \leq \inf \sigma(H_0)$, i.e. (1.2) holds with $\mathcal{H} = H$, $\mathcal{H}_0 = H_0$, and $\gamma = 1$, and, hence, the SSF $\xi(.; H, H_0)$ exists.
The main goal of the present article is to study the asymptotic behaviour as $b \to \infty$ of the SSF $\xi(\cdot; H(b), H_0(b))$. Note that the distance between the Landau levels grows linearly with b. We study the SSF in the scale adapted to this rate of growth. Namely, we fix a parameter $\mathcal{E} \in \mathbb{R}$ and consider $\xi(\mathcal{E}b + \lambda; H(b), H_0(b))$ as a function of $\lambda \in \mathbb{R}$. We distinguish two different types of asymptotics. The first type which we call *asymptotics far from the Landau levels*, concerns the case $\mathcal{E} \in \mathbb{R}$, $\mathcal{E}/2 \notin \mathbb{Z}_+$. The second asymptotic regime called *asymptotics of the SSF near a Landau level* occurs when $\mathcal{E} = 2bq_0$, $q_0 \in \mathbb{Z}_+$. In both cases, convergence of $\xi(\mathcal{E}b + \lambda; H(b), H_0(b))$ to its limiting value takes place in $L^\infty_{\rm loc}$ with respect to the variable λ.
The paper is organized as follows. Section 2 is devoted to the formulation of the main results and brief comments on them. In Section 3 we recall some basic properties of linear compact operators, and the representation formula of the SSF due to F. Gesztesy, K. Makarov [15]. Various auxiliary results utilized further in the proofs of the main ones, are established in Section 4. The proof of Theorems 2.1 and 2.4 containing SSF asymptotics of order $\sqrt{b}$, can be found in Section 5. The properties of the limiting quantity occurring in the asymptotic relations established in Theorem 2.3, are investigated in Section 6. Section 7 is devoted to asymptotic trace formulae for compact operators of Birman-Schwinger type. Section 8 contains the proof of Theorem 2.3 dealing with the case where the asymptotics of the SSF near a given Landau level is of order b. Finally, in Section 9, we reveal some spectral properties of the Landau Hamiltonian (see (4.1)). In particular, we prove the unitary equivalence of Toeplitz operators corresponding to different Landau levels.

2 Main Results

2.1 Formulation of the main results

In this subsection we formulate our main results as Theorems 2.1 – 2.4. Our first theorem treats the asymptotics of $\xi(\cdot; H(b), H_0(b))$ far from the Landau levels.

Theorem 2.1. *Let* (1.5) *hold. Assume that* $\mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+$, *and let* $\Delta \subset \mathbb{R}$ *be a bounded interval. Then*

$$\operatorname*{ess\,sup}_{\lambda \in \Delta} \left| \xi(\mathcal{E}b + \lambda; H(b), H_0(b)) - \frac{b^{1/2}}{4\pi^2} \sum_{q=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2q)^{-1/2} \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x} \right| = O(1), \quad b \to \infty, \tag{2.1}$$

where $[\mathcal{E}/2]$ *denotes the integer part of the real number* $\mathcal{E}/2$.

The proof of Theorem 2.1 is contained in Subsection 5.2.

The following two theorems concern the asymptotics of the SSF near a Landau level. In order to formulate our Theorem 2.3, we introduce the following family of self-adjoint operators in $L^2(\mathbb{R}, dx_3)$:

$$\chi_0 := -d^2/dx_3^2, \quad \chi(X_\perp) := \chi_0 + V(X_\perp, \cdot), \quad X_\perp \in \mathbb{R}^2,$$

which are defined on the Sobolev space $\mathrm{H}^2(\mathbb{R})$, and depend on the parameter $X_\perp \in \mathbb{R}^2$. Then (1.5) easily implies $(\chi(X_\perp) - \lambda_0)^{-1} - (\chi_0 - \lambda_0)^{-1} \in S_1$ for each $X_\perp \in \mathbb{R}^2$ and $\lambda_0 < \inf \sigma(\chi(X_\perp)) \cup \sigma(\chi_0)$. Hence, the SSF $\xi(\cdot;\chi(X_\perp),\chi_0)$ is well-defined as an element of $L^1((\lambda_0,\infty),(\lambda-\lambda_0)^{-2}d\lambda)$. It is well known that under the assumption (1.5), one can choose $\xi(\cdot;\chi(X_\perp),\chi_0)$ to be continuous for $\lambda > 0$. For $\lambda < 0$, let us choose $\xi(\cdot;\chi(X_\perp),\chi_0)$ to be right continuous.

Proposition 2.2. *For all $\lambda \neq 0$, one has $\xi(\lambda;\chi(\cdot),\chi_0) \in L^1(\mathbb{R}^2, dX_\perp)$. Moreover, the integral $\int_{\mathbb{R}^2} \xi(\lambda;\chi(X_\perp),\chi_0)dX_\perp$ is continuous in $\lambda > 0$. This integral is continuous in λ at $\lambda = \lambda_0 < 0$ if and only if*

$$\left|\{X_\perp \in \mathbb{R}^2 \mid \lambda \in \sigma(\chi(X_\perp))\}\right| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$.

Proof of Proposition 2.2 is given in Section 6.

Theorem 2.3. *Assume that* (1.5) *holds, and let $q_0 \in \mathbb{Z}_+$. Then:*
(i) For any compact interval $\Delta_1 \subset (0,\infty)$, one has

$$\operatorname*{ess\,sup}_{\lambda\in\Delta_1} \left| b^{-1}\xi(2bq_0+\lambda; H(b), H_0(b)) - \frac{1}{2\pi}\int_{\mathbb{R}^2} \xi(\lambda;\chi(X_\perp),\chi_0)\, dX_\perp \right| \to 0, \quad b\to\infty. \tag{2.2}$$

(ii) For any compact interval $\Delta_2 \subset (-\infty,0)$, one has

$$\limsup_{b\to\infty} \operatorname*{ess\,sup}_{\lambda\in\Delta_2} \left\{ b^{-1}\xi(2bq_0+\lambda; H(b), H_0(b)) - \frac{1}{2\pi}\lim_{\epsilon\to+0}\int_{\mathbb{R}^2} \xi(\lambda-\epsilon;\chi(X_\perp),\chi_0)\, dX_\perp \right\} \leq 0 \tag{2.3}$$

$$\liminf_{b\to\infty} \operatorname*{ess\,inf}_{\lambda\in\Delta_2} \left\{ b^{-1}\xi(2bq_0+\lambda; H(b), H_0(b)) - \frac{1}{2\pi}\lim_{\epsilon\to+0}\int_{\mathbb{R}^2} \xi(\lambda+\epsilon;\chi(X_\perp),\chi_0)\, dX_\perp \right\} \geq 0 \tag{2.4}$$

The proof of Theorem 2.3 can be found in Section 8. In particular, it follows from Theorem 2.3(ii) that if the integral $\int_{\mathbb{R}^2} \xi(\lambda;\chi(X_\perp),\chi_0)dX_\perp$ is continuous on a compact interval $\Delta_1 \subset (-\infty,0)$, then one has (2.2).
Denote

$$\Lambda := \min_{X_\perp\in\mathbb{R}^2} \inf \sigma(\chi(X_\perp)).$$

Evidently $\Lambda \in [-C_0, 0]$ and $\Lambda = 0$ for non-negative V. By (1.3), we have $\xi(\lambda;\chi(X_\perp),\chi_0) = 0$ for all $X_\perp$ if $\lambda < \Lambda$. Thus, the integrals in (2.3), (2.4) vanish if $\lambda < \Lambda$. In this case there exists a more precise version of (2.2) contained in our last theorem.

Theorem 2.4. *Let* (1.5) *hold. Assume that the partial derivatives of $\langle x_3\rangle^{m_3}V$ with respect to the variables $X_\perp \in \mathbb{R}^2$ exist and are uniformly bounded in $\mathbb{R}^3$. Let $q_0 \in \mathbb{Z}_+$ and let $\Delta \subset (-\infty,\Lambda)$ be a compact interval. Then*

$$\operatorname*{ess\,sup}_{\lambda\in\Delta} \left| b^{-1/2}\xi(2bq_0+\lambda; H(b), H_0(b)) - \frac{1}{4\pi^2}\sum_{q=0}^{q_0-1}(2(q_0-q))^{-1/2}\int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x} \right| \to 0, \quad b\to\infty. \tag{2.5}$$

Remark: Here and in the sequel we set the sum $\sum_{q=0}^{q_0-1}$ equal to zero if $q_0 = 0$.

The proof of Theorem 2.4 is contained in Subsection 5.3.
Note that for $q_0 = 0$ the above theorem is trivial. Indeed, in this case the sum over q vanishes. On the other hand, one has

$$\Lambda = \lim_{b\to\infty} \inf \sigma(H(b)); \tag{2.6}$$

this follows from [3, Theorem 5.8]. Thus, by (1.3), for all sufficiently large $b > 0$ the SSF $\xi(2bq_0 + \lambda; H(b), H_0(b))$ in (2.5) also vanishes.

Remark: One can check that for $\lambda > \Lambda$, a sufficient condition for the non-vanishing of the limit coefficient in (2.2), is for V to be of a definite sign. In this case the coefficient $\int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0)\, dX_\perp$ has the same sign as V.

2.2 Continuity of the SSF

As a by-product of our construction, we obtain a result on the continuity of the SSF. This result might be of an independent interest, so we decided to include it in the paper:

Proposition 2.5. *Assume that* (1.5) *holds. Then the SSF* $\xi(E; H(b), H_0(b))$ *is bounded on every compact subset of* $\mathbb{R} \setminus 2b\mathbb{Z}_+$ *and continuous on* $\mathbb{R} \setminus (\sigma_p(H(b)) \cup 2b\mathbb{Z}_+)$*, where* $\sigma_p(H(b))$ *denotes the set of the eigenvalues of the operator* $H(b)$*.*

Note that $H(b)$ may have not only isolated negative eigenvalues but also eigenvalues embedded into the continuous spectrum $[0, \infty)$ (see [3, Theorem 5.1]). The following proposition concerns the location of the eigenvalues of $\sigma_p(H(b))$.

Proposition 2.6. *Assume that* (1.5) *holds. Then*

$$\sigma_p(H(b)) \subset \cup_{q=0}^{\infty} [2bq - C(m_3)^2 C_0^2, 2bq + C(m_3)^2 C_0^2]$$

where C_0 *is the constant from* (1.5) *and* $C(m_3) = (1/2) \int_{\mathbb{R}} \langle x \rangle^{-m_3} dx$*.*

Remark: Generically, the only possible points of accumulation of the eigenvalues of $H(b)$ are the Landau levels $2bq$, $q \in \mathbb{Z}_+$ (see [3, Theorem 4.7], [14, Theorem 3.5.3 (iii)]). We will not use this fact here.

2.3 Comments

In this subsection we comment briefly on our main results and compare them with the results existing in the literature.

- Generically, asymptotic relations (2.1), (2.2), and (2.5) can be unified into a single formula. Indeed, it is well-known (see e.g. [11]) that

  $$\lim_{E \to \infty} E^{1/2} \xi(E; \chi(X_\perp), \chi_0) = \frac{1}{2\pi} \int_{\mathbb{R}} V(X_\perp, x_3)\, dx_3, \quad X_\perp \in \mathbb{R}^2.$$

  Thus, each of relations (2.1) with $0 < \mathcal{E}/2 \notin \mathbb{Z}_+$, or (2.2)–(2.5) with $2q_0 = \mathcal{E}$, $q_0 \in \mathbb{Z}_+$, implies

  $$\xi(\mathcal{E}b + \lambda; H(b), H_0(b)) = \frac{b}{2\pi} \sum_{q=0}^{[\mathcal{E}/2]} \int_{\mathbb{R}^2} \xi(b(\mathcal{E} - 2q) + \lambda; \chi(X_\perp), \chi_0) dX_\perp\, (1 + o(1)), \quad b \to \infty, \tag{2.7}$$

  in $L^\infty_{\text{loc}}(\mathbb{R} \setminus \{0\})$. On its turn (2.7) can be re-written as

  $$\xi(\mathcal{E}b + \lambda; H(b), H_0(b)) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \xi(\mathcal{E}b + \lambda - s; \chi(X_\perp), \chi_0) dX_\perp d\nu_b(s)\, (1 + o(1)),\ b \to \infty,$$

  where $\nu_b(s) := \frac{b}{2\pi} \sum_{q=0}^{\infty} \Theta(s - 2bq)$, $s \in \mathbb{R}$, and $\Theta(s) := \begin{cases} 0 & \text{if} \quad s \leq 0, \\ 1 & \text{if} \quad s > 0, \end{cases}$ is the Heaviside function. It is well-known that ν is *the integrated density of states* for the two-dimensional Landau Hamiltonian (see (4.1)).

- By (1.3) for $\lambda < 0$ we have $\xi(\lambda; H(b), H_0) = -N(\lambda; H(b))$. The asymptotics as $b \to \infty$ of the counting function $N(\lambda; H(b))$ with $\lambda < 0$ fixed, has been investigated in [25] under considerably less restrictive assumptions on V than in Theorems 2.1 – 2.4. The asymptotic properties as $\lambda \uparrow 0$ and as $\lambda \downarrow \Lambda$ of the asymptotic coefficient $-\frac{1}{2\pi}\int_{\mathbb{R}^2} N(\lambda; \chi(X_\perp))dX_\perp$ which appears in (2.3), (2.4) in the case of a negative perturbation, have been studied in [26].

- As mentioned in the Introduction, on the absolutely continuous spectrum of $H_0(b)$ the SSF is related to the scattering determinant by the Birman-Krein formula (1.4). The existence and completeness of the wave operators for the pair $((i\nabla + \mathbf{A})^2 + U, (i\nabla + \mathbf{A})^2)$ has been established in [3] for quite general electromagnetic potentials $(\mathbf{A}, U)$ including the case of constant magnetic fields. More recent results on the scattering theory for the pair $(H_0(b) + V, H_0(b))$ can be found in the monograph [14]. As far as the authors are informed, the asymptotics as $b \to \infty$ of the scattering phase arg det $S(\lambda; H(b), H_0(b))$ has not been studied before in the literature.

- Finally, we would like to mention several possible extensions of our results.
  We believe that the assumptions about the continuity and boundedness of V and its derivatives could be lifted by using an appropriate approximation procedure. In particular, we expect that some local integrable singularities can be allowed. We do not include this approximation procedure in the present article in order to avoid the unreasonable growth of its size.
  Furthermore, it is natural to try to obtain an optimal remainder estimate in (2.2) and (2.5), in the cases where such an estimate exists. Hopefully, the theory of the pseudo-differential operators (ΨDO) with operator-valued symbols, developed in [5], [18], and [12] could be useful in this respect.

3 Representation of the SSF as an averaged index of projections of compact operators

In this section we recall the representation formula of the SSF (see formula (3.15) below) in terms of an integral of an index for a Fredholm pair of self-adjoint spectral projections. This formula was obtained in [15] for the case of trace class perturbation and generalized in [23] to the case of relatively trace class perturbations. The formula generalizes earlier results of [21], [30] and can be regarded as a far going extension of the Birman-Schwinger principle (see [7]).
In order to write down this formula, we need to recall basic properties of compact operators and the notion of an index of a pair of projections.

3.1 Basic properties of linear compact operators

We denote by S_∞ the class of linear compact operators acting in a fixed Hilbert space. Let $T = T^* \in S_\infty$. Denote by $E_T(I)$ the spectral projection of T associated with the interval $I \subset \mathbb{R}$. For $s > 0$ set

$$n_\pm(s; T) := \text{rank } E_{\pm T}((s, \infty)).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$ put

$$n_*(s; T) := n_+(s^2; T^*T), \quad s > 0. \tag{3.1}$$

If $T = T^*$, then evidently

$$n_*(s; T) = n_+(s, T) + n_-(s; T), \quad s > 0. \tag{3.2}$$

Moreover, if $T_j = T_j^* \in S_\infty$, $j = 1, 2$, the Weyl's inequalities

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2) \tag{3.3}$$

hold for each s_1, $s_2 > 0$.
Further, we denote by S_p, $p \in (0, \infty)$, the Schatten-von Neumann class of compact operators for which the functional $\|T\|_{S_p} := \left(p \int_0^\infty s^{p-1} n_*(s;T)\, ds\right)^{1/p}$ is finite. If $T \in S_p$, $p \in (0,\infty)$, then the following elementary inequality

$$n_*(s;T) \le s^{-p} \|T\|_{S_p}^p \tag{3.4}$$

holds for every $s > 0$. If $T = T^* \in S_p$, $p \in (0, \infty)$, then (3.1) and (3.2) imply

$$n_\pm(s;T) \le s^{-p} \|T\|_{S_p}^p, \quad s > 0. \tag{3.5}$$

Finally, we define the self-adjoint operators $\operatorname{Re} T := \frac{1}{2}(T + T^*)$ and $\operatorname{Im} T := \frac{1}{2i}(T - T^*)$. Evidently,

$$n_\pm(s; \operatorname{Re} T) \le 2n_*(s;T), \quad n_\pm(s; \operatorname{Im} T) \le 2n_*(s;T). \tag{3.6}$$

3.2 Index of a pair of projections

A pair of orthogonal projections P, Q in a Hilbert space is called Fredholm, if

$$\{1, -1\} \cap \sigma_{\text{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q$ is compact, then the pair P, Q is Fredholm. The index of a Fredholm pair is given by the formula

$$\operatorname{index}(P, Q) = \dim \operatorname{Ker}(P - Q - I) - \dim \operatorname{Ker}(P - Q + I).$$

In particular, if $P - Q \in S_1$, then $\operatorname{index}(P,Q) = \operatorname{Tr}(P - Q)$. Clearly, one has

$$\operatorname{index}(P, Q) = -\operatorname{index}(Q, P) = -\operatorname{index}(I - P, I - Q).$$

If both P, Q and Q, R are Fredholm pairs and either $P - Q$ or $Q - R$ is compact, then the pair P, R is also Fredholm and the following 'chain rule' holds true:

$$\operatorname{index}(P, R) = \operatorname{index}(P, Q) + \operatorname{index}(Q, R). \tag{3.7}$$

See e.g. [4] for the proof of the last statement and the details.
Let M and $\widetilde{M}$ be bounded self-adjoint operators. If the spectral projections $E_M((-\infty, 0))$, $E_{\widetilde{M}}((-\infty, 0))$ are a Fredholm pair, we will use the shorthand notation

$$\operatorname{ind}(\widetilde{M}, M) := \operatorname{index}(E_{\widetilde{M}}((-\infty, 0)), E_M((-\infty, 0))).$$

We will mostly use this notation in the case $\widetilde{M} = M + A$, where A is a compact self-adjoint operator and M is a bounded self-adjoint operator such that 0 is not in the essential spectrum of M. In this case, representing the spectral projections by Riesz integrals and using the resolvent identity, it is easy to see that the difference $E_{\widetilde{M}}((-\infty, 0)) - E_M((-\infty, 0))$ is compact and therefore the above pair of spectral projections is Fredholm.
Below we list the properties of ind that we will need in the paper.
(a) If $A \in S_1$ and 0 is not in the essential spectrum of M (and, hence, of $M + A$), then

$$\operatorname{ind}(M + A, M) = \lim_{\epsilon \to +0} \xi(-\epsilon; M, M + A). \tag{3.8}$$

This follows from the trace formula (1.1) after approximating the characteristic function of the interval $(-\infty, 0)$ by smooth functions.

(b) If $0 \le A \in S_\infty$, and 0 is not in the spectrum of M, then one has a simple Birman-Schwinger type formula for computing $\mathrm{ind}(M+A,M)$:

$$\mathrm{ind}(M+A,M) = -\sum_{s\in[0,1]} \dim \mathrm{Ker}(M+sA) = -\operatorname{rank} E_{A^{1/2}M^{-1}A^{1/2}}((-\infty,-1]); \tag{3.9}$$

see [15, Corollary 4.8] and [23, Lemma 5.2] for the proof.
(c) One has a stability result for ind (see e.g. [15, Theorem 3.12] for the proof):

$$0 \in \rho(M+A), \quad \lim_{n\to\infty} \|A_n - A\| = 0 \ \Rightarrow \lim_{n\to\infty} \mathrm{ind}(M+A_n,M) = \mathrm{ind}(M+A,M). \tag{3.10}$$

(d) If A is a finite rank operator, then

$$-\operatorname{rank} A_+ \le \mathrm{ind}(M+A,M) \le \operatorname{rank} A_-,$$

where $A_\pm = \frac{1}{2}(|A| \pm A)$. This follows from the property (a) and the analogous bound for the SSF (see e.g. [31]).
(e) Let $\widetilde{M} \ge M$; then $\mathrm{ind}(\widetilde{M},M) \le 0$.

Proof. It suffices to prove that

$$\mathrm{Ker}(E_{\widetilde{M}}((-\infty,0)) - E_M((-\infty,0)) - I) = \{0\}.$$

Suppose to the contrary that for a vector $\psi \ne 0$ one has $(E_{\widetilde{M}}((-\infty,0)) - E_M((-\infty,0)) - I)\psi = 0$; then $E_{\widetilde{M}}((-\infty,0))\psi = \psi$ and $E_M((-\infty,0))\psi = 0$. It follows that $(\widetilde{M}\psi,\psi) < 0$ and $(M\psi,\psi) \ge 0$ — contradiction. □

(f) Let M be a bounded self-adjoint operator such that $[-a,a] \subset \rho(M)$ for some $a > 0$. Let A_0 and A be compact self-adjoint operators and suppose that $\|A\| \le a$. Then

$$\mathrm{ind}(M+A_0+a, M+a) \le \mathrm{ind}(M+A_0+A, M) \le \mathrm{ind}(M+A_0-a, M-a).$$

Proof. By the chain rule,

$$\mathrm{ind}(M+A_0+A,M) = \mathrm{ind}(M+A_0+A, M+A_0-a) + \mathrm{ind}(M+A_0-a, M).$$

By the property (e), $\mathrm{ind}(M+A_0+A, M+A_0-a) \le 0$. This gives the second of the two required inequalities. The first one can be obtained in a similar fashion. □

(g) Combining the properties (d) and (f), one obtains the following bound. Let M be a bounded self-adjoint operator such that $[-a,a] \subset \rho(M)$ for some $a > 0$. Let A_0 and A be compact self-adjoint operators. Then

$$\mathrm{ind}(M+A_0+a, M+a) - n_+(a,A) \le \mathrm{ind}(M+A_0+A,M) \le \mathrm{ind}(M+A_0-a, M-a) + n_-(a,A). \tag{3.11}$$

3.3 Representation of the SSF

Now we are ready to describe the required representation for the SSF.
Let $\mathcal{H}$ and $\mathcal{H}_0$ be two lower-bounded self-adjoint operators acting in the same Hilbert space. Let $\lambda_0 < \inf \sigma(\mathcal{H}) \cup \sigma(\mathcal{H}_0)$. First of all, assume that (1.2) holds for some $\gamma > 0$. Further, let

$$\mathcal{V} := (\mathcal{H} - \mathcal{H}_0) = \mathcal{K}^* \mathcal{J} \mathcal{K}, \tag{3.12}$$

such that $\mathcal{J}$ is a bounded self-adjoint operator with a bounded inverse, and

$$\mathcal{K}(\mathcal{H}_0 - \lambda_0)^{-1/2} \in S_\infty. \tag{3.13}$$

Finally, suppose that

$$\mathcal{K}(\mathcal{H}_0-\lambda_0)^{-\gamma'} \in S_2 \tag{3.14}$$

holds for some $\gamma' > 0$. For $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ set $\mathcal{T}(z) := \mathcal{K}(\mathcal{H}_0 - z)^{-1}\mathcal{K}^*$.

Lemma 3.1. [8] *Let* (3.12) – (3.14) *hold. Then for almost every $E \in \mathbb{R}$ the operator norm limit $\mathcal{T}(E+i0) := \mathrm{n}-\lim_{\delta\downarrow 0}\mathcal{T}(E+i\delta)$ exists, and by* (3.13) *we have $\mathcal{T}(E+i0) \in S_\infty$. Moreover, $\operatorname{Im}\mathcal{T}(E+i0) \in S_1$.*

Theorem 3.2. [15], [23] *Let* (1.2) *and* (3.12) – (3.14) *hold. Then for almost every $E \in \mathbb{R}$ we have*

$$\xi(E;\mathcal{H},\mathcal{H}_0) = \int_{-\infty}^{\infty} \operatorname{ind}(\mathcal{J}^{-1} + \operatorname{Re}\mathcal{T}(E+i0) + t\operatorname{Im}\mathcal{T}(E+i0), \mathcal{J}^{-1})d\mu(t), \tag{3.15}$$

where $d\mu(t) = \pi^{-1}(1+t^2)^{-1}dt$.

The representation (3.15) was found in [15] for the case of trace class $\mathcal{V}$. A generalisation for relatively trace class perturbations was proven in [23].
Let us comment on the convergence of the integral in (3.15). Below we mimick the proof of [21, Lemma 2.1]. Choose $s > 0$ sufficiently small so that $[-s, s]$ does not contain the spectrum of $\mathcal{J}^{-1}$. Using the property (g) of ind, we obtain:

$$|\operatorname{ind}(\mathcal{J}^{-1} + \operatorname{Re}\mathcal{T}(E+i0) + t\operatorname{Im}\mathcal{T}(E+i0), \mathcal{J}^{-1})| \le n_*(s, \operatorname{Re}\mathcal{T}(E+i0) + t\operatorname{Im}\mathcal{T}(E+i0)). \tag{3.16}$$

Applying Weyl's inequality (3.3), we obtain for any $s_1, s_2 \in (0,1)$, $s_1 + s_2 = s$:

$$\int_{-\infty}^{\infty} n_*(s, \operatorname{Re}\mathcal{T}(E+i0)+t\operatorname{Im}\mathcal{T}(E+i0))d\mu(t) \le n_*(s_1, \operatorname{Re}\mathcal{T}(E+i0)) + \int_0^{\infty} n_*(s_2, t\operatorname{Im}\mathcal{T}(E+i0))d\mu(t)$$

$$\le n_*(s_1, \operatorname{Re}\mathcal{T}(E+i0)) + \frac{1}{\pi s_2}\|\operatorname{Im}\mathcal{T}(E+i0)\|_{S_1}. \tag{3.17}$$

This proves absolute convergence of the integral in (3.15) and provides a bound which we will use in the sequel.
Suppose now that V satisfies (1.5). Then relations (3.12) – (3.14) hold with $\mathcal{V} = V$, $\mathcal{H}_0 = H_0$, $\mathcal{K} = |V|^{1/2}$, $\mathcal{J} = \operatorname{sign} V$ and $\gamma = \gamma' = 1$. For $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, set $T(z) := |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. By Lemma 3.1, for almost every $E \in \mathbb{R}$ the operator norm limit

$$T(E+i0) := \mathrm{n} - \lim_{\delta\downarrow 0} T(E+i\delta) \tag{3.18}$$

exists, and

$$0 \le \operatorname{Im} T(E+i0) \in S_1. \tag{3.19}$$

It follows from Lemma 4.2 below that the limit in (3.18) exists and relation (3.19) is valid for *every* $E \notin 2b\mathbb{Z}_+$. Denote

$$A(E) = \operatorname{Re} T(E+i0), \quad B(E) = \operatorname{Im} T(E+i0), \quad J = \operatorname{sign} V. \tag{3.20}$$

Then (3.15) becomes

$$\xi(E; H(b), H_0(b)) = \int_{-\infty}^{\infty} \operatorname{ind}(J + A(E) + tB(E), J)d\mu(t), \quad \text{a.e. } E \in \mathbb{R}, \tag{3.21}$$

the r.h.s. being well-defined for every $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$.

4 Preliminary uniform estimates

In Subsection 4.1 we obtain estimates of the spectrum of the sandwiched resolvent $T(E + i0)$ which will be essential for the proofs of Theorems 2.1, 2.3, 2.4 in Sections 5 and 8. Moreover, these estimates allow us to prove Propositions 2.5 and 2.6, which is done in Subsection 4.2.

4.1 Estimates for the sandwiched resolvent

Recall (see e.g. [3]) the formula for the resolvent of $H_0(b)$:

$$(H_0(b) - z)^{-1} = \sum_{q=0}^{\infty} p_q \otimes r(z - 2bq), \qquad \operatorname{Im} z > 0.$$

Here p_q is the orthogonal projection onto the eigenspace corresponding to the Landau level $2bq$ of the two-dimensional magnetic Hamiltonian

$$h(b) := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \tag{4.1}$$

and $r(z)$ is the resolvent of $-d^2/dx_3^2$ in $L^2(\mathbb{R}, dx_3)$. Due to the assumption (1.5) on V, one can write $|V(\mathbf{x})| = M(\mathbf{x})\langle X_\perp\rangle^{-m_\perp}\langle x_3\rangle^{-m_3}$, where $0 \le M \in L^\infty(\mathbb{R}^3)$.
For $z \in \mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, consider the sandwiched resolvent

$$T(z) = \sum_{q=0}^{\infty} T_q(z),$$
$$T_q(z) = M^{1/2}((\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2}) \otimes t(z - 2bq))M^{1/2}, \quad t(z) = \langle x_3\rangle^{-m_3/2} r(z) \langle x_3\rangle^{-m_3/2}. \tag{4.2}$$

Bounds on $t(z)$. Recall that $m_3 > 1$. Under this assumption, it is well known (and can be immediately seen from the explicit form of the integral kernel of $r(z)$) that for $z \in \mathbb{C}_+$, the operator $t(z)$ belongs to the Hilbert-Schmidt class S_2, is continuous in $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ in the Hilbert-Schmidt norm, and obeys the bound

$$\|t(E + i0)\|_{S_2} \le \frac{C(m_3)}{\sqrt{|E|}}, \quad E \in \mathbb{R} \setminus \{0\}, \tag{4.3}$$

where $C(m_3) = (1/2)\int_{\mathbb{R}} \langle x\rangle^{-m_3} dx$. It is less evident that similar properties hold true in the trace class S_1. This can be seen as follows. According to the spectral theorem, write

$$t(z) = \int_0^\infty F(\nu)^* F(\nu) \frac{d\nu}{\nu - z}, \qquad z \in \mathbb{C}_+,$$

where $F(\nu) : L^2(\mathbb{R}) \to \mathbb{C}^2$ is given by

$$F(\nu) : u(x) \mapsto \left(\frac{1}{2\sqrt{\pi}}\nu^{-1/4}\int_{-\infty}^{\infty} e^{+i\sqrt{\nu}x}\langle x\rangle^{-m_3/2}u(x)dx, \frac{1}{2\sqrt{\pi}}\nu^{-1/4}\int_{-\infty}^{\infty} e^{-i\sqrt{\nu}x}\langle x\rangle^{-m_3/2}u(x)dx\right).$$

It is easy to see that $F(\nu)$ belongs to the Hilbert-Schmidt class S_2, and is Hölder continuous in ν in the Hilbert-Schmidt norm. It follows that for any $z \in \mathbb{C}_+$, the operator $t(z)$ belongs to the trace class, is continuous in $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ in the trace norm and satisfies the bound

$$\|t(E + i0)\|_{S_1} \le \frac{C}{\sqrt{|E|}}(1 + E_+^{1/4}), \quad E \in \mathbb{R} \setminus \{0\}. \tag{4.4}$$

The r.h.s. in (4.4) can be replaced by $\frac{C}{\sqrt{|E|}}(1+E_+^\delta)$ for any $\delta>0$, but for our purposes it suffices to take $\delta = 1/4$. Finally, a direct inspection of the integral kernel shows that for all $E\neq 0$, $\operatorname{Im} t(E+i0)$ has rank two and one has the estimate

$$\|\operatorname{Im} t(E+i0)\|_{S_p}^p \le C|E|^{-p/2} \tag{4.5}$$

for any $0<p\le 1$.
A bound on $\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2}$. One has an explicit formula for the integral kernel of p_q:

$$K_{q,b}(\mathbf{x},\mathbf{x}') = \frac{b}{2\pi} L_q^{(0)}\left(\frac{b|\mathbf{x}-\mathbf{x}'|^2}{2}\right)\exp\left(-\frac{b}{4}(|\mathbf{x}-\mathbf{x}'|^2 + 2i(x_1'x_2 - x_1x_2'))\right) \tag{4.6}$$

(see e.g. [20] or [27, Subsection 2.3.2]) where the Laguerre polynomials $L_q^{(0)}$ are defined in (9.7) below. Note that $K_{q,b}(\mathbf{x},\mathbf{x}) = \frac{b}{2\pi}$ for each $q\in\mathbb{Z}_+$ and $\mathbf{x}\in\mathbb{R}^2$. Using this fact, we immediately obtain

$$\begin{aligned}\|\langle X_\perp\rangle^{-m_3} p_q\|_{S_2}^2 = \|\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2}\|_{S_1} &= \operatorname{Tr}(\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2})\\ &= \frac{b}{2\pi}\int_{\mathbb{R}^2}\langle X_\perp\rangle^{-m_\perp} dX_\perp = Cb. \end{aligned}\tag{4.7}$$

Later on, we will prove a stronger bound (see Lemma 8.1).
Bounds on $T_q(z)$. Putting together the above bounds in (4.2), we obtain

Lemma 4.1. *For any $z\in\mathbb{C}_+$, the operator $T_q(z)$ belongs to the trace class S_1, and continuously depends on $z\in\overline{\mathbb{C}_+}\setminus\{2bq\}$ in the trace class norm. Moreover,*

$$\|T_q(E+i0)\| \le \frac{C_1}{|E-2bq|^{1/2}}, \qquad E\in\mathbb{R}\setminus\{2bq\}, \tag{4.8}$$

$$\|T_q(E+i0)\|_{S_1} \le \frac{C_2 b}{|E-2bq|^{1/2}}(1+|E-2bq|^{1/4}), \qquad E\in\mathbb{R}\setminus\{2bq\}. \tag{4.9}$$

In the sequel for $E\in\mathbb{R}\setminus\{2bq\}$ and for $\lambda\in\mathbb{R}\setminus\{0\}$ we denote

$$T_q(E) = M^{1/2}((\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2})\otimes t(E-2bq))M^{1/2}, \quad t(\lambda) = \langle x_3\rangle^{-m_3/2} r(\lambda+i0)\langle x_3\rangle^{-m_3/2}. \tag{4.10}$$

Finally, we get a bound for $T(z)$.

Lemma 4.2. *For any $z\in\mathbb{C}_+$, the operator $T(z)$ is compact, continuously depends on $z\in\overline{\mathbb{C}_+}\setminus 2b\mathbb{Z}_+$ in the operator norm, and obeys the estimate*

$$\|T(E+i0)\| \le \frac{C_0 C(m_3)}{\operatorname{dist}(E, 2b\mathbb{Z}_+)^{1/2}}, \quad E\in\mathbb{R}\setminus 2b\mathbb{Z}_+. \tag{4.11}$$

Moreover, for $E\in\mathbb{R}\setminus 2b\mathbb{Z}_+$, the operator $B(E) = T(E+i0)$ is non-negative, and vanishes if $E<0$. Further, $B(E)$ belongs to the trace class S_1, and continuously depends on $E\in\mathbb{R}\setminus 2b\mathbb{Z}_+$ in the trace norm. Finally, for $2b(q_0-1)<E<2bq_0$, $q_0\in\mathbb{Z}_+$, $q_0\ge 1$, and $r=0,1$, we have

$$\operatorname{Tr}(J^r B(E)) = \frac{b}{4\pi}\sum_{q=0}^{q_0-1}(E-2bq)^{-1/2}\int_{\mathbb{R}^3}(\operatorname{sign} V(\mathbf{x}))^r |V(\mathbf{x})| d\mathbf{x}. \tag{4.12}$$

Proof. Compactness follows from the diamagnetic inequality; in fact, one has $T(z)\in S_2$, but we do not need this fact here. Next, fix $q_0\in\mathbb{Z}_+$. For $E<2bq_0$, $E\in\mathbb{R}\setminus 2b\mathbb{Z}_+$, continuity in the operator norm can be seen by representing

$$T(E+i0) = \sum_{q=0}^{q_0-1} T_q(E) + T_{q_0}^+(E). \tag{4.13}$$

Here each $T_q(E)$ is continuous by Lemma 4.1 and $T^+_{q_0}(E)$ is even analytic in $E \in \mathbb{C} \setminus [2bq_0, \infty)$ by the spectral theorem. Using (4.3), one obtains

$$\|T(E+i0)\| \leq C_0 \|\sum_{q=0}^{\infty} p_q \otimes t(E-2bq)\| \leq C_0 \sup_{q\in\mathbb{Z}_+} \|t(E-2bq)\| \leq \frac{C_0 C(m_3)}{\mathrm{dist}(E, 2b\mathbb{Z}_+)^{1/2}}.$$

Taking imaginary parts in (4.13) gives

$$B(E) = \sum_{q=0}^{q_0-1} B_q(E), \qquad E < 2bq_0, \qquad E \in \mathbb{R} \setminus 2b\mathbb{Z}_+,$$

where $B_q(E) = \mathrm{Im}\, T_q(E)$ is trace norm continuous by Lemma 4.1. Note that the relations $B(E) \geq 0$ for $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$, and $B(E) = 0$ for $E < 0$, follow easily from the definition of the operator $T(z)$ (see (4.2)).

Finally, from (4.10), (4.6) and the explicit formula for the integral kernel of $\mathrm{Im}\, t(\lambda)$,

$$\langle x_3 \rangle^{-m_3/2} \frac{\cos(\sqrt{\lambda}\,(x_3 - x_3'))}{2\sqrt{\lambda}} \langle x_3' \rangle^{-m_3/2}, \quad x_3, x_3' \in \mathbb{R},$$

we obtain (4.12). □

4.2 Proof of Propositions 2.5 and 2.6

1. *Proof of Proposition 2.5.*

Let us identify the SSF with the r.h.s. of representation (3.21) and prove its continuity on the set $\mathbb{R} \setminus (\sigma_p(H(b)) \cup 2b\mathbb{Z}_+)$. According to the stability result of [15, Theorem 3.12], the r.h.s. of (3.21) is continuous in E at the point $E = E_0$ if the following conditions are satisfied:

$$\lim_{E\to E_0} \|A(E) - A(E_0)\| = 0, \qquad \lim_{E\to E_0} \|B(E) - B(E_0)\|_{S_1} = 0. \tag{4.14}$$

$$0 \in \rho(J + A(E_0) + \tau B(E_0)) \text{ for some } \tau \in \mathbb{R}. \tag{4.15}$$

The limiting relations (4.14) are met by Lemma 4.2. By the analytic Fredholm alternative, (4.15) will follow from $0 \in \rho(J + A(E_0) + iB(E_0))$ or, equivalently, from $-1 \in \rho(JT(E_0 + i0))$. By mimicking Agmon's 'bootstrap argument' [2], one obtains that for $E_0 \in \mathbb{R} \setminus 2b\mathbb{Z}_+$,

$$-1 \in \sigma(JT(E_0 + i0)) \Leftrightarrow E_0 \in \sigma_p(H(b)). \tag{4.16}$$

Thus, condition (4.15) holds true for all $E_0 \in \mathbb{R} \setminus (\sigma_p(H(b)) \cup 2b\mathbb{Z}_+)$.

Let us now prove that the r.h.s. of representation (3.21) is bounded on any closed interval which does not contain Landau levels. By the bound (3.17), one has

$$|\int_{-\infty}^{\infty} \mathrm{ind}(J + A(E) + tB(E), J) d\mu(t)| \leq n(1/3, A(E)) + \frac{3}{\pi}\|B(E)\|_{S_1}.$$

By Lemma 4.2, the r.h.s. is bounded on any closed interval of $\mathbb{R} \setminus 2b\mathbb{Z}_+$.

2. *Proof of Proposition 2.6.*

By (4.16), the problem reduces to an estimate on the norm of $T(E + i0)$. The estimate (4.11) shows that for $\mathrm{dist}(E, 2b\mathbb{Z}_+) > C_0^2 C(m_3)^2$, one has $\|T(E + i0)\| < 1$ and therefore $E \notin \sigma_p(H(b))$. □

5 SSF asymptotics of order $\sqrt{b}$

In this section we study the asymptotic behaviour, as $b \to +\infty$, of the SSF $\xi(E; H, H_0)$ in two situations. First to prove Theorem 2.1 we consider energies far from Landau levels, that is for $E = \mathcal{E}b + \lambda$ with $\mathcal{E} \in \mathbb{R} \setminus 2\mathbb{Z}_+$. Then to prove Theorem 2.4 we consider energies under Landau levels, that is $E = 2q_0 b + \lambda$, with $\lambda < \Lambda$ ($\Lambda := \min_{X_\perp \in \mathbb{R}^2} \inf \sigma(\chi(X_\perp))$).

5.1 Abstract Lemma

First let us prove a simple lemma of an abstract nature. This lemma might be of an independent interest. Note that formulae similar to (5.17) can be found in [15].

Lemma 5.1. *Let $A = A^*$ be a compact operator, $0 \le B \in S_1$, and let $J = J^* = J^{-1}$. Suppose that $0 \in \rho(J + A)$. Then*

$$\int_{-\infty}^{\infty} \operatorname{ind}(J + A + tB, J) d\mu(t) = \operatorname{ind}(J + A, J) + \pi^{-1} \operatorname{Tr} \arctan(B^{1/2}(J + A)^{-1} B^{1/2}) \tag{5.17}$$

and

$$\left| \pi^{-1} \operatorname{Tr} \arctan(B^{1/2}(J + A)^{-1} B^{1/2}) - \pi^{-1} \operatorname{Tr}(JB) \right| \le \frac{1}{3} \|(J + A)^{-1}\|^3 \operatorname{Tr} B^3 + |\operatorname{Tr}(BJA(J + A)^{-1})|. \tag{5.18}$$

Proof. Using the 'chain rule' (3.7) and integrating, we get

$$\int_{-\infty}^{\infty} \operatorname{ind}(J + A + tB, J) d\mu(t) = \operatorname{ind}(J + A, J) + \int_{-\infty}^{\infty} \operatorname{ind}(J + A + tB, J + A) d\mu(t).$$

Let us split the integral in the r.h.s. into the sum of integrals over $(-\infty, 0)$ and $(0, \infty)$. Using (3.9), we obtain:

$$\int_{-\infty}^{\infty} \operatorname{ind}(J+A+tB, J+A) d\mu(t) = \int_{0}^{\infty} \operatorname{ind}(J+A+tB, J+A) d\mu(t) + \int_{0}^{\infty} \operatorname{ind}(J+A-tB, J+A) d\mu(t)$$
$$= -\int_0^\infty n_-(1, tB^{1/2}(J + A)^{-1} B^{1/2}) d\mu(t) + \int_0^\infty n_+(1, tB^{1/2}(J + A)^{-1} B^{1/2}) d\mu(t)$$
$$= \pi^{-1} \operatorname{Tr} \arctan(B^{1/2}(J + A)^{-1} B^{1/2}).$$

This proves formula (5.17).
Let us prove the bound (5.18). Denoting $M = B^{1/2}(J + A)^{-1} B^{1/2}$, we get

$$\operatorname{Tr} \arctan M - \operatorname{Tr} M = \int_0^\infty \frac{n_+(t, M)}{1 + t^2} dt - \int_0^\infty \frac{n_-(t, M)}{1 + t^2} dt - \int_0^\infty n_+(t, M) dt + \int_0^\infty n_-(t, M) dt,$$

and therefore

$$|\operatorname{Tr} \arctan M - \operatorname{Tr} M| \le \int_0^\infty \frac{t^2 n_*(t, M)}{1 + t^2} dt \le \int_0^\infty t^2 n_*(t, M) dt = \frac{1}{3} \|M\|_{S_3}^3$$
$$\le \frac{1}{3} (\|B^{1/2}\|_{S_6}^2 \|(J + A)^{-1}\|)^3 = \frac{1}{3} \|(J + A)^{-1}\|^3 \operatorname{Tr} B^3.$$

Finally,

$$|\operatorname{Tr} M - \operatorname{Tr} JB| = \left| \operatorname{Tr}(B^{1/2}((J + A)^{-1} - J) B^{1/2}) \right| = |\operatorname{Tr}(BJA(J + A)^{-1})|.$$

This proves the estimate (5.18). □

Now, using notation (3.18), (3.20) and (4.10), we will employ Lemma 5.1 to prove Theorem 2.1 and Theorem 2.4.

5.2 Proof of Theorem 2.1

We will use the representation (3.21) and Lemma 5.1. Denote $q_0 = [\mathcal{E}/2]$. Suppose that b is sufficiently large so that $\mathcal{E}b + \lambda < 2(q_0+1)b$ for all $\lambda \in \Delta$. Then by (4.12)

$$\begin{aligned}\pi^{-1}\operatorname{Tr} JB(\mathcal{E}b+\lambda) &= \frac{b}{4\pi^2}\sum_{q=0}^{q_0}(\mathcal{E}b+\lambda-2bq)^{-1/2}\int_{\mathbb{R}^3}V(\mathbf{x})d\mathbf{x}\\ &= \frac{b^{1/2}}{4\pi^2}\sum_{q=0}^{q_0}(\mathcal{E}-2q)^{-1/2}\int_{\mathbb{R}^3}V(\mathbf{x})d\mathbf{x}+O(1), \quad b\to+\infty, \end{aligned}\tag{5.19}$$

uniformly over $\lambda \in \Delta$. Next, by (4.11),

$$\sup_{\lambda\in\Delta}\|T(\mathcal{E}b+\lambda+i0)\| = O(b^{-1/2}), \quad b\to\infty. \tag{5.20}$$

It follows that for all sufficiently large b,

$$\sup_{\lambda\in\Delta}\|A(\mathcal{E}b+\lambda+i0)\| < 1/2 \tag{5.21}$$

and therefore $\operatorname{ind}(J + A(\mathcal{E}b+\lambda), J) = 0$, $\lambda \in \Delta$.
Let us estimate the error terms given by the r.h.s. of (5.18). By (5.21), for all sufficiently large b one has $\sup_{\lambda\in\Delta}\|(J + A(\mathcal{E}b+\lambda))^{-1}\| \le 2$. Using this and (5.19), (5.20) and Lemma 4.2 we obtain:

$$\begin{gathered}\|(J+A)^{-1}\|^3\operatorname{Tr}B^3 \le 2\|B\|^2\operatorname{Tr}B = O(b^{-1})O(b^{1/2}) = O(b^{-1/2}), \quad b\to\infty;\\ |\operatorname{Tr}(BJA(J+A)^{-1})| \le \operatorname{Tr}B\|A\|\|(J+A)^{-1}\| = O(b^{1/2})O(b^{-1/2}) = O(1), \quad b\to\infty,\end{gathered}$$

uniformly over $\lambda \in \Delta$. This completes the proof of Theorem 2.1.

5.3 Proof of Theorem 2.4

Lemma 5.2. *Assume that* (1.5) *hold and that the partial derivatives of* $\langle x_3\rangle^{m_3}V$ *with respect to the variables* $X_\perp \in \mathbb{R}^2$ *exist and are uniformly bounded in* $\mathbb{R}^3$*. Then we have* $\lim_{b\to\infty}\|(1-P_{q_0})\langle x_3\rangle^{m_3}VP_{q_0}\| = 0$*, where* $P_q = p_q\otimes I$ *in* $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2, dX_\perp)\otimes L^2(\mathbb{R}, dx_3)$*.*

The proof of Lemma 5.2 can be found in Subsection 9.2.

Proof of Theorem 2.4. We will use the representation (3.21) and Lemma 5.1.
1. First observe that for any $\lambda < 0$, the operator $T_{q_0}(2q_0b+\lambda)$ is selfadjoint and non-negative and belongs to the trace class. Let us prove that

$$\operatorname{ind}(J + T_{q_0}(2q_0b+\lambda), J) = 0, \quad \forall\lambda\in\Delta, \tag{5.22}$$

$$\sup_{\lambda\in\Delta}\|(J+T_{q_0}(2q_0b+\lambda))^{-1}\| \le C, \text{ where } C \text{ does not depend on } b>0. \tag{5.23}$$

Consider the operator $I\otimes\chi_0+V$ in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2, dX_\perp)\otimes L^2(\mathbb{R}, dx_3)$. Our assumption $\Delta\subset(-\infty,\Lambda)$ means that $\sup\Delta < \Lambda = \inf\sigma(I\otimes\chi_0+V)$ and therefore, by the Birman-Schwinger principle,

$$\sup_{\lambda\in\Delta}\|(I\otimes\chi_0-\lambda)^{-1/2}V(I\otimes\chi_0-\lambda)^{-1/2}\| < 1.$$

It follows that

$$\sup_{\lambda\in\Delta}\|(p_{q_0}\otimes(\chi_0-\lambda)^{-1/2})V(p_{q_0}\otimes(\chi_0-\lambda)^{-1/2})\| < 1,$$

or equivalently,

$$\sup_{\lambda\in\Delta}\|(T_{q_0}(2q_0b+\lambda))^{1/2}J(T_{q_0}(2q_0b+\lambda))^{1/2}\|<1. \tag{5.24}$$

¿From here, using the formula

$$(J+T_{q_0})^{-1}=J-JT_{q_0}^{1/2}(I+T_{q_0}^{1/2}JT_{q_0}^{1/2})^{-1}T_{q_0}^{1/2}J$$

and the estimate (4.8), we obtain (5.23). Using (3.9) and (5.24), we get (5.22).
2. Next, similarly to (4.11) we have

$$\sup_{\lambda\in\Delta}\|A(2bq_0+\lambda)-T_{q_0}(2bq_0+\lambda)\|\le C_0\sup_{\lambda\in\Delta}\|\langle x_3\rangle^{-m_3/2}\big(\sum_{q\neq q_0}p_q\otimes r(2b(q_0-q)+\lambda+i0)\big)\langle x_3\rangle^{-m_3/2}\|$$
$$\le C_0C(m_3)\sup\{|2b(q_0-q)+\lambda|^{-1/2}\mid q\in\mathbb{Z}_+,\ q\neq q_0,\lambda\in\Delta\}=O(b^{-1/2}),\quad b\to\infty. \tag{5.25}$$

¿From here and (5.22) by the stability of index, one has

$$\operatorname{ind}(J+A(2bq_0+\lambda),J)=0,\quad \lambda\in\Delta, \tag{5.26}$$

for all sufficiently large b. Also, from (5.23) and (5.25) we get

$$\sup_{\lambda\in\Delta}\|(J+A(2bq_0+\lambda))^{-1}\|=O(1),\quad b\to\infty. \tag{5.27}$$

3. Let us apply Lemma 5.1. For the leading term, we have formula (5.19) which gives the limiting expression in (2.5). Now let us check that the remainder terms given by the r.h.s. of (5.18) can be estimated as $o(b^{1/2})$. Consider the term $\|(J+A)^{-1}\|^3\operatorname{Tr}B^3$. For $\lambda<0$,

$$B(2bq_0+\lambda)=\sum_{q=0}^{q_0-1}\operatorname{Im}T_q(2bq_0+\lambda), \tag{5.28}$$

and so by (5.19) and (4.11),

$$\sup_{\lambda\in\Delta}\operatorname{Tr}B^3(2bq_0+\lambda)=O(b^{-\frac{1}{2}}). \tag{5.29}$$

Combining this with (5.27), we get the required bound for $\|(J+A)^{-1}\|^3\operatorname{Tr}B^3$.
4. Finally, consider the term $\operatorname{Tr}(BJA(J+A)^{-1})$. Let us prove that

$$\sup_{\lambda\in\Delta}\|B(2q_0b+\lambda)JA(2q_0b+\lambda)\|_{S_1}=o(b^{1/2}),\quad b\to\infty. \tag{5.30}$$

By (5.25) and (5.19), it suffices to prove that

$$\sup_{\lambda\in\Delta}\|B(2q_0b+\lambda)JT_{q_0}(2q_0b+\lambda)\|_{S_1}=o(b^{1/2}),\quad b\to\infty.$$

By (5.28), it suffices to consider $\operatorname{Im}(T_q(2q_0b+\lambda))JT_{q_0}(2q_0b+\lambda)$ for any fixed $q<q_0$. Using the notation $M(\mathbf{x})=|V(\mathbf{x})|\langle X_\perp\rangle^{m_\perp}\langle x_3\rangle^{m_3}$ and $V_\perp(X_\perp,x_3):=\langle x_3\rangle^{m_3}V(X_\perp,x_3)$, we obtain

$$\operatorname{Im}(T_q(2q_0b+\lambda))JT_{q_0}(2q_0b+\lambda)$$
$$=M^{1/2}(\langle X_\perp\rangle^{-m_\perp/2}p_q\otimes\operatorname{Im}t(2b(q_0-q)+\lambda))P_qV_\perp P_{q_0}(p_{q_0}\langle X_\perp\rangle^{-m_\perp}\otimes t(\lambda))M^{1/2}.$$

By Lemma 5.2, as b tends to infinity, we have $\|P_qV_\perp P_{q_0}\|=o(1)$. Next, by (4.3) and (4.7), we have

$$\|\langle X_\perp\rangle^{m_\perp/2}p_q\otimes\operatorname{Im}t(2b(q_0-q)+\lambda)\|_{S_2}\le Cb^{1/2}|2b(q_0-q)-\lambda|^{-1/2}=O(1),\quad b\to\infty,$$
$$\|p_{q_0}\langle X_\perp\rangle^{m_\perp/2}\otimes t(\lambda)\|_{S_2}=O(b^{1/2}),\quad b\to\infty,$$

uniformly over $\lambda\in\Delta$. Combining the above bounds, we get (5.30). □

6 Properties of $\xi(\lambda;\chi(X_\perp),\chi_0)$

Here we prepare some auxiliary statements for the proof of Theorem 2.4. These statements concern the SSF $\xi(\lambda;\chi(X_\perp),\chi_0)$. We will use the representation (3.15) for $\xi(\lambda;\chi(X_\perp),\chi_0)$. With the notation

$$\tau(X_\perp,\lambda)=|V(X_\perp,\cdot)|^{1/2}(\chi_0-\lambda-i0)^{-1}|V(X_\perp,\cdot)|^{1/2}, \tag{6.1}$$

$$\iota(X_\perp)=\operatorname{sign}V(X_\perp,\cdot),\quad \alpha(X_\perp,\lambda)=\operatorname{Re}\tau(X_\perp,\lambda),\quad \beta(X_\perp,\lambda)=\operatorname{Im}\tau(X_\perp,\lambda), \tag{6.2}$$

this representation reads

$$\xi(\lambda;\chi(X_\perp),\chi_0)=\int_{-\infty}^{\infty}\operatorname{ind}(\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda),\iota(X_\perp))d\mu(t),\qquad \text{a.e. }\lambda\in\mathbb{R}. \tag{6.3}$$

Along with the operator $\chi(X_\perp)$, we consider its modification: for $\eta\in\mathbb{R}$, $|\eta|\neq 1$, let

$$\chi(X_\perp,\eta)=\chi_0+|V(X_\perp,\cdot)|(\iota(X_\perp)-\eta)^{-1}.$$

According to (3.15), for this operator we have

$$\xi(\lambda;\chi(X_\perp,\eta),\chi_0)=\int_{-\infty}^{\infty}\Xi_\lambda(\eta,X_\perp,t)d\mu(t),\qquad \text{a.e. }\lambda\in\mathbb{R}, \tag{6.4}$$

where

$$\Xi_\lambda(\eta,X_\perp,t):=\operatorname{ind}(\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)-\eta,\iota(X_\perp)-\eta),\quad |\eta|\neq 1. \tag{6.5}$$

We will need some continuity and measurability properties of $\Xi_\lambda(\eta,X_\perp,t)$.

Lemma 6.1. *(i) The set*

$$\Omega=\{(\lambda,\eta,X_\perp,t)\mid \eta\notin\sigma(\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)),\lambda\neq 0,|\eta|\neq 1\}$$

is open in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}$ and the function $\Xi_\lambda(\eta,X_\perp,t)$ is constant on connected components of this set.
(ii) The function $\Xi_\lambda(\eta,X_\perp,t)$ is lower semicontinuous on the set

$$\{(\lambda,\eta,X_\perp,t)\mid \lambda\neq 0,|\eta|\neq 1,X_\perp\in\mathbb{R}^2,t\in\mathbb{R}\}.$$

Proof. 1. First consider the difference of resolvents

$$(\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)-z)^{-1}-(\iota(X_\perp)-z)^{-1}$$

for a fixed z away from the spectra of $\iota(X_\perp)$ and $\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)$. Let us prove that this difference continuously depends on λ, $X_\perp$, t in the trace norm. Here the only non-trivial issue is continuous dependence on $X_\perp$ (observe that $\iota(X_\perp)$ is not continuous in $X_\perp$ even in the operator norm). In order to prove continuous dependance on $X_\perp$, first observe that the operators $\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)$, $\iota(X_\perp)(\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))$, $(\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))\iota(X_\perp)$, and $\iota(X_\perp)(\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))\iota(X_\perp)$ continuously depend on $X_\perp$ in the trace norm. Using the identity

$$(\iota+\alpha+t\beta-z)^{-1}-(\iota-z)^{-1}=-(I+(\iota-z)^{-1}(\alpha+t\beta))^{-1}(\iota-z)^{-1}(\alpha+t\beta)(\iota-z)^{-1},$$

and expanding $(\iota-z)^{-1}$ in norm convergent series, we obtain the desired statement.
2. From the result of the previous step we obtain that the eigenvalues of $\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)$ continuously depend on λ, $X_\perp$, t. This proves that Ω is open. As $\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)$ is of the trace class, one has

$$\begin{aligned}\Xi_\lambda(\eta,X_\perp,t)&=\operatorname{index}(E_{\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)}((-\infty,\eta)),E_{\iota(X_\perp)}((-\infty,\eta)))\\&=\operatorname{Tr}(E_{\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda)}((-\infty,\eta))-E_{\iota(X_\perp)}((-\infty,\eta))).\end{aligned}$$

Representing the above spectral projections as Riesz integrals, and using the result of the first step of the proof, we see that $\Xi_\lambda(\eta, X_\perp, t)$ is continuous on Ω. As $\Xi_\lambda(\eta, X_\perp, t)$ is integer-valued, it is constant on connected components of Ω. Precisely the same reasoning also shows that $\Xi_\lambda(\eta, X_\perp, t)$ is lower semicontinuous on the whole range of its variables. □

Recall that lower semicontinuous functions are measurable. Thus, Lemma 6.1 ensures that $\Xi_\lambda(\eta, X_\perp, t)$ is measurable with respect to any combination of its variables. We will need to integrate $\Xi_\lambda(\eta, X_\perp, t)$ with respect to some of its variables; integrability is provided by the following

Lemma 6.2. *(i) For any $p \in (0, 1)$, one has the bound*

$$\sup_{|\eta|\le 1/2} |\Xi_\lambda(\eta, X_\perp, t)| \le C(1 + \lambda_+^{1/4})|\lambda|^{-1/2}\langle X_\perp\rangle^{-m_\perp} + C|\lambda|^{-p/2}t^p\langle X_\perp\rangle^{-pm_\perp}. \tag{6.6}$$

(ii) For any $t \in \mathbb{R}$ and any $\Delta \subset \mathbb{R} \setminus \{0\}$, one has

$$\sup_{\lambda\in\Delta} \int_{-\infty}^{\infty} d\eta \int_{\mathbb{R}^2} dX_\perp |\Xi_\lambda(\eta, X_\perp, t)| < \infty. \tag{6.7}$$

Proof. (i) Recall that $\tau(X_\perp, \lambda)$ is a trace class operator which satisfies (4.4) and $\beta(X_\perp, \lambda)$ has rank two and satisfies (4.5). Thus, by the estimate (1.5) on V, we have

$$\|\alpha(X_\perp, \lambda)\|_{S_1} \le \frac{C}{\sqrt{|\lambda|}}(1 + \lambda_+^{1/4})\langle X_\perp\rangle^{-m_\perp}, \qquad \|\beta(X_\perp, \lambda)\|_{S_p}^p \le C|\lambda|^{-p/2}\langle X_\perp\rangle^{-pm_\perp}. \tag{6.8}$$

Using these bounds and arguing similarly to (3.16), (3.17), we obtain for any $p \in (0, 1)$:

$$\begin{aligned}\sup_{|\eta|\le 1/2} |\Xi_\lambda(\eta, X_\perp, t)| &\le n_*(1/4, \alpha(X_\perp, \lambda) + t\beta(X_\perp, \lambda)) \le n_*(1/8, \alpha(X_\perp, \lambda)) + n_*(1/8, t\beta(X_\perp, \lambda)) \\ &\le \|8\alpha(X_\perp, \lambda)\|_{S_1} + \|8t\beta(X_\perp, \lambda)\|_{S_p}^p \le C(1 + \lambda_+^{1/4})|\lambda|^{-1/2}\langle X_\perp\rangle^{-m_\perp} + C|\lambda|^{-p/2}t^p\langle X_\perp\rangle^{-pm_\perp}.\end{aligned}$$

(ii) First note that by the property (a) of ind, one has

$$\Xi_\lambda(\eta, X_\perp, t) = -\lim_{\epsilon\to+0} \xi(\eta - \epsilon; \iota(X_\perp) + \alpha(X_\perp, \lambda) + t\beta(X_\perp, \lambda), \iota(X_\perp)). \tag{6.9}$$

Using this fact and Krein's bound for the SSF (see e.g. [31, Theorem 8.2.1]), we obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} d\eta \int_{\mathbb{R}^2} dX_\perp |\xi(\eta; \iota(X_\perp) + \alpha(X_\perp, \lambda) + t\beta(X_\perp, \lambda), \iota(X_\perp))| \\ \le \int_{\mathbb{R}^2} \|\alpha(X_\perp, \lambda) + t\beta(X_\perp, \lambda)\|_{S_1} dX_\perp < \infty.\end{aligned}$$

This completes the proof. □

Lemma 6.3. *(i) For any $\eta \in [-1/2, 1/2]$ and any $X_\perp \in \mathbb{R}^2$, the function $\int_{-\infty}^{\infty} \Xi_\lambda(\eta, X_\perp, t)d\mu(t)$ is continuous in $\lambda > 0$ and right continuous in $\lambda < 0$.*
(ii) The function $\int_{\mathbb{R}^2} dX_\perp \int_{-\infty}^{\infty} d\mu(t)\, \Xi_\lambda(\eta, X_\perp, t)$ is continuous in $\lambda > 0$ and $\eta \in [-1/2, 1/2]$.

Proof. (i) Let us first prove continuity in $\lambda > 0$. We shall use the dominated convergence theorem; Lemma 6.2(i) gives an integrable bound, and Lemma 6.1(i) provides continuity of the integrand away from the set $S := \{t \mid \eta \in \sigma(\iota(X_\perp) + \alpha(X_\perp, \lambda) + t\beta(X_\perp, \lambda))\}$. We only need to prove that this set has zero measure in $\mathbb{R}$. By the analytic Fredholm alternative, one has $t \in S$ either for countably many t or for all $t \in \mathbb{C}$. Let us show that the latter is impossible. Indeed, if $i \in S$, then $-1 \in$

$\sigma((\iota(X_\perp)-\eta)^{-1}(\alpha(X_\perp,\lambda)+i\beta(X_\perp,\lambda)))$, and by the classical Agmon's 'bootstrap argument', it follows that $\lambda>0$ is an eigenvalue of the one-dimensional Schrödinger operator $\chi(X_\perp,\eta)$, which is impossible. This proves continuity in $\lambda>0$.
Lower semicontinuity of $\Xi_\lambda(\eta,X_\perp,t)$ is retained by its integral over t. Also, $\Xi_\lambda(\eta,X_\perp,t)$, and therefore its integral, is non-increasing in $\lambda<0$. It follows that the integral of $\Xi_\lambda(\eta,X_\perp,t)$ is right continuous for $\lambda<0$.
(ii) can be proved by following the same argument. □

Proof of Proposition 2.2. 1. Lemma 6.3(i) shows that with our choice of $\xi(\lambda;\chi(X_\perp),\chi_0)$ (continuous in $\lambda>0$ and right continuous in $\lambda<0$), the equality (6.3) holds true for all $\lambda\neq 0$. Then integrability of $\xi(\lambda;\chi(X_\perp),\chi_0)$ is ensured by the estimate (6.6) with $2/m_\perp<p<1$.
2. In view of (1.3), the second statement is obvious. □

7 Asymptotic trace formulae

In this section we establish the key limiting relation used in the proof of Theorem 2.3. Let $T_q(E)$ be as above (see (4.10)) and denote

$$A_q=\operatorname{Re}T_q(2qb+\lambda),\qquad B_q=\operatorname{Im}T_q(2qb+\lambda),\qquad J=\operatorname{sign}V. \tag{7.1}$$

Moreover, we use the notation introduced in (6.1) – (6.2).

Proposition 7.1. *Let* (1.5) *hold and let* Δ *be a compact interval in* $\mathbb{R}\setminus\{0\}$. *Then for every* $t\in\mathbb{R}$ *and each integers* $p\geq 1$, $r\geq 0$ *we have*

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ ((A_{q_0}+tB_{q_0})^pJ^r)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{Tr}\ ((\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))^p\iota(X_\perp)^r)\,dX_\perp, \tag{7.2}$$

where convergence is uniform in $\lambda\in\Delta$.

Proof. Since $J^2=I$ and $\iota(X_\perp)^2=I$, we have only to consider the cases $r=0$ and $r=1$. For $r=0$ or $r=1$, we have

$$\mathrm{Tr}\ ((A_{q_0}+tB_{q_0})^pJ^r)=$$
$$\int_{\mathbb{R}^{2p}}\int_{\mathbb{R}^p}(\operatorname{sign}V(X_{\perp,1},x_{3,1}))^r\Pi_{j=1}^p|V(X_{\perp,j},x_{3,j})|\times$$
$$\Pi'^p_{j=1}K_{q_0,b}(X_{\perp,j},X_{\perp,j+1})\mathcal{R}_{\lambda,t}(x_{3,j}-x_{3,j+1})\Pi_{j=1}^p dX_{\perp,j}dx_{3,j}$$

where

$$\mathcal{R}_{\lambda,t}(x):=\begin{cases}-\frac{\sin(\sqrt{\lambda}|x|)}{2\sqrt{\lambda}}+t\ \frac{\cos(\sqrt{\lambda}\,x)}{2\sqrt{\lambda}} & \text{if}\quad \lambda>0,\\ \frac{e^{-\sqrt{-\lambda}|x|}}{2\sqrt{-\lambda}} & \text{if}\quad \lambda<0,\end{cases}\qquad t\in\mathbb{R},\quad x\in\mathbb{R},$$

and the notation $\Pi'^p_{j=1}$ means that in the product of p factors the variables $X_{\perp,p+1}$ and $x_{3,p+1}$ should be set equal respectively to $X_{\perp,1}$ and $x_{3,1}$. For $p=1$ the claimed asymptotic is an equality because $K_{q,b}(X_\perp,X_\perp)=\frac{b}{2\pi}$. For $p\geq 2$, let us change the variables

$$X_{\perp,1}=X'_{\perp,1},\quad X_{\perp,j}=X'_{\perp,1}+b^{-1/2}X'_{\perp,j},\quad j=2,\dots,p. \tag{7.3}$$

Thus we obtain

$$\mathrm{Tr}\ ((A_{q_0}+tB_{q_0})^pJ^r)=$$
$$b\int_{\mathbb{R}^{2p}}\int_{\mathbb{R}^p}(\operatorname{sign}V(X_{\perp,1},x_{3,1}))^r|V(X'_{\perp,1},x_{3,1})|\Pi_{j=2}^p|V(X'_{\perp,1}+b^{-1/2}X'_{\perp,j},x_{3,j})|\times$$

$$K_{q_0,1}(0,X'_{\perp,2})\Pi_{j=2}^{p-1}K_{q_0,1}(X'_{\perp,j},X'_{\perp,j+1})K_{q_0,1}(X'_{\perp,p},0)\,\Pi'^{p}_{j=1}\mathcal{R}_{\lambda,t}(x_{3,j}-x_{3,j+1})\Pi_{j=1}^{p}dX'_{\perp,j}dx_{3,j}. \quad (7.4)$$

Here and in the sequel, if $p=2$, then the product $\Pi_{j=2}^{p-1}K_{q_0,b}(X_{\perp,j},X_{\perp,j+1})$ should be set equal to one. The modulus of the integrand on the r.h.s of (7.4) is upper-bounded by the $L^1(\mathbb{R}^{3p})$-function

$$\left(\frac{1+|t|}{2\sqrt{|\lambda|}}\right)^p \langle X'_{\perp,1}\rangle^{-m_\perp}\mathcal{P}(X'_{\perp,2},\ldots,X'_{\perp,p})e^{-g\sum_{j=2}^{p}|X'_{\perp,j}|^2}\Pi_{j=1}^{p}\langle x_{3,j}\rangle^{-m_3}$$

where $\mathcal{P}$ is a polynomial, and g is a positive constant. Moreover, for each $({X_\perp}'_{,1},x_{3,1},\ldots,{X_\perp}'_{,p},x_{3,p})$ we have

$$\lim_{b\to\infty}|V(X'_{\perp,1},x_{3,1})|\Pi_{j=2}|V(X'_{\perp,1}+b^{-1/2}X'_{\perp,j},x_{3,j})| = \Pi_{j=1}^{p}|V(X'_{\perp,1},x_{3,j})|.$$

Applying the dominated convergence theorem, we find that (7.4) entails

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ ((A_{q_0}+tB_{q_0})^pJ^r) =$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^p}\Pi_{j=1}^{p}|V(X_{\perp,1},x_{3,j})|\Pi'^{p}_{j=1}\mathcal{R}_{\lambda,t}(x_{3,j}-x_{3,j+1})(\mathrm{sign}\,V(X_{\perp,1},x_{3,1}))^r dX_{\perp,1}\Pi_{j=1}^{p}dx_{3,j}\ \times$$

$$\int_{\mathbb{R}^{2(p-1)}}K_{q_0,1}(0,X_{\perp,2})\Pi_{j=2}^{p-1}K_{q_0,1}(X_{\perp,j},X_{\perp,j+1})K_{q_0,1}(X_{\perp,p},0)\Pi_{j=2}^{p}dX_{\perp,j} =$$

$$\int_{\mathbb{R}^2}\mathrm{Tr}\ \left((\alpha(X_{\perp,1},\lambda)+t\beta(X_{\perp,1},\lambda))^p\iota(X_{\perp,1})^r\right)dX_{\perp,1}\ \times$$

$$\int_{\mathbb{R}^{2(p-1)}}K_{q_0,1}(0,X_{\perp,2})\Pi_{j=2}^{p-1}K_{q_0,1}(X_{\perp,j},X_{\perp,j+1})K_{q_0,1}(X_{\perp,p},0)\Pi_{j=2}^{p}dX_{\perp,j}, \quad (7.5)$$

uniformly in $\lambda\in\Delta$. In order to conclude that (7.5) is equivalent to (7.2), it remains to note that

$$\int_{\mathbb{R}^{2(p-1)}}K_{q_0,1}(0,X_{\perp,2})\Pi_{j=2}^{p-1}K_{q_0,1}(X_{\perp,j},X_{\perp,j+1})K_{q_0,1}(X_{\perp,p},0)\Pi_{j=2}^{p}dX_{\perp,j} = K_{q_0,1}(0,0)=\frac{1}{2\pi}.$$

□

Corollary 7.2. *For any polynomial ϕ and any $t\in\mathbb{R}$, one has*

$$\lim_{b\to\infty}b^{-1}\,\mathrm{Tr}(\phi(J+A_{q_0}+tB_{q_0})-\phi(J)) = \frac{1}{2\pi}\int_{\mathbb{R}^2}dX_\perp\ \mathrm{Tr}(\phi(\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))-\phi(\iota(X_\perp))),$$

where the convergence is uniform in λ on compact subsets of $\mathbb{R}\setminus\{0\}$.

Proof. By the linearity of the trace and of the integral, it is sufficient to prove the lemma for $\phi(x)=x^k$, $k\in\mathbb{Z}_+$. Using the cyclicity of the trace, we have:

$$\mathrm{Tr}\left((J+A_{q_0}+tB_{q_0})^k-J^k\right) = \sum_{p=1}^{k}\frac{k!}{(k-p)!\,p!}\,\mathrm{Tr}\left((A_{q_0}+tB_{q_0})^pJ^{k-p}\right).$$

Then applying again the cyclicity of the trace and Proposition 7.1, we deduce:

$$\lim_{b\to\infty}b^{-1}\,\mathrm{Tr}\left((J+A_{q_0}+tB_{q_0})^k-J^k\right) =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\sum_{p=1}^{k}\frac{k!}{(k-p)!\,p!}\,\mathrm{Tr}\ \left((\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))^p\iota(X_\perp)^{k-p}\right)dX_\perp =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{Tr}\ \left((\iota(X_\perp)+\alpha(X_\perp,\lambda)+t\beta(X_\perp,\lambda))^k-\iota(X_\perp)^k\right)dX_\perp.$$

This completes the proof of Corollary 7.2. □

8 SSF asymptotics of order b: Proof of Theorem 2.3

In this section we use the above results and the notation (3.18), (3.20), (4.10), (7.1) to prove Theorem 2.3.

Lemma 8.1. *For any $q \in \mathbb{Z}_+$ and any $p > 2/m_\perp$, one has*

$$\|p_q \langle X_\perp\rangle^{-m_\perp} p_q\|_{S_p}^p = O(b), \quad b \to \infty.$$

The proof of the lemma can be found in Subsection 9.4.
Fix $q_0 \in \mathbb{Z}_+$. All 'constants' C appearing in the proof, may depend on V and q_0.

Lemma 8.2. *Assume (1.5) and let Δ be a compact interval in $\mathbb{R} \setminus \{0\}$. Then:*
(i) For all $b > 0$ and all $q \in \mathbb{Z}_+$, one has $T_q(2bq_0 + \lambda) \in S_1$ and

$$\sup_{\lambda\in\Delta} \|T_q(2bq_0 + \lambda)\|_{S_1} = O(b^{3/4}), \quad b \to \infty, \qquad \text{if } q \neq q_0,$$

$$\sup_{\lambda\in\Delta} \|T_{q_0}(2bq_0 + \lambda)\|_{S_1} = O(b), \quad b \to \infty.$$

(ii) For any $p > 2/m_\perp$ and $\lambda \neq 0$, one has $B_{q_0}(2bq_0 + \lambda) \in S_p$ and

$$\sup_{\lambda\in\Delta} \|B_{q_0}(2bq_0 + \lambda)\|_{S_p}^p = O(b), \quad b \to \infty.$$

Proof. First note that by (1.5), the proof reduces to the case $V(x) = \langle X_\perp\rangle^{-m_\perp}\langle x_3\rangle^{-m_3}$. Let us consider this case. One has

$$T_q(2bq_0 + \lambda) = \langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2} \otimes t(2b(q_0 - q) + \lambda).$$

Since the S_p-norms of the operators $p_q\langle X_\perp\rangle^{-m_\perp} p_q$ and $\langle X_\perp\rangle^{-m_\perp/2} p_q \langle X_\perp\rangle^{-m_\perp/2}$ coincide, the statement follows from Lemma 8.1, the bounds (4.4), (4.5), and the fact that $t(\lambda)$ is of rank two.

□

Proof of Theorem 2.3
1. Let us first prove that for any $\eta \in (0, 1)$,

$$\limsup_{b\to\infty} \sup_{\lambda\in\Delta_1\cup\Delta_2} \left\{ b^{-1} \int_{-\infty}^{\infty} \operatorname{ind}(J + A + tB, J) d\mu(t) - \int_{-\infty}^{\infty} b^{-1} \operatorname{ind}(J - \eta + A_{q_0} + tB_{q_0}, J - \eta) d\mu(t) \right\} \leq 0, \tag{8.1}$$

By (3.11) and (3.17),

$$\int_{-\infty}^{\infty} \operatorname{ind}(J + A + tB, J) d\mu(t) \leq$$

$$\int_{-\infty}^{\infty} \operatorname{ind}(J - \eta + A_{q_0} + tB_{q_0}, J - \eta) d\mu(t) + \int_{-\infty}^{\infty} n_-(\eta, A - A_{q_0} + t(B - B_{q_0})) d\mu(t) \leq$$

$$\int_{-\infty}^{\infty} \operatorname{ind}(J - \eta + A_{q_0} + tB_{q_0}, J - \eta) d\mu(t) + n_-(\eta/2, A - A_{q_0}) + \frac{2}{\pi\eta}\|B - B_{q_0}\|_{S_1}.$$

By a calculation similar to (5.25), we get

$$\sup_{\lambda\in\Delta_1\cup\Delta_2} \|A(2q_0 b + \lambda) - A_{q_0}(2q_0 b + \lambda)\| = O(b^{-1/2}),$$

and therefore $n_-(\eta/2, A - A_{q_0}) = 0$, $\lambda \in \Delta_1 \cup \Delta_2$ for all sufficiently large b. Next, by Lemma 8.2(i),

$$\sup_{\lambda\in\Delta_1\cup\Delta_2} \|B(2q_0b+\lambda) - B_{q_0}(2q_0b+\lambda)\|_{S_1} = \sup_{\lambda\in\Delta_1\cup\Delta_2} \|\sum_{q=0}^{q_0-1} B_q(2q_0b+\lambda)\|_{S_1} = O(b^{3/4}), \quad b\to\infty.$$

This proves the bound (8.1).

2. Below we prove that for any $\eta \in (0, 1/2)$,

$$\limsup_{b\to\infty} \sup_{\lambda\in\Delta_1\cup\Delta_2} \left\{ b^{-1}\int_{-\infty}^{\infty} \operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta)d\mu(t) - \frac{1}{2\pi}\int_{\mathbb{R}^2} dX_\perp \int_{-\infty}^{\infty} d\mu(t)\Xi_\lambda(2\eta, X_\perp, t) \right\} \le 0. \tag{8.2}$$

Fix $t \in \mathbb{R}$. Applying Corollary 7.2, Krein's trace formula (1.1), and (3.8) we get for any polynomial ϕ

$$\sup_{\lambda\in\Delta_1\cup\Delta_2} \left| \int_{-\infty}^{\infty} b^{-1} \operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta)\phi(\eta)d\eta - \frac{1}{2\pi}\int_{\mathbb{R}^2} dX_\perp \int_{-\infty}^{\infty} d\eta\, \Xi_\lambda(\eta, X_\perp, t)\phi(\eta) \right| \to 0 \tag{8.3}$$

as $b \to \infty$. Further, the bounds

$$\sup_{b>0} \sup_{\lambda\in\Delta_1\cup\Delta_2} \|T_{q_0}(2bq_0+\lambda)\| < \infty, \qquad \sup_{X_\perp\in\mathbb{R}^2} \sup_{\lambda\in\Delta_1\cup\Delta_2} \|\tau(X_\perp, \lambda)\| < \infty,$$

ensure that for all $\lambda \in \Delta_1 \cup \Delta_2$, the functions $\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta)$ and $\Xi_\lambda(\eta, X_\perp, t)$ vanish when η is outside some fixed interval $(-R, R)$, where R may depend on q_0, V, and t, but not on b or $X_\perp$. Together with the inclusion $\Xi_\lambda(\cdot,\cdot,t) \in L^1(\mathbb{R}\times\mathbb{R}^2, d\eta dX_\perp)$ (see Lemma 6.2(ii)) this allows us to change the order of integration in the second integral in (8.3). This yields

$$\sup_{\lambda\in\Delta_1\cup\Delta_2} \left| \int_{-R}^{R} d\eta\phi(\eta) \left(b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta) - \frac{1}{2\pi}\int_{\mathbb{R}^2} \Xi_\lambda(\eta, X_\perp, t)dX_\perp \right) \right| \to 0 \tag{8.4}$$

as $b \to \infty$.

3. Let us show that (8.4) extends to any $\phi \in C(-R, R)$. By (3.8), Krein's inequality (see e.g. [31, Theorem 8.2.1]), and Lemma 8.2(i),

$$\sup_{\lambda\in\Delta_1\cup\Delta_2} \int_{-R}^{R} |b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta)|d\eta = \sup_{\lambda\in\Delta_1\cup\Delta_2} \int_{-R}^{R} |b^{-1}\xi(\eta; J+A_{q_0}+tB_{q_0}, J)|d\eta$$
$$\le b^{-1} \sup_{\lambda\in\Delta_1\cup\Delta_2} \|A_{q_0}+tB_{q_0}\|_{S_1} = O(1),$$

as $b \to \infty$. Thus, the norms of $b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0}, J-\eta)$ are uniformly bounded in $L^1(-R, R)$, and so using the Weierstrass theorem, we can extend (8.4) by continuity from polynomials onto all $\phi \in C(-R, R)$.

4. Consider (8.4) with any $\phi \in C(-R, R)$ such that $\operatorname{supp}\phi \in (-1/2, 1/2)$. Let us apply the dominated convergence theorem to show that (8.4) can be integrated over $d\mu(t)$. For any $\eta \in (-1/2, 1/2)$, proceeding similarly to (3.16), (3.17), and applying Lemma 8.2(ii), one obtains

$$|b^{-1}\xi(\eta; J+A_{q_0}+tB_{q_0}, J)| \le b^{-1}n_*(1/2, A_{q_0}+tB_{q_0}) \le b^{-1}\|4A_{q_0}\|_{S_1} + b^{-1}\|4tB_{q_0}\|_{S_p}^p \le C + Ct^p, \tag{8.5}$$

uniformly over $\eta \in (-1/2, 1/2)$ and $\lambda \in \Delta_1 \cup \Delta_2$. Choosing $2/m_\perp < p < 1$, we get an integrable bound for the first term in (8.4). An integrable bound for the second term in (8.4) is provided by Lemma 6.2.

Thus, (8.4) can be integrated over $d\mu(t)$ and so we obtain

$$\sup_{\lambda\in\Delta_1\cup\Delta_2}\left|\int_{-\infty}^{\infty}d\mu(t)\int_{-1/2}^{1/2}d\eta\phi(\eta)\left(b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0},J-\eta)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\Xi_\lambda(\eta,X_\perp,t)dX_\perp\right)\right|$$
$$\leq\int_{-\infty}^{\infty}d\mu(t)\sup_{\lambda\in\Delta_1\cup\Delta_2}\left|\int_{-1/2}^{1/2}d\eta\phi(\eta)\left(b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0},J-\eta)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\Xi_\lambda(\eta,X_\perp,t)dX_\perp\right)\right|$$
$$\to 0 \quad (8.6)$$

as $b\to\infty$. The bounds (8.5) and (6.6) show that one can interchange the order of integration in the l.h.s. of (8.6), which yields

$$\sup_{\lambda\in\Delta_1\cup\Delta_2}\left|\int_{-1/2}^{1/2}d\eta\phi(\eta)\int_{-\infty}^{\infty}d\mu(t)\left(\frac{1}{b}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0},J-\eta)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\Xi_\lambda(\eta,X_\perp,t)dX_\perp\right)\right|\to 0 \quad (8.7)$$

as $b\to\infty$.
5. Using the chain rule (3.7) and the property (e) of ind, it is straightforward to see that the functions $\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0},J-\eta)$ and $\Xi_\lambda(\eta,X_\perp,t)$ are non-decreasing in $\eta\in(-1,1)$. For a given $\eta_0\in(0,1/2)$, choose a continuous nonnegative function ϕ with support in $(\eta_0,2\eta_0)$ such that $\int_{\eta_0}^{2\eta_0}\phi(\eta)d\eta=1$. Then

$$\int_{-1/2}^{1/2}d\eta\phi(\eta)\int_{-\infty}^{\infty}d\mu(t)\left(b^{-1}\operatorname{ind}(J-\eta+A_{q_0}+tB_{q_0},J-\eta)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\Xi_\lambda(\eta,X_\perp,t)\right)$$
$$\geq\int_{-\infty}^{\infty}d\mu(t)\left(b^{-1}\operatorname{ind}(J-\eta_0+A_{q_0}+tB_{q_0},J-\eta_0)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\Xi_\lambda(2\eta_0,X_\perp,t)\right).$$

Combining this with (8.7), we get (8.2).
6. Combining (8.1), (8.2) and (3.21) we obtain

$$\limsup_{b\to\infty}\operatorname*{ess\,sup}_{\lambda\in\Delta_1\cup\Delta_2}\left\{b^{-1}\xi(2bq_0+\lambda;H(b),H_0(b))-\frac{1}{2\pi}\int_{\mathbb{R}^2}dX_\perp\int_{-\infty}^{\infty}d\mu(t)\Xi_\lambda(2\eta,X_\perp,t)\right\}\leq 0. \quad (8.8)$$

Similarly, one obtains

$$\liminf_{b\to\infty}\operatorname*{ess\,inf}_{\lambda\in\Delta_1\cup\Delta_2}\left\{b^{-1}\xi(2bq_0+\lambda;H(b),H_0(b))-\frac{1}{2\pi}\int_{\mathbb{R}^2}dX_\perp\int_{-\infty}^{\infty}d\mu(t)\Xi_\lambda(-2\eta,X_\perp,t)\right\}\geq 0. \quad (8.9)$$

Further arguments are different for Δ_1 and Δ_2. Let us first consider the interval Δ_1. By Lemma 6.3(ii), the integral $\int_{\mathbb{R}^2}dX_\perp\int_{-\infty}^{\infty}d\mu(t)\Xi_\lambda(\eta,X_\perp,t)$ is continuous in $(\lambda,\eta)\in\Delta_1\times[-1/2,1/2]$ and therefore is uniformly continuous on this set. It follows that

$$\sup_{\lambda\in\Delta_1\cup\Delta_2}\left|\int_{\mathbb{R}^2}dX_\perp\int_{-\infty}^{\infty}d\mu(t)\left(\Xi_\lambda(\eta,X_\perp,t)-\Xi_\lambda(0,X_\perp,t)\right)\right|\to 0 \text{ as } \eta\to 0. \quad (8.10)$$

Combining (8.8), (8.9), recalling that η can be taken arbitrary small, and using (8.10) and (6.4), we obtain the desired limiting relation (2.2).
7. Finally, using (8.8), (8.9), let us prove the relations (2.3), (2.4). Note that here $\lambda<0$ and therefore $\beta(X_\perp,\lambda)=0$. For any $\eta\in(-1,1)$, $X_\perp\in\mathbb{R}^2$ by the representation (3.15), formula (1.3), and a continuity argument we obtain

$$\Xi_\lambda(\eta,X_\perp,t)=\operatorname{ind}(\iota(X_\perp)+\alpha(X_\perp,\lambda)-\eta,\iota(X_\perp)-\eta)=-\lim_{\epsilon\to+0}N(\lambda+\epsilon;\chi_0+|V(X_\perp,\cdot)|(\iota(X_\perp)-\eta)^{-1}).$$

Next, for $|\eta| \leq 1/2$ we have

$$N(\lambda+\epsilon; \chi_0 + |V(X_\perp, \cdot)|(\iota(X_\perp) - \eta)^{-1}) = N(\lambda+\epsilon; \chi_0 + V(X_\perp, \cdot) + |V(X_\perp, \cdot)|\eta(1-\iota(X_\perp)\eta)^{-1})$$
$$\geq N(\lambda+\epsilon - 2C_0|\eta|, \chi_0 + V(X_\perp, \cdot)),$$

and therefore we get

$$\Xi_\lambda(\eta, X_\perp, t) \leq \Xi_{\lambda - 2C_0|\eta|}(0, X_\perp, t). \tag{8.11}$$

Similarly, for $|\eta| \leq 1/2$, $2C_0\eta < -\lambda$, we obtain

$$\Xi_\lambda(\eta, X_\perp, t) \geq \Xi_{\lambda + 2C_0|\eta|}(0, X_\perp, t). \tag{8.12}$$

Together with (8.8), (8.9), and (6.4) this completes the proof.

9 Spectral properties of the Landau Hamiltonian

9.1 Creation and annihilation operators. Angular momentum eigenbases

In this subsection we construct orthonormal bases of the subspaces $\mathcal{H}_q := \text{Ker}\,(h(g) - 2bq) = p_q L^2(\mathbb{R}^2)$, $q \in \mathbb{Z}_+$ (see Subsection 4.1), known in the physics literature as *the angular-momentum eigenbases* (see [13], [17]). For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ introduce the complex variable $\zeta := x_1 + ix_2$, set

$$\partial := \frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \overline{\partial} := \frac{\partial}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

and define the annihilation and the creation operators

$$a = a(b) := -2i\left(\overline{\partial} + \frac{b}{4}\zeta\right) = -2ie^{-b|\zeta|^2/4}\,\overline{\partial}\,e^{b|\zeta|^2/4}, \tag{9.1}$$

$$a^* = a(b)^* := -2i\left(\partial - \frac{b}{4}\overline{\zeta}\right) = -2ie^{b|\zeta|^2/4}\,\partial\,e^{-b|\zeta|^2/4}, \tag{9.2}$$

defined originally on the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, and then closed in $L^2(\mathbb{R}^2)$. It is easy to see that the operators a and a^* are mutually adjoint. Moreover, on $\mathcal{S}(\mathbb{R}^2)$ we have

$$[a(b), a(b)^*] = 2b, \tag{9.3}$$

and $h(b) = a^*a$. Therefore, the common domain of both closed operators a and a^* coincides with the domain of $h(b)^{1/2}$. A standard argument from the representation theory of the Heisenberg algebra yields $\mathcal{H}_0 = \text{Ker}\, a$, $\mathcal{H}_q = (a^*)^q \mathcal{H}_0$, $q \geq 1$, (see e.g. [6, Section 5.2]). Moreover, Ker $a = \left\{f \in L^2(\mathbb{R}^2) | f = ge^{-b|\mathbf{x}|^2/4},\ \overline{\partial}g = 0\right\}$. Hence, it is easy to check that the functions

$$\varphi_{0,k}(\mathbf{x}) := \frac{1}{\sqrt{\pi k!}}\left(\frac{b}{2}\right)^{(k+1)/2}(x_1 + ix_2)^k e^{-b|\mathbf{x}|^2/4}, \quad \mathbf{x} \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+, \tag{9.4}$$

form an orthonormal basis of $\mathcal{H}_0$, while the functions

$$\varphi_{q,k}(\mathbf{x}) := \frac{1}{\sqrt{(2b)^q q!}}\left((a^*)^q \varphi_{0,k}\right)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \quad k \in \mathbb{Z}_+, \tag{9.5}$$

form an orthonormal basis in $\mathcal{H}_q$, $q \geq 1$. The functions $\varphi_{q,k}$ can be written in polar coordinates (ρ, θ) as

$$\varphi_{q,k}(\rho\cos\theta, \rho\sin\theta) = (-i)^q \sqrt{\frac{q!}{\pi k!}}\left(\frac{b}{2}\right)^{\frac{k-q+1}{2}} e^{i(k-q)\theta}\rho^{k-q} L_q^{(k-q)}\left(b\rho^2/2\right) e^{-b\rho^2/4} \tag{9.6}$$

where

$$L_q^{(k-q)}(\xi) := \sum_{l=\max\{0,q-k\}}^{q} \frac{k!}{(k-q+l)!(q-l)!} \frac{(-\xi)^l}{l!}, \quad \xi \geq 0, \quad q \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+, \tag{9.7}$$

are the generalized Laguerre polynomials.

9.2 Proof of Lemma 5.2

Lemma 5.2 is an immediate corollary from the following

Lemma 9.1. *Let* $v \in \mathrm{Lip}(\mathbb{R}^2)$ *such that* $\sup_{x\in\mathbb{R}^2}(|v(x)| + |\nabla v(x)|) < \infty$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\|(1-p_q)vp_q\| \leq b^{-1/2}\sqrt{2}\left(\sqrt{q+1}+\sqrt{q}\right) \sup_{\mathbf{x}\in\mathbb{R}^2} |\nabla v(\mathbf{x})|. \tag{9.8}$$

Proof. Without any loss of generality we can assume that $v \in C^\infty(\mathbb{R}^2)$, and all its derivatives are uniformly bounded on $\mathbb{R}^2$. We have $(1-p_q)vp_q = (1-p_q)[v,p_q]p_q$ where $[v,p_q] := vp_q - p_q v$ denotes the commutator of the operators v and p_q. Suppose that $2b > 1$. Then $p_q = \frac{1}{2\pi i}\int_{\Gamma_q}(z-h)^{-1}dz$ where Γ_q is the circle $\{z \in \mathbb{C} \mid |z-2bq| = 1\}$ run over in the positive direction. Therefore,

$$[v,p_q] = -\frac{1}{2\pi i}\int_{\Gamma_q}[(z-h)^{-1},v]dz = -\frac{1}{2\pi i}\int_{\Gamma_0}(z+2bq-h)^{-1}[h,v](z+2bq-h)^{-1}dz.$$

Since $h = a^*a$, we get $[h,v] = [a^*a,v] = a^*[a,v] + [a^*,v]a = -2i(a^*\bar{\partial}V + \partial V a)$. Hence,

$$(1-p_q)[v,p_q]p_q = 2i(h-2bq)^{-1}(1-p_q)a^*\bar{\partial}Vp_q + 2i(h-2bq)^{-1}(1-p_q)\partial V a p_q. \tag{9.9}$$

It is easy to check that

$$\|(h-2bq)^{-1}(1-p_q)a^*\| = \|a(h-2bq)^{-1}(1-p_q)\| = \sqrt{\frac{q+1}{2b}}, \quad \|p_q\| = 1, \tag{9.10}$$

$$\|(h-2bq)^{-1}(1-p_q)\| = \frac{1}{2b}, \quad \|ap_q\| = \sqrt{2bq}. \tag{9.11}$$

Combining (9.9) with (9.10) – (9.11), we immediately get (9.8). ☐

9.3 Unitary equivalence of Toeplitz operators corresponding to different Landau levels

By $\langle f,g\rangle := \int_{\mathbb{R}^2} f\overline{g}d\mathbf{x}$, $f,g \in L^2(\mathbb{R}^2)$, we denote the scalar product in $L^2(\mathbb{R}^2)$. Moreover, we use the notation $(f,g) := \langle f,\overline{g}\rangle$, $f,g \in L^2(\mathbb{R}^2)$. If $g \in \mathcal{S}(\mathbb{R}^2)$ is fixed, then the functionals (f,g) and $\langle f,g\rangle$ can be extended by continuity to $f \in \mathcal{S}'(\mathbb{R}^2)$. Note also that if $f \in \mathcal{S}'(\mathbb{R}^2)$, $h \in \mathcal{S}(\mathbb{R}^2)$, then the product fh is a well-defined element of $\mathcal{S}'(\mathbb{R}^2)$, acting according to $(fh,g) = (f,hg)$.

Lemma 9.2. *Let* $F \in \mathcal{S}'(\mathbb{R}^2)$. *Let* $j,k,q \in \mathbb{Z}_+$. *Then we have*

$$\langle F\varphi_{q,k},\varphi_{q,j}\rangle = \langle(\mathcal{D}_q F)\varphi_{0,k},\varphi_{0,j}\rangle \tag{9.12}$$

where

$$(\mathcal{D}_q F)(\mathbf{x}) = \sum_{s=0}^{q} d_{s,q}(\Delta^s F)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{9.13}$$

and

$$d_{s,q} := \frac{q!}{(s!)^2(q-s)!}(2b)^{-s}. \tag{9.14}$$

Proof. First of all note that the functions $\varphi_{q,k}\overline{\varphi}_{q,j}$ as well as $\varphi_{0,k}\overline{\varphi}_{0,j}$ are in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. Therefore, we can assume without loss of generality that $F \in \mathcal{S}(\mathbb{R}^2)$ since $\mathcal{S}(\mathbb{R}^2)$ is dense in $\mathcal{S}'(\mathbb{R}^2)$, and $\mathcal{D}_q : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ is a continuous mapping. Then all the functions $(D^\alpha F)\varphi_{s,k} \equiv \frac{1}{\sqrt{s!(2b)^s}}(D^\alpha F)(a^*)^s\varphi_{0,k}$ with $\alpha \in \mathbb{Z}_+^2$, $s \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$ are in $\mathcal{S}(\mathbb{R}^2)$, and hence in the domain of the operators a and a^*. Further, (9.1) - (9.2) imply

$$[a, F] = -[F, a] = -2i\overline{\partial}F, \quad [a^*, F] = -[F, a^*] = -2i\partial F, \tag{9.15}$$

and, consequently,

$$[[a, F], a^*] = -2i\left[\overline{\partial}F, a^*\right] = 4\partial\overline{\partial}F = \Delta F. \tag{9.16}$$

Fix $j, k, q \in \mathbb{Z}_+$. First we will prove by induction with respect to $q \in \mathbb{Z}_+$, that (9.12) is valid with coefficients $d_{s,q}$ which depend only on s, q, and b.
Fix $Q \in \mathbb{Z}_+$. Assume that (9.12) is valid with coefficients $d_{s,q}$ which depend only on s, q, and b for every $q = 0, \dots, Q$. We will prove that the analogous relations hold for $q = Q + 1$. We have

$$\langle F\varphi_{Q+1,k}, \varphi_{Q+1,j}\rangle = \frac{1}{(Q+1)!(2b)^{Q+1}}\langle F(a^*)^{Q+1}\varphi_{0,k}, (a^*)^{Q+1}\varphi_{0,j}\rangle. \tag{9.17}$$

Utilizing the commutation relations (9.3), (9.16), and (9.15), and taking into account the fact that $\varphi_{0,j}$ is in Ker a, we get

$$\langle F(a^*)^{Q+1}\varphi_{0,k}, (a^*)^{Q+1}\varphi_{0,j}\rangle = 2b(Q+1)\langle F(a^*)^Q\varphi_{0,k}, (a^*)^Q\varphi_{0,j}\rangle + \langle [a, F](a^*)^{Q+1}\varphi_{0,j}, (a^*)^Q\varphi_{0,k}\rangle =$$

$$2b(Q+1)\langle F(a^*)^Q\varphi_{0,k}, (a^*)^Q\varphi_{0,j}\rangle + \langle \Delta F(a^*)^Q\varphi_{0,k}, (a^*)^Q\varphi_{0,j}\rangle + 2bQ\langle [a, F](a^*)^Q\varphi_{0,k}, (a^*)^{Q-1}\varphi_{0,j}\rangle$$

if $Q > 0$; if $Q = 0$, the last term should be set to be equal to zero. By recurrence we obtain

$$\langle F(a^*)^{Q+1}\varphi_{0,k}, (a^*)^{Q+1}\varphi_{0,j}\rangle = 2b(Q+1)\langle F(a^*)^Q\varphi_{0,k}, (a^*)^Q\varphi_{0,j}\rangle +$$

$$Q!(2b)^Q\sum_{q=0}^{Q}\frac{1}{(2b)^q q!}\langle \Delta F(a^*)^q\varphi_{0,k}, (a^*)^q\varphi_{0,j}\rangle. \tag{9.18}$$

Combining (9.17) with (9.18), we get

$$\langle F\varphi_{Q+1,k}, \varphi_{Q+1,j}\rangle = \langle F\varphi_{Q,k}, \varphi_{Q,j}\rangle + \frac{1}{2b(Q+1)}\sum_{q=0}^{Q}\langle \Delta F\varphi_{q,k}, \varphi_{q,j}\rangle = \sum_{s=0}^{Q+1} d_{s,Q+1}\langle \Delta^s F\varphi_{0,k}, \varphi_{0,j}\rangle$$

with

$$d_{s,Q+1} := \begin{cases} d_{0,Q} \quad \text{if} \quad s = 0, \\ \frac{1}{2b(Q+1)}\sum_{n=s}^{Q+1} d_{s-1,n-1} + d_{s,Q} \quad \text{if} \quad 1 \le s \le Q, \\ \frac{1}{2b(Q+1)} d_{Q,Q} \text{ if } s = Q+1. \end{cases}$$

Evidently, $d_{s,Q+1}$ depends only on s, Q and b. In order to demonstrate that the numerical values of the coefficients $d_{s,q}$ are given by (9.13) we proceed in the following way. Set $F_p(\mathbf{x}) := |\mathbf{x}|^{2p}$, $p = 0, \dots, q$. Then the identities

$$\langle F_p\varphi_{q,0}, \varphi_{q,0}\rangle = \langle (\mathcal{D}_q F_p)\varphi_{0,0}, \varphi_{0,0}\rangle = \sum_{s=0}^{q} d_{s,q}\langle (\Delta^s F_p)\varphi_{0,0}, \quad p = 0, \dots, q. \tag{9.19}$$

are special cases of (9.12). A straightforward calculation yields

$$\langle F_p\varphi_{q,0}, \varphi_{q,0}\rangle = \frac{(p+q)!}{q!}(b/2)^{-p},$$

$$\langle (\Delta^s F_p)\, \varphi_{0,0}, \varphi_{0,0}\rangle = \begin{cases} 4^s \frac{(p!)^2}{(p-s)!}(b/2)^{s-p} & \text{if} \quad 0 \le s \le p, \\ 0 \quad \text{if} \quad p < s \le q, \end{cases}$$

Therefore, it follows from (9.19) that $d_{s,q}$, $s = 0, \dots, q$, is be the unique solution of the linear system

$$\sum_{s=0}^{p} \frac{4^s (p!)^2}{(p-s)!}(b/2)^{s-p}\, d_{s,q} = \frac{(p+q)!}{q!}(b/2)^{-p}, \quad p = 0, \dots, q. \tag{9.20}$$

Setting

$$d_{s,q} = (2b)^{-s} h_{s,q}, \quad s = 0, \dots, q, \tag{9.21}$$

we find that (9.20) is equivalent to the system

$$\sum_{s=0}^{p} \frac{p!}{(p-s)!}\, h_{s,q} = \binom{p+q}{p}, \quad p = 0, \dots, q. \tag{9.22}$$

Taking into account the elementary combinatorial identity $\sum_{s=0}^{p} \binom{q}{s}\binom{p}{p-s} = \binom{p+q}{p}$ (see e.g. [16, Eq. 0.165]), we find that the solution of (9.22) is given by

$$h_{s,q} = \binom{q}{s}\frac{1}{s!}, \quad s = 0, \dots, q. \tag{9.23}$$

The combination of (9.21) and (9.23) yields (9.14). □

Corollary 9.3. *Let $q \in \mathbb{Z}_+$ and F be the multiplication operator by a real function $F \in C^{2q}(\mathbb{R}^2)$. Assume that $\Delta^s F \in L^\infty(\mathbb{R}^2)$, $s = 0, \dots, q$. Then the operator $p_q(b) F p_q(b) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is unitarily equivalent to the operator $p_0(b)(\mathcal{D}_q F)p_0(b) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, the differential operation $\mathcal{D}_q$ being defined in* (9.13) - (9.14).

9.4 Proof of Lemma 8.1

For $\theta > 0$ and $q \in \mathbb{Z}_+$ define the operator $\mathcal{G}_{q,\theta,b} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by

$$\mathcal{G}_{q,\theta,b} := p_q(b)\; W_\theta\; p_q(b).$$

where W_θ is the multiplication operator by the function $\langle \mathbf{x}\rangle^{-2/\theta}$, $\mathbf{x} \in \mathbb{R}^2$. It is easy to check that $\mathcal{G}_{q,\theta,b}$ is compact, self-adjoint and non-negative (see [24, Lemma 5.1]).
Now Lemma 8.1 is a direct corollary of the following

Lemma 9.4. *Let $\theta > 0$, $q \in \mathbb{Z}_+$, $s > 0$, and $b \ge 1/2$. Then we have*

$$n_+(s; \mathcal{G}_{q,\theta,b}) \le C_3 + C_4 b s^{-\theta} \tag{9.24}$$

where $C_3 = C_3(\theta)$ and $C_4 = C_4(q, \theta)$ are independent of s and b.

Proof. By Lemma 9.3 the non-zero eigenvalues of the operator $\mathcal{G}_{q,\theta,b}$ coincide with the non-zero eigenvalues of the operator $\tilde{\mathcal{G}}_{q,\theta,b} = p_0(b)\; (\mathcal{D}_q W_\theta)\; p_0(b)$. Evidently, there exists a function $w_{q,\theta,b} : [0,\infty) \to \mathbb{R}$ different from zero almost everywhere such that $(\mathcal{D}_q W_\theta)(\mathbf{x}) = w_{q,\theta,b}(|\mathbf{x}|^2)$, $\mathbf{x} \in \mathbb{R}^2$. By [28, Lemma 3.3] the non-zero eigenvalues ν_k of the operator $\tilde{\mathcal{G}}_{q,\theta,b}$ can be written as

$$\nu_k = \nu_k(q, \theta, b) := \frac{1}{k!}\int_0^\infty w_{q,b,\theta}(2\eta/b) e^{-\eta} \eta^k d\eta, \quad k \in \mathbb{Z}_+. \tag{9.25}$$

Since $b/2 \geq 1$ we have $d_{s,q} \leq q!$ (see (9.13) – (9.14)). Hence, $w_{q,b,\theta}(t) \leq C_5\langle t\rangle^{-1/\theta}$, $t \in \mathbb{R}$, with C_5 which may depend on q and θ but not on b and t. Therefore, (9.25) implies

$$\nu_k(q,b,\theta) = \frac{1}{k!}\int_0^\infty w_{q,b,\theta}(2\eta/b)e^{-\eta}\eta^k d\eta \leq \frac{C_5}{k!}\int_0^\infty \langle 2\eta/b\rangle^{-1/\theta}e^{-\eta}\eta^k d\eta \leq$$

$$C_5\left(\frac{b}{2}\right)^{1/\theta}\frac{1}{k!}\int_0^\infty e^{-\eta}\eta^{k-1/\theta}d\eta = \frac{C_5}{2^{1/\theta}}\, b^{1/\theta}\,\frac{\Gamma(k+1-1/\theta)}{\Gamma(k+1)}, \quad k \in \mathbb{Z}_+, \quad k > -1+1/\theta.$$

Since $\lim_{k\to\infty} k^{1/\theta}\frac{\Gamma(k+1-1/\theta)}{\Gamma(k+1)} = 1$ (see [1, Eq. (6.1.46)]), there exists $k_0 = k_0(\theta) \geq 1$ such that $k \geq k_0$ implies $\nu_k(q,\theta,b) \leq \frac{2C_5}{2^{1/\theta}}(k/b)^{-1/\theta}$. Therefore,

$$n_+(s;\mathcal{G}_{q,b,\theta}) = n_+(s;\tilde{\mathcal{G}}_{q,b,\theta}) = \#\{k \in \mathbb{Z}_+ | \nu_k(q,\theta,b) > s\} \leq$$

$$k_0 + \#\left\{k > k_0 \mid \frac{2C_5}{2^{1/\theta}}(k/b)^{-1/\theta} > s\right\} \leq k_0 + \frac{1}{2}\,(2C_5)^\theta\, bs^{-\theta}$$

which is equivalent to (9.24) with $C_3 = k_0$, $C_4 = (2C_5)^\theta/2$. □

Acknowledgements. V. Bruneau and G. Raikov gratefully acknowledge the financial support of the Chilean Science Foundation *Fondecyt* under Grants 1020737 and 7020737. A. Pushnitski was supported by the EPSRC grant no. GR/R53210/01.

References

[1] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Eds. M.Abramowitz, I.Stegun, National Bureau of Standards, Applied Mathematics Series **55**, 1964.

[2] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa **2** (1975), no.4, 151-218.

[3] J.Avron, I.Herbst, B.Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), 847-883.

[4] J.Avron, R.Seiler, B.Simon, *The index of a pair of projections*, J. Funct. Anal. **120** (1994), no. 1, 220–237.

[5] A. Balazard-Konlein, *Calcul fonctionnel pour des opérateurs h-admissibles à symbole opérateur et applications*, Thèse de Docteur de 3ème cycle, Université de Nantes, 1985.

[6] F.A.Berezin, M.A.Shubin *The Schrödinger Equation.* Kluwer Academic Publishers, Dordrecht, 1991.

[7] M.Š.Birman, *On the spectrum of singular boundary value problems*, Mat. Sbornik **55** (1961) 125-174 [in Russian]; English translation in: Amer. Math. Soc. Transl., (2) **53** (1966), 23-80.

[8] M. Š. Birman, S. B. Èntina, *Stationary approach in abstract scattering theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 401–430 [in Russian]; English translation: Math. USSR-Izv. **1**, no. 1 (1967), 391-420.

[9] M.Š.Birman, M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144** (1962), 475–478 [in Russian]; English translation in Soviet Math. Doklady **3** (1962).

[10] M.Š.Birman, D.R.Yafaev, *The spectral shift function. The papers of M. G. Kreĭn and their further development*, (Russian) Algebra i Analiz **4** (1992), 1–44; English translation in St. Petersburg Math. J. **4** (1993), 833–870.

[11] V.S. Buslaev, L.D. Faddeev, *Formulas for traces for a singular Sturm-Liouville differential operator*, Dokl. Akad. Nauk SSSR **132** 13–16 (Russian); English translation in Soviet Math. Dokl.**1** (1960), 451–454

[12] M. Dimassi, J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit.* LMS Lecture Note Series, **268**, Cambridge University Press Cambridge, 1999.

[13] V.Fock, *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*, Z. Physik **47** (1928), 446-448.

[14] C. Gérard, I. Łaba, *Multiparticle Quantum Scattering in Constant Magnetic Fields*, Mathematical Surveys and Monographs, **90**, AMS, Providence, RI, 2002.

[15] F. Gesztesy, K. Makarov, *The* Ξ *operator and its relation to Krein's spectral shift function*, J. Anal. Math. **81** (2000), 139–183.

[16] I. S.Gradshteyn, I. M.Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[17] T.Hupfer, H.Leschke, S.Warzel, *Upper bounds on the density of states of single Landau levels broadened by Gaussian random potentials*, J. Math. Phys. **42** (2001), 5626-5641.

[18] V. Ivrii, *Microlocal analysis and precise spectral asymptotics*, Springer monographs in Math. Springer, Berlin, 1998.

[19] M. G. Krein, *On the trace formula in perturbation theory*, Mat. Sb. **33 (75)**, no. 3 (1953), 597–626 [in Russian]

[20] L. Landau, *Diamagnetismus der Metalle*, Z. Physik **64** (1930), 629-637.

[21] A. Pushnitskiĭ, *A representation for the spectral shift function in the case of perturbations of fixed sign*, Algebra i Analiz **9** (1997), 197–213 [in Russian]; English translation in St. Petersburg Math. J. **9** (1998), 1181–1194.

[22] A. Pushnitski, *Estimates for the spectral shift function of the polyharmonic operator*, J. Math. Phys. **40** (1999), 5578–5592.

[23] A. Pushnitski, *The spectral shift function and the invariance principle*, J. Funct. Anal. **183** (2001), 269–320.

[24] G. D. Raikov, *Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips*, Commun. P.D.E. **15** (1990), 407-434; Errata: Commun. P.D.E. **18** (1993), 1977-1979.

[25] G.D.Raikov, *Eigenvalue asymptotics for the Schrödinger operator in strong constant magnetic fields*, Comm. Partial Differential Equations 23 (1998), no. 9-10, 1583–1619.

[26] G. D. Raikov, *Asymptotic properties of the magnetic integrated density of states*, Electron. J. Diff. Eq. **1999**, No. 13, 27 pp. (1999).

[27] G.D.Raikov, M. Dimassi, *Spectral asymptotics for quantum Hamiltonians in strong fields*, Cubo Mat. Educ. **3** (2001), 317 - 391.

[28] G.D.Raikov, S.Warzel, *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials*, Rev. Math. Phys. **14** (2002), 1051–1072.

[29] M.Reed, B.Simon, *Methods of Modern Masthematical Physics. 4. Analysis of Operators*, Academic Press, New York – London, 1978.

[30] A. V. Sobolev, *Efficient bounds for the spectral shift function*, Ann. Inst. H. Poincare Phys. Theor. **58**, no. 1 (1993), 55–83.

[31] D. R. Yafaev, *Mathematical scattering theory. General theory.* Translations of Mathematical Monographs, **105** American Mathematical Society, Providence, RI, 1992.

Vincent Bruneau
Mathématiques Appliquées de Bordeaux
Université Bordeaux I
351, Cours de la Libération
33405 Talence, France
E-mail: vbruneau@math.u-bordeaux.fr

Alexander Pushnitski
Department of Mathematical Sciences
Loughborough University
Loughborough, Leicestershire, LE11 3TU
United Kingdom
E-mail: A.B.Pushnitski@lboro.ac.uk

Georgi Raikov
Departamento de Matemáticas
Universidad de Chile
Las Palmeras 3425, Casilla 653
Santiago, Chile
E-mail: graykov@uchile.cl