Spectral shift function of the Schrödinger operator in the large coupling constant limit

A. B. Pushnitski and M. V. Ruzhansky

1. Introduction. Let H_0 and H be selfadjoint operators in a Hilbert space. If the difference $H - H_0$ is a trace class operator, then there exists a function $\xi \in L^1(\mathbb{R})$ such that the following trace formula due to I. M. Lifshitz and M. G. Krein holds true:

$$\mathrm{Tr}(\phi(H) - \phi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)\phi'(\lambda)d\lambda, \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

The function $\xi(\lambda) = \xi(\lambda; H, H_0)$ is called the spectral shift function for the pair H_0, H. A detailed exposition of the spectral shift function theory can be found in the book [9]; see also the survey [3].

Let us consider the (selfadjoint) Laplace operator Δ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Define the operator $H_0 = h(-\Delta)$, where the function $h : [0, +\infty) \to \mathbb{R}$ satisfies

$$\begin{gathered} h \in C^2(\mathbb{R}), \quad h(0) = 0, \quad h'(r) > 0 \quad \forall r > 0, \\ \text{there exists the limit} \quad \lim_{r\to+\infty} r^{-m}h(r) = h_\infty > 0, \quad m > 0. \end{gathered} \tag{1}$$

Next, let the perturbation V be the operator of multiplication by a real valued potential $V(x)$, which satisfies the estimate

$$|V(x)| \leq \frac{C}{(1 + |x|)^l}, \quad l > d. \tag{2}$$

Let $H = H_0 + V$. The most important case is the Schrödinger operator, which corresponds to the choice $h(r) = r$. However, we consider a fairly wide class of functions h in order to demonstrate the dependence of our results on the symbol of the (pseudo)differential operator H_0.

Although the difference $H - H_0$ is not of the trace class, condition (2) ensures that the difference of sufficiently high powers of the resolvents of H and H_0 is of the trace class. This allows one to define the spectral shift function $\xi(\lambda; H, H_0)$ on the basis of the invariance principle (cf. [3]).

Various results about the high energy ($\lambda \to +\infty$) or semiclassical ($h(r) = h_\infty r$, $h_\infty \to 0$) asymptotic behaviour of the spectral shift function $\xi(\lambda; H_0 + \alpha V, H_0)$ are known. In the present paper we address the question of the asymptotic behaviour of the spectral shift function $\xi(\lambda; H_0 + \alpha V, H_0)$ in the large coupling constant limit: $\alpha \to +\infty$.

2. Results. It turns out that the asymptotic behaviour of the spectral shift function depends heavily on the sign of the potential V. For non-positive potentials one has

Theorem 1 *Let h satisfy conditions (1). Let $H_0 = h(-\Delta)$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Assume that the potential $V \leq 0$ satisfies estimate (2) for $l > \max\{d, 2m\}$. Then for almost all $\lambda \in \mathbb{R}$ the following asymptotic formula holds true:*

$$\begin{aligned} \xi(\lambda; H_0 + \alpha V, H_0) &= -\alpha^{d/(2m)} C_1 (1 + o(1)), \quad \alpha \to +\infty, \\ C_1 &= (2\pi)^{-d} h_\infty^{-d/(2m)} \mathrm{vol}\{x \in \mathbb{R}^d \mid |x| < 1\} \int_{\mathbb{R}^d} |V(x)|^{d/(2m)} dx. \end{aligned} \tag{3}$$

Let us now discuss the case of non-negative potentials. In this case one has to consider potentials with power asymptotics at infinity. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Assume that for some non-negative function $\Psi \in C(\mathbb{S}^{d-1})$ one has

$$\sup_{\omega \in \mathbb{S}^{d-1}} |V(\rho\omega) - \Psi(\omega)\rho^{-l}| = o(\rho^{-l}), \ \rho \to \infty. \tag{4}$$

Theorem 2 *Let h satisfy conditions (1). Let $H_0 = h(-\Delta)$ in $L^2(\mathbb{R}^d)$, $d \geq 1$. Assume that the potential $V \geq 0$ is bounded and satisfies the condition (4) with some function $\Psi \in C(\mathbb{S}^{d-1})$, $\Psi \geq 0$, and some $l > d$. Then for all $\lambda > 0$ the following asymptotic formula holds true:*

$$\begin{aligned} \xi(\lambda; H_0 + \alpha V, H_0) &= \alpha^{d/l} C_2 (1 + o(1)), \quad \alpha \to +\infty, \\ C_2 &= (2\pi)^{-d} d^{-1} \int_{h(|p|^2) < \lambda} (\lambda - h(|p|^2))^{-d/l} dp \int_{\mathbb{S}^{d-1}} \Psi^{d/l}(\hat{x}) d\hat{x}. \end{aligned} \tag{5}$$

Theorem 1 with $h(r) = r$ has been proven by the first author in [5]. The case of general h can be easily dealt with by combining the techniques of [5] and [6]. Theorem 2 is a joint result of the authors [6].

The main ingredients of the proof of Theorem 2 are a representation for the spectral shift function from [4], the asymptotic formula for the spectrum of pseudo-differential operators [1, 2], the variational quotients technique, and several facts about the boundary value problems for elliptic pseudodifferential equations. All the difficulties of the proof appear already in the case $h(r) = r$; the generalization to the case of arbitrary h is not difficult.

3. Discussion. 1. For $\lambda < 0$ under the hypothesis of Theorem 1 the spectral shift function $\xi(\lambda)$ is the negative of the number of eigenvalues of the operator $H_0 + \alpha V$ in the interval $(-\infty, \lambda)$. Therefore, for $\lambda < 0$, $V \leq 0$ and $h(r) = r$, formula (3) turns into the well known Weyl asymptotic formula for the counting function for the spectrum of the Schrödinger operator (cf. [7, Theorem XIII.80]).

2. It is clear from (3) that the order of the leading term of the asymptotics of the spectral shift function depends on the growth order of the symbol h at infinity but does not depend of the potential V. For $V \geq 0$ the situation is opposite and the roles of the coordinate and momentum variables are reversed.

3. Asymptotic coefficients C_1 and C_2 can be interpreted in terms of the phase space volume. Indeed, one readily checks that

$$\begin{aligned} C_1 &= \lim_{\alpha\to+\infty} \alpha^{-d/(2m)} \mathrm{vol}\{(x,p)\in\mathbb{R}^{2d} \mid h(|p|^2)+\alpha V(x)<\lambda<h(|p|^2)\}, \\ C_2 &= \lim_{\alpha\to+\infty} \alpha^{-d/l} \mathrm{vol}\{(x,p)\in\mathbb{R}^{2d} \mid h(|p|^2)<\lambda<h(|p|^2)+\alpha V(x)\}. \end{aligned}$$

4. The paper [5] contains a statement (Theorem 1.7) about potentials V of a variable sign. More precise results can be obtained by combining the techniques of papers [5], [6] and [8]. Let us note briefly that in the case of a potential V of variable sign and $l \neq 2m$ the leading term of the asymptotics of the spectral shift function is $C\alpha^{\nu}$, where $\nu = \max\{d/(2m), d/l\}$, and constant C can be explicitely expressed in terms of the potential V.

5. Finally we note that condition (1) can be relaxed in several directions.

References

[1] M. Sh. Birman, M. Z. Solomyak, Vestnik Leningrad Univ. Math. **10**, 3 (1982), 237–247.

[2] M. Sh. Birman, M. Z. Solomyak, Vestnik Leningrad Univ. Math. **12**, 3 (1980), 155–161.

[3] M. Sh. Birman, D. R. Yafaev, St. Petersburg Math. J. **4**, no. 5 (1993), 833–870.

[4] A. B. Pushnitski, St. Petersburg Math. J., **9**, no. 6 (1998), 1181–1194.

[5] A. Pushnitski, Comm. in PDE, **25**, no. 3-4 (2000), 703–736.

[6] A. Pushnitski, M. Ruzhansky, to appear in Comm. PDE, 2002.

[7] M. Reed, B. Simon, *Methods of modern mathematical physics*, vol. 4, Academic Press, 1978.

[8] O. Safronov, J. Funct. Anal., **182**, no. 1 (2001), 151-169.

[9] D. R. Yafaev, *Mathematical scattering theory. General theory*, Amer. Math. Soc., Providence, RI, 1992.

A. B. Pushnitski, Loughborough University, A.B.Pushnitski@lboro.ac.uk
M. V. Ruzhansky, Imperial College, University of London, ruzh@ic.ac.uk