Estimates for the spectral shift function of the polyharmonic operator

Alexander Pushnitski
School of Mathematical Sciences
University of Sussex
Falmer BN1 9QH
United Kingdom

Abstract

The Lifshits–Krein spectral shift function is considered for the pair of operators $H_0 = (-\triangle)^l$, $l > 0$ and $H = H_0 + V$ in $L_2(\mathbb{R}^d)$, $d \geq 1$; here V is a multiplication operator. The estimates for this spectral shift function $\xi(\lambda; H, H_0)$ are obtained in terms of the spectral parameter $\lambda > 0$ and the integral norms of V. These estimates are in a good agreement with the ones predicted by the classical phase space volume considerations.

I. Introduction

The main object of study of this paper is the I. M. Lifshits–M. G. Krein *spectral shift function* (SSF). For an exposition of the SSF theory, see, e.g., [1] or [2]. For a general pair of selfadjoint operators H_1, H_2 in a Hilbert space $\mathcal{H}$, satisfying some trace class condition (see §2.3 below), the SSF $\xi(\lambda; H_2, H_1)$ appears in connection with the *trace formula*:

$$\operatorname{Tr}\left(\psi(H_2) - \psi(H_1)\right) = \int_{-\infty}^{\infty} \xi(\lambda; H_2, H_1)\psi'(\lambda)d\lambda, \quad \psi \in C_0^\infty(\mathbb{R}). \tag{1.1}$$

Besides, the SSF is related to the scattering matrix $S(\lambda)$ for the pair H_1, H_2 by the *Birman–Krein formula*:

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda; H_2, H_1)},$$

for almost every λ on the absolutely continuous spectrum of H_1. This formula allows one to interpret the SSF as the scattering phase. Moreover, sometimes it is treated as the definition of the SSF. See [1] for references and discussion.

Let $H_0 = (-\triangle)^l$, $l > 0$ in $L_2(\mathbb{R}^d)$, $d \geq 1$, and let $V = V(x)$, $x \in \mathbb{R}^d$, be a (real valued) perturbation potential, which decays sufficiently fast as $|x| \to \infty$. In this paper we obtain bounds on $\xi(\lambda; H_0 + V, H_0)$ in terms of λ and integral norms of V. The most interesting case is $l = 1$ (Schrödinger operator). Nevertheless, the technique of this paper allows us to obtain bounds on the SSF in a uniform way for all $l \in (0, \infty)$ (and all dimensions d), and thus we consider the problem in its natural generality. The main result of this paper (Theorem 2.2) is formulated in §2, and its corollaries — in §3. Our estimates are very close to the ones predicted by the semiclassical intuition — see discussion in §3. Our results extend the estimates of [3] and are closely related to the results of [4] — see discussion in the end of §3. Our technique is based on the new representation for the SSF obtained in [5].

II. Preliminaries. Statement of the main result

2.1 Notation. For a closable linear operator T in a Hilbert space $\mathcal{H}$, by $\overline{T}$ we denote the closure of T. For a selfadjoint operator A, the symbols $\sigma(A)$, $\rho(A)$ denote its spectrum and resolvent set; $R(z, A) = (A - zI)^{-1}$ (for $z \in \rho(A)$) and $E_A(\delta)$ is the spectral projection associated to a Borel set $\delta \subset \mathbb{R}$. By $\mathbf{S}_\infty(\mathcal{H})$ we denote the space of all compact operators in $\mathcal{H}$. For $T = T^* \in \mathbf{S}_\infty(\mathcal{H})$ we introduce the counting functions of the spectrum by $n_\pm(s, T) := \operatorname{rank} E_{\pm T}((s, +\infty))$, $s > 0$. Note that (see, e.g., [6])

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2), \quad s_1, s_2 > 0. \tag{2.1}$$

For $p \ge 1$ the Neumann–Schatten classes $\mathbf{S}_p$ are defined in a usual way:

$$T \in \mathbf{S}_p \quad \text{if} \quad \|T\|^p_{\mathbf{S}_p} := \sum_n s_n^p(T) < \infty,$$

where $\{s_n(T)\}$ is the sequence of singular numbers of T.

Integral without the domain of integration explicitly specified implies integration over $\mathbb{R}^d$. Formulas and statements with double indices ($\pm$ and $\mp$) should be read as pairs of statements, in one of which all the indices take upper values and in another — the lower ones. By $C(d)$, $C(l)$, etc. (possibly with sub- and superscripts) we denote various constants which depend only on d, l, etc. and whose particular values are of no importance. A constant which first appears in formula $(i.j)$ is denoted by $C_{i.j}$.

Remind the definition of the lattice space $l_1(L_r) \subset L_r(\mathbb{R}^d)$, $r \ge 1$:

$$u \in l_1(L_r) \quad \text{if} \quad \|u\|_{l_1(L_r)} := \sum_{j \in \mathbb{Z}^d} \left(\int_{\mathbf{Q}^d + j} |u|^r dx \right)^{1/r} < \infty, \quad \mathbf{Q}^d = (0,1)^d \subset \mathbb{R}^d.$$

Everywhere $\mathcal{H} = L_2(\mathbb{R}^d)$, $d \ge 1$ and $H_0 = (-\triangle)^l$, $l > 0$; $\varkappa = d/(2l)$. We shall need a notation for a logarithmic weight function. Namely, for $x \in \mathbb{R}^d$ and $\gamma > 0$ let

$$F_\gamma(x) = 1 + (\log_+ |x|)^\gamma. \tag{2.2}$$

2.2 Assumptions on V. Below by V we denote both a function $V : \mathbb{R}^d \to \mathbb{R}$ and the (selfadjoint) operator of multiplication by $V(x)$ in $L_2(\mathbb{R}^d)$. In various places we shall use some of the following assumptions on V:

$$V \text{ is } H_0\text{–form compact}; \tag{2.3}$$
$$V \in L_1(\mathbb{R}^d); \tag{2.4}$$
$$V \in L_\varkappa(\mathbb{R}^d); \tag{2.5}$$
$$\int |V(x)| F_\gamma(x) dx < \infty, \quad \gamma > 2. \tag{2.6}$$

Note that for $\varkappa < 1$ (2.3) follows from (2.4), and for $\varkappa > 1$ — from (2.5). Clearly, under the assumption (2.3) the operators $H_0 + V$, $H_0 + V_+$, $H_0 - V_-$ are well defined via the corresponding quadratic forms. The following assumption will appear only for $\varkappa \geq 2$:

$$\text{if } \varkappa \geq 2, \text{ then } V \in l_1(L_2). \tag{2.7}$$

2.3 Existence of the SSF. Let H_1, H_2 be a pair of selfadjoint operators in a Hilbert space $\mathcal{H}$ such that

$$H_2 - H_1 \in \mathbf{S}_1(\mathcal{H}). \tag{2.8}$$

Then the SSF $\xi(\lambda; H_2, H_1)$ exists and is given by the *Krein formula* [7]

$$\xi(\lambda; H_2, H_1) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \arg \Delta_{H_2/H_1}(\lambda + i\varepsilon), \text{ a.e. } \lambda \in \mathbb{R}, \tag{2.9}$$

where $\Delta_{H_2/H_1}(z) = \det(H_2 - zI)(H_1 - zI)^{-1}$ and the branch of the argument is fixed by

$$\arg \Delta_{H_2/H_1}(z) \to 0 \quad \text{as} \quad \operatorname{Im} z \to +\infty.$$

The SSF obeys the monotonicity property [7]:

$$\pm((H_2 - H_1)\chi, \chi) \geq 0 \quad \forall \chi \in \mathcal{H} \quad \Rightarrow \quad \pm\xi(\lambda; H_2, H_1) \geq 0. \tag{2.10}$$

For a triple H_0, H_1, H_2 of operators, such that $H_1 - H_0 \in \mathbf{S}_1$ and $H_2 - H_0 \in \mathbf{S}_1$, one has

$$\xi(\lambda; H_2, H_0) = \xi(\lambda; H_2, H_1) + \xi(\lambda; H_1, H_0). \tag{2.11}$$

In applications, instead of (2.8), it is usually possible to check the inclusion $f(H_2) - f(H_1) \in \mathbf{S}_1$, where f is some monotone smooth enough function. In this case one can first take $\tilde{H}_1 = f(H_1)$, $\tilde{H}_2 = f(H_2)$ and define $\xi(\lambda; \tilde{H}_2, \tilde{H}_1)$ according to (2.9) and then put

$$\xi(\lambda; H_2, H_1) := \operatorname{sign} f' \cdot \xi(f(\lambda); f(H_2), f(H_1)). \tag{2.12}$$

Thus defined, $\xi(\lambda; H_2, H_1)$ still obeys the trace formula (1.1). See [1, 2] for the details.

Proposition 2.1 (i) *Let* $\varkappa < 2$ *and let* V *obey* (2.3), (2.4)*. Then, for any* $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_0 + V))$*, the relation*

$$R(\lambda_0, H_0) - R(\lambda_0, H_0 + V) \in \mathbf{S}_1 \tag{2.13}$$

holds. Thus, the spectral shift functions $\xi(\lambda; H_0+V, H_0)$, $\xi(\lambda; H_0+V_+, H_0)$, $\xi(\lambda; H_0-V_-, H_0)$ *are well defined by* (2.12) *with* $f(\lambda) = (\lambda - \lambda_0)^{-1}$. *The following inequalities hold:*

$$\xi(\lambda; H_0 - V_-, H_0) \leq 0 \leq \xi(\lambda; H_0 + V_+, H_0), \tag{2.14}$$

$$\xi(\lambda; H_0 - V_-, H_0) \leq \xi(\lambda; H_0 + V, H_0) \leq \xi(\lambda; H_0 + V_+, H_0). \tag{2.15}$$

(ii) *Let* $\varkappa \geq 2$ *and let* V *obey* (2.3), (2.7). *Then, for any* $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_0 + V))$ *with large enough absolute value, and for any integer* $k > \varkappa - \frac{1}{2}$, *the relation*

$$R^k(\lambda_0, H_0) - R^k(\lambda_0, H_0 + V) \in \mathbf{S}_1 \tag{2.16}$$

holds. Thus, the spectral shift functions $\xi(\lambda; H_0+V, H_0)$, $\xi(\lambda; H_0+V_+, H_0)$, $\xi(\lambda; H_0-V_-, H_0)$ *are well defined by* (2.12) *with* $f(\lambda) = (\lambda - \lambda_0)^{-k}$. *The inequalities* (2.14), (2.15) *hold.*

Similar statements appeared in the literature in many different versions in connection with the trace class scattering theory. Nevertheless, for the sake of completeness, we give the proof of Proposition 2.1 in the end of this section. The inequalities (2.15) reduce the problem of estimating the SSF to the case of perturbations of a definite sign. Thus, below we shall always assume that $V \geq 0$ and consider the pair of functions $\xi(\lambda; H_0 \pm V, H_0)$.

Finally, note that, since $\sigma(H_0) = [0, \infty)$, one has

$$\xi(-\lambda; H_0 - V, H_0) = -\operatorname{rank} E_{H_0 - V}((-\infty, -\lambda)), \quad \lambda > 0. \tag{2.17}$$

2.4 Estimates for the SSF. The main result of this paper is

Theorem 2.2 *Let* $V \geq 0$. *Under the assumptions* (2.3), (2.6), (2.7), *the following estimates hold for* $\lambda > 0$ *and* $\gamma > 2$*:*

$$\xi(\lambda; H_0 + V, H_0) \leq C(d, l, \gamma)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d, l)\lambda^{\varkappa-1}(\log_+ \lambda)\|V\|_{L_1}, \tag{2.18}$$

$$\begin{aligned} |\xi(\lambda; H_0 - V, H_0)| \leq\ & C(d, l, \gamma)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d, l)\lambda^{\varkappa-1}(\log_+ \lambda)\|V\|_{L_1} \\ & + |\xi(-\lambda; H_0 - 6V, H_0)|. \end{aligned} \tag{2.19}$$

The proof is given in §§4–8. The factor 6 in (2.19) is chosen this way in order to simplify the constants appearing in the proof. Actually, this factor can be replaced by any number greater than 1; at the same time, the constants $C(d, l, \gamma)$ and $C(d, l)$ may have to be increased. Using bounds for $|\xi(-\lambda; H_0 - 6V, H_0)|$ (which, by (2.17), reduce to the bounds on the number of eigenvalues of $H_0 - 6V$), one can estimate the SSF $\xi(\lambda; H_0 - V, H_0)$ entirely in terms of λ and integral norms of V. This will be done in §3. In §3 we also discuss Theorem 2.2 and related results. **2.5 Proof of Proposition 2.1** (i) By (2.4), one has $\sqrt{|V|}R(\lambda_0, H_0) \in \mathbf{S}_2$. Therefore, (2.13) follows from the identity

$$\begin{aligned} R(\lambda_0, H_0 + V) = R(\lambda_0, H_0) \quad - \quad & (\operatorname{sign} V \sqrt{|V|}R(\lambda_0, H_0))^*(I + \sqrt{|V|}R(\lambda_0, H_0)\sqrt{|V|}\operatorname{sign} V)^{-1} \\ & \times(\sqrt{|V|}R_0(\lambda_0, H_0)). \end{aligned}$$

Clearly,

$$H_0 - V_- \le H_0 \le H_0 + V_+, \quad H_0 - V_- \le H_0 + V \le H_0 + V_+$$

in the quadratic form sense. Therefore,

$$R(\lambda_0, H_0 - V_-) \ \ge \ R(\lambda_0, H_0) \ge R(\lambda_0, H_0 + V_+), \tag{2.20}$$

$$R(\lambda_0, H_0 - V_-) \ \ge \ R(\lambda_0, H_0 + V) \ge R(\lambda_0, H_0 + V_+). \tag{2.21}$$

From here, by (2.10) and (2.11), we get

$$\begin{aligned} \xi((\lambda-\lambda_0)^{-1}; R(\lambda_0, H_0 - V_-), R(\lambda_0, H_0)) &\ge 0 \\ &\ge \xi((\lambda-\lambda_0)^{-1}; R(\lambda_0, H_0 + V_+), R(\lambda_0, H_0)), \\ \xi((\lambda-\lambda_0)^{-1}; R(\lambda_0, H_0 - V_-), R(\lambda_0, H_0)) &\ge \xi((\lambda-\lambda_0)^{-1}; R(\lambda_0, H_0 + V), R(\lambda_0, H_0)) \\ &\ge \xi((\lambda-\lambda_0)^{-1}; R(\lambda_0, H_0 + V_+), R(\lambda_0, H_0)). \end{aligned}$$

From here, by the definition (2.12), the relations (2.14), (2.15) follow. (ii) By (2.7), one has

$$VR^{k+\frac{1}{2}}(\lambda_0, H_0) \in \mathbf{S}_1, \quad k > \varkappa - \frac{1}{2}$$

(see, e.g., [8, Theorem 11.1]). Therefore, by [9, Theorem XI.12], the inclusion (2.16) holds. Due to the results of [10] (see also [2, §8.10]), the inequalities (2.20), (2.21) together with (2.16), (2.11) imply

$$\begin{aligned} \xi((\lambda-\lambda_0)^{-k}; R^k(\lambda_0, H_0 - V_-), R^k(\lambda_0, H_0)) &\ge 0 \\ &\ge \xi((\lambda-\lambda_0)^{-k}; R^k(\lambda_0, H_0 + V_+), R^k(\lambda_0, H_0)), \\ \xi((\lambda-\lambda_0)^{-k}; R^k(\lambda_0, H_0 - V_-), R^k(\lambda_0, H_0)) &\ge \xi((\lambda-\lambda_0)^{-k}; R^k(\lambda_0, H_0 + V), R^k(\lambda_0, H_0)) \\ &\ge \xi((\lambda-\lambda_0)^{-k}; R^k(\lambda_0, H_0 + V_+), R^k(\lambda_0, H_0)). \end{aligned}$$

From here, by the definition (2.12), the relations (2.14), (2.15) follow. ∎

III. Corollaries and discussion

3.1 Semiclassical considerations. Let us consider a "classical analogue" of the SSF, expressed in terms of the phase space volumes, corresponding to the systems with the Hamiltonians $h_0(p,x) = p^{2l}$ and $h_\pm(p,x) = p^{2l} \pm V(x)$, where $V \ge 0$. Let ω_d be the volume of a unit ball in $\mathbb{R}^d$; define

$$\begin{aligned} \xi^{cl}(\lambda; H_0 + V, H_0) &= (2\pi)^{-d}\mathrm{vol}\{(p,x) \in \mathbb{R}^{2d} \mid h_0(p,x) < \lambda < h_+(p,x)\} \\ &= (2\pi)^{-d}\omega_d \int_{\mathbb{R}^d} (\lambda^\varkappa - (\lambda - V(x))_+^\varkappa) dx, \quad \lambda > 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \xi^{cl}(\lambda; H_0 - V, H_0) &= -(2\pi)^{-d}\mathrm{vol}\{(p,x) \in \mathbb{R}^{2d} \mid h_-(p,x) < \lambda < h_0(p,x)\} \\ &= -(2\pi)^{-d}\omega_d \int_{\mathbb{R}^d} ((\lambda + V(x))^\varkappa - \lambda^\varkappa) dx, \quad \lambda > 0. \end{aligned} \tag{3.2}$$

It is well known that $\xi^{cl}(\lambda; H_0 \pm V, H_0)$ behaves in many respects like $\xi(\lambda; H_0 \pm V, H_0)$; for example, it has the same asymptotics in most asymptotical regimes — see the review [11] and references therein. The integrands in (3.1), (3.2) admit the following elementary bounds:

$$\lambda^\varkappa - (\lambda - V)_+^\varkappa \leq \max\{\varkappa, 1\}\lambda^{\varkappa-1}V, \quad \varkappa > 0; \tag{3.3}$$
$$(\lambda + V)^\varkappa - \lambda^\varkappa \leq \varkappa\lambda^{\varkappa-1}V, \quad \varkappa \leq 1; \tag{3.4}$$
$$(\lambda + V)^\varkappa - \lambda^\varkappa \leq C_1 V^\varkappa + C_2\lambda^{\varkappa-1}V, \quad \varkappa > 1. \tag{3.5}$$

It is easy to give concrete explicit values for C_1, C_2 in (3.5); for example, $C_1 = 2^\varkappa$, $C_2 = 2^\varkappa - 1$. Substituting (3.3)–(3.5) into (3.1), (3.2), we get the following bounds for ξ^{cl}:

$$\xi^{cl}(\lambda; H_0 + V, H_0) \leq (2\pi)^{-d}\omega_d \max\{\varkappa, 1\}\lambda^{\varkappa-1}\|V\|_{L_1}, \quad \varkappa > 0; \tag{3.6}$$
$$|\xi^{cl}(\lambda; H_0 - V, H_0)| \leq (2\pi)^{-d}\omega_d\varkappa\lambda^{\varkappa-1}\|V\|_{L_1}, \quad \varkappa \leq 1; \tag{3.7}$$
$$|\xi^{cl}(\lambda; H_0 - V, H_0)| \leq C_1(2\pi)^{-d}\omega_d\|V\|_{L_\varkappa}^\varkappa + C_2(2\pi)^{-d}\omega_d\lambda^{\varkappa-1}\|V\|_{L_1}, \quad \varkappa > 1. \tag{3.8}$$

Estimates (3.6)–(3.8) are in good agreement with the asymptotics of ξ^{cl} for high energy and large coupling constant:

$$\xi^{cl}(\lambda; H_0 \pm V, H_0) \sim \pm(2\pi)^{-d}\omega_d\varkappa\lambda^{\varkappa-1}\int V(x)dx, \quad \lambda \to \infty, \tag{3.9}$$

$$\xi^{cl}(\lambda; H_0 - gV, H_0) \sim -(2\pi)^{-d}\omega_d g^\varkappa \int V^\varkappa(x)dx, \quad g \to \infty. \tag{3.10}$$

We consider (3.6)–(3.8) as the model estimates. Clearly, (2.18) is in agreement with (3.6) up to the constants, the logarithmic weight F_γ and the term $\log_+ \lambda$. In order to compare (2.19) with (3.7), (3.8), one has to estimate the term $|\xi(-\lambda; H_0 - 6V, H_0)|$. This will be done below differently for $\varkappa > 1$, $\varkappa < 1$ and $\varkappa = 1$.

3.2 The case $\varkappa > 1$. In this case, we use the Cwikel–Lieb–Rozenblum bound [12, 13, 14]:

$$|\xi(-\lambda; H_0 - V, H_0)| \leq C(d, l)\|V\|_{L_\varkappa}^\varkappa. \tag{3.11}$$

Substituting (3.11) into (2.19), we get

Corollary 3.1 *Let $\varkappa > 1$. Assume the hypothesis of Theorem 2.2 and the inclusion* (2.5)*; then, for any $\lambda > 0$ and $\gamma > 2$,*

$$|\xi(\lambda; H_0 - V, H_0)| \leq C(d, l, \gamma)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d, l)\lambda^{\varkappa-1}(\log_+ \lambda)\|V\|_{L_1} + C(d, l)\|V\|_{L_\varkappa}^\varkappa. \tag{3.12}$$

3.3 The case $\varkappa < 1$. For $\varkappa < 1$ let us use the Birman–Schwinger principle and estimate $|\xi(-\lambda; H_0 - V, H_0)|$ (for $V \geq 0$) in the following way:

$$|\xi(-\lambda; H_0 - V, H_0)| \leq \mathrm{Tr}\,(\sqrt{V}R(-\lambda, H_0)\sqrt{V}) = C(d, l)\lambda^{\varkappa-1}\|V\|_{L_1}. \tag{3.13}$$

Substituting (3.13) into (2.19) and taking into account the obvious estimate $\|V\|_{L_1} \leq \|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2}$, we get the following corollary.

Corollary 3.2 *Let $V \geq 0$ and $\varkappa < 1$; assume* (2.6)*. Then, for any $\lambda > 0$ and $\gamma > 2$*

$$|\xi(\lambda; H_0 - V, H_0)| \leq C(d, l, \gamma)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d, l)\lambda^{\varkappa-1}(\log_+ \lambda)\|V\|_{L_1}. \quad (3.14)$$

3.4 The case $\varkappa = 1$. Now

$$\xi^{cl}(-\lambda; H_0 - V, H_0) = -(2\pi)^{-d}\omega_d \int V(x)dx, \quad \forall \lambda \in \mathbb{R}.$$

Nevertheless, as it is well known, the "naive" estimate

$$|\xi(-\lambda; H_0 - V, H_0)| \leq C\|V\|_{L_1}, \quad \lambda > 0 \quad (3.15)$$

is wrong; see, e.g., [15] for the discussion. Instead, there are numerous estimates of $\xi(-\lambda; H_0 - V, H_0)$, which are worse than (3.15) by a logarithmic term of some kind; see [16, 17]. Such estimates are a bit cumbersome as compared to (3.11), (3.13). Instead of discussing them, we consider two rough but simple estimates. The first one is (see, e.g., [18, Proposition 5.5])

$$|\xi(-\lambda; H_0 - V, H_0)| \leq \|\sqrt{V}R(-\lambda, H_0)\sqrt{V}\|_{\mathbf{S}_q}^q \leq C(d, q)\lambda^{1-q}\|V\|_{L_q}^q, \quad q > 1; \quad (3.16)$$

it is of an "almost correct" order in V and λ. Substituting (3.16) into (2.19), we get

Corollary 3.3 *Let $\varkappa = 1$; assume* (2.3), (2.6) *and let $V \in L_q(\mathbb{R}^d)$ for some $q > 1$. Then, for any $\lambda > 0$ and $\gamma > 2$*

$$|\xi(\lambda; H_0 - V, H_0)| \leq C(d, \gamma)\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d)(\log_+ \lambda)\|V\|_{L_1} + C(d, q)\lambda^{1-q}\|V\|_{L_q}^q. \quad (3.17)$$

Another simple estimate is valid (see, e.g., [18, Proposition 5.4]) for λ bounded away from zero:

$$|\xi(-\lambda; H_0 - V, H_0)| \leq C(d, r)\|V\|_{l_1(L_r)}, \quad r > 1, \quad \lambda \geq 1. \quad (3.18)$$

Note that $V \in l_1(L_r)$, $r > 1$, implies (2.3). Thus, substituting (3.18) into (2.19), we obtain

Corollary 3.4 *Let $\varkappa = 1$; assume* (2.6) *and let $V \in l_1(L_r)$ for some $r > 1$. Then, for any $\lambda \geq 1$ and $\gamma > 2$*

$$|\xi(\lambda; H_0 - V, H_0)| \leq C(d, \gamma)\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2} + C(d)(\log_+ \lambda)\|V\|_{L_1} + C(d, r)\|V\|_{l_1(L_r)}. \quad (3.19)$$

Substituting the estimates of [16, 17] into (2.19), one can obtain more precise statements.

3.5 Comparison with the results of [3]. In [3], the following result has been obtained. Let $d \geq 2$, $l = 1$ and let $V = V(x) \geq 0$ satisfy the estimate

$$V(x) \leq C_{3.20}(1 + |x|)^{-\rho}, \quad \rho > d. \quad (3.20)$$

For all $\lambda \geq C > 0$ and all coupling constants $g > 0$ the following bounds have been established (see [3, Theorem 4.2]):

$$\xi(\lambda; H_0 + gV, H_0) \leq C_{3.21}g\lambda^{\varkappa-1}(|\log \lambda| + 1), \quad (3.21)$$

$$|\xi(\lambda; H_0 - gV, H_0)| \leq C_{3.22}(g\lambda^{\varkappa-1}(|\log \lambda| + 1) + g^{\varkappa}). \quad (3.22)$$

The constants $C_{3.21}$, $C_{3.22}$ may depend on d and $C_{3.20}$. Clearly, (3.21) follows from (2.18) and (3.22) — from (3.12), (3.19). The basic difference between our results and the ones of [3] is in the fact that the dependence on V is explicit in (2.18), (3.12), (3.19) but not in (3.21), (3.22). Besides, Theorem 2.2 and Corollaries 3.1–3.4 extend the results of [3] in some other respects:

1)Theorem 2.2 and Corollary 3.2 deal with the case $\varkappa < 1$, which has not been considered in [3]; 2) the class of potentials V in Theorem 2.2 and Corollaries 3.1–3.4 is broader than the one given by (3.20);

3) Theorem 2.2 and Corollaries 3.1–3.3 concern all $\lambda > 0$ whereas in [3] λ is assumed to be bounded away from zero.

3.6 Integral estimates for the SSF. In [4], the *integral* estimates for the SSF $\xi(\lambda, H_0 \pm V, H_0)$ have been obtained (for the same operators H_0, $V \geq 0$ as in the present paper). Before discussing them, let us write down the estimates for ξ^{cl}, which easily follow from the definition (3.1), (3.2):

$$\int_0^R \xi^{cl}(\lambda; H_0 + V, H_0) d\lambda \leq (2\pi)^{-d} \omega_d R^{\varkappa} \|V\|_{L_1}, \quad R > 0,\ \varkappa > 0, \tag{3.23}$$

$$\int_0^R |\xi^{cl}(\lambda; H_0 - V, H_0)| d\lambda \leq (2\pi)^{-d} \omega_d R^{\varkappa} \|V\|_{L_1}, \quad R > 0,\ \varkappa \leq 1, \tag{3.24}$$

$$\int_0^R |\xi^{cl}(\lambda; H_0 - V, H_0)| d\lambda \leq C_1(d,l) R^{\varkappa} \|V\|_{L_1} + C_2(d,l) R \|V\|_{L_\varkappa}^{\varkappa}, \quad R > 0,\ \varkappa > 1. \tag{3.25}$$

It appears that the estimates (3.23), (3.25) can be carried over to the "real" SSF (see [4], estimates (6.2) and (6.10) respectively); it is interesting that even the constant in [4, (6.2)] coincides with the classical one, given by (3.23). For $\varkappa < 1$, the estimate

$$\int_0^R |\xi(\lambda; H_0 - V, H_0)| d\lambda \leq C(d,l) R^{\varkappa} \|V\|_{L_1}, \quad R > 0,\ \varkappa < 1,$$

(with the constant $C(d,l)$ different from the classical one, given by (3.24)) follows from [4, (6.7)] and (3.13). For $\varkappa = 1$ the estimate [4, (6.7)] together with (3.16) implies

$$\int_0^R |\xi(\lambda; H_0 - V, H_0)| d\lambda \leq C(d) R \|V\|_{L_1} + C(d,q) R^{2-q} \|V\|_{L_q}^q, \quad R > 0,\ \varkappa = 1$$

for any $q > 1$, and together with (3.18) it implies

$$\int_0^R |\xi(\lambda; H_0 - V, H_0)| d\lambda \leq C(d) R \|V\|_{L_1} + C(d,r) R \|V\|_{l_1(L_r)}, \quad R > 1,\ \varkappa = 1$$

for any $r > 1$.

3.7 Remarks. 1. Note that in [3] some estimates for the SSF have been found which have better order in λ and g, than (3.21), (3.22), depending on the exponent ρ in (3.20).

These estimates, however, are of a conditional character, since they depend on some hypothesis on the boundary values of the resolvent of H_0, which has not been proved yet.

2. The proof of Theorem 2.2 borrows some elements of [3]. But the operator theoretic part of our approach (Propositions 4.2, 4.3) is completely different.

3. It is natural to compare the estimates for the SSF with its asymptotics. Note that formula (3.9) for the "real" SSF and its various extensions is well known (see, e.g., [11] and references therein). The relation (3.10) for the "real" SSF is a well known fact for $\lambda < 0$ (see, e.g., [18]) and has been proved in [19] for $\lambda > 0$ and $l = 1$. Comparing the estimates (2.18), (3.12), (3.14), (3.19) with the asymptotics (3.9), (3.10), we see that the estimates are of a correct order in the coupling constant g (as $g \to \infty$) and of an almost correct (up to the logarithmic terms) order in λ as $\lambda \to \infty$.

4. In the case $d = l = 1$, a pointwise estimate on the SSF, which is somewhat different from (3.14), has been obtained in [20]. See the end of §8 for the discussion of this estimate.

IV. Representation for the SSF

Assume (2.3), (2.4) and denote $W := \sqrt{V}$. Assumption (2.3) means that

$$W(H_0 + I)^{-1/2} \in \mathbf{S}_\infty. \tag{4.1}$$

For $\operatorname{Im} z > 0$ consider the "sandwiched resolvent"

$$T(z) := (W(H_0 + I)^{-1/2})(H_0 + I)R_0(z)(W(H_0 + I)^{-1/2})^* = \overline{WR_0(z)W};$$

by (4.1), $T(z) \in \mathbf{S}_\infty$. Denote

$$A(z) = \operatorname{Re} T(z), \quad K(z) = \operatorname{Im} T(z).$$

By (2.4), for any bounded interval $\delta \subset \mathbb{R}$ one has

$$WE_{H_0}(\delta) \in \mathbf{S}_2. \tag{4.2}$$

From here follows

Proposition 4.1 *Assume* (2.3), (2.4). *For a.e.* $\lambda \in \mathbb{R}$,

$$\exists \lim_{\varepsilon \to 0+} T(\lambda + i\varepsilon) =: T(\lambda + i0) \in \mathbf{S}_\infty \quad \textit{and} \quad K(\lambda + i0) \in \mathbf{S}_1. \tag{4.3}$$

Proof For any $\delta \subset \mathbb{R}$, denote

$$\begin{gathered} T_\delta(z) := (W(H_0 + I)^{-1/2}E_{H_0}(\delta))(H_0 + I)R_0(z)(W(H_0 + I)^{-1/2}E_{H_0}(\delta))^*, \\ A_\delta(z) = \operatorname{Re} T_\delta(z), \quad K_\delta(z) = \operatorname{Im} T_\delta(z). \end{gathered} \tag{4.4}$$

Now let $\delta \subset \mathbb{R}$ be some open *bounded* interval. It is one of the fundamental results of the trace class scattering theory (see [21] or [2]) that (4.2) implies

$$\exists\, T_\delta(\lambda + i0) \in \mathbf{S}_2 \quad \text{and} \quad K_\delta(\lambda + i0) \in \mathbf{S}_1, \quad \text{a.e. } \lambda \in \mathbb{R}. \tag{4.5}$$

On the other hand, the operator $T_{\mathbb{R}\setminus\delta}(z)$ is analytically extendable through δ; obviously,

$$T_{\mathbb{R}\setminus\delta}(\lambda) \in \mathbf{S}_\infty \quad \text{and} \quad K_{\mathbb{R}\setminus\delta}(\lambda) = 0, \quad \lambda \in \delta. \tag{4.6}$$

Finally, writing $T(z) = T_\delta(z) + T_{\mathbb{R}\setminus\delta}(z)$ and taking into account (4.5), (4.6), we get (4.3) for a.e. $\lambda \in \delta$. Since $\delta \subset \mathbb{R}$ is arbitrary, this implies the required statement. ∎

It will follow from the reasoning of §7 that under the additional assumption (2.6), the condition (4.3) actually holds for *all* $\lambda \neq 0$ and $T(\lambda + i0)$ depends continuously on λ in the operator norm and $K(\lambda + i0)$ — in the trace norm.

Let λ be such that (4.3) holds; denote

$$\mathcal{N}_\pm(\lambda) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\pm(1, A(\lambda + i0) + tK(\lambda + i0)). \tag{4.7}$$

One easily checks that (4.3) implies convergence of the integral in (4.7).

Proposition 4.2 [5, 4] *Assume* (2.3), (2.4), (2.7)*; then for a.e.* $\lambda \in \mathbb{R}$

$$\xi(\lambda; H_0 \pm V, H_0) = \pm\mathcal{N}_\mp(\lambda). \tag{4.8}$$

Representation (4.7), (4.8) for the SSF has been established in [5] (see also [22] for a generalisation) as an abstract operator theoretic fact; application to the polyharmonic operator was considered in [4, Theorems 6.2, 6.3]. Formula (4.8) can be considered as the *Birman-Schwinger principle on the continuous spectrum*; see [5] for the discussion.

A straightforward analysis of the r.h.s. of (4.7) gives the following

Proposition 4.3 [5] *Let, for some* $\lambda > 0$*, the condition* (4.3) *hold. Then for any* $\theta \in (0, 1)$ *the following estimate holds:*

$$\mathcal{N}_\pm(\lambda) \le n_\pm(1-\theta, A(\lambda + i0)) + \theta^{-1}\pi^{-1}\|K(\lambda + i0)\|_{\mathbf{S}_1}. \tag{4.9}$$

In the proof of Theorem 2.2, we shall use (4.9) together with the following decomposition:

$$A(\lambda + i0) = A_\delta(\lambda + i0) + A_{\mathbb{R}\setminus\delta}(\lambda), \quad \lambda > 0, \quad \delta = (0, 2\lambda), \tag{4.10}$$

where the operators in the r.h.s. are defined by (4.4). Substituting (4.10) into (4.9), fixing $\theta = 1/4$ and using (2.1), we get the following inequality:

$$\mathcal{N}_\pm(\lambda) \le n_\pm(1/2, A_{\mathbb{R}\setminus\delta}(\lambda)) + n_\pm(1/4, A_\delta(\lambda + i0)) + 4\pi^{-1}\|K(\lambda + i0)\|_{\mathbf{S}_1}. \tag{4.11}$$

The relation (4.11) plays the key role in the proof of Theorem 2.2. In what follows we estimate each of the three terms in the r.h.s. of (4.11). Note that $A_{\mathbb{R}\setminus\delta}(\lambda) \ge 0$; thus, $n_-(1/2, A_{\mathbb{R}\setminus\delta}(\lambda)) = 0$. Certainly, there is much freedom in choosing the constants θ in (4.9) and $s_{1,2}$ in (2.1), but this choice affects only the constants in the resulting formulas (2.18), (3.12), (3.14), (3.17), (3.19).

V. Estimate for $A_{\mathbb{R}\setminus\delta}$

Assume (2.3); as in the end of the previous section, let us fix some $\lambda > 0$, denote $\delta = (0, 2\lambda)$ and define the operator $A_{\mathbb{R}\setminus\delta}$ according to (4.4).

Proposition 5.1 *Under the above assumptions, for any $s > 0$,*

$$n_+(s, A_{\mathbb{R}\setminus\delta}(\lambda + i0)) \leq |\xi(-\lambda; H_0 - 3s^{-1}V, H_0)|. \tag{5.1}$$

Proof: A straightforward calculation shows that

$$R(\lambda, H_0)E_{H_0}(\mathbb{R} \setminus \delta) \leq 3R(-\lambda, H_0)$$

in the quadratic form sense. It follows that $A_{\mathbb{R}\setminus\delta}(\lambda + i0) \leq 3T(-\lambda)$. From here, using the Birman-Schwinger principle, we get:

$$\begin{aligned} n_+(s, A_{\mathbb{R}\setminus\delta}(\lambda + i0)) &\leq n_+(s, 3T(-\lambda)) = \operatorname{rank} E_{H_0 - 3s^{-1}V}((-\infty, -\lambda)) \\ &= |\xi(-\lambda; H_0 - 3s^{-1}V, H_0)|. \quad \blacksquare \end{aligned}$$

VI. Estimate for K

Assume (2.3), (2.4). For $\lambda > 0$ consider the operator

$$(WE((0, \lambda)))(WE((0, \lambda)))^*. \tag{6.1}$$

By (4.2), the operator (6.1) belongs to the trace class. We will need a well known representation for the derivative of (6.1) with respect to λ. In order to write down this representation, for every $t > 0$ define the operator $J(t) : L_2(\mathbb{R}^d) \to L_2(\mathbb{S}^{d-1})$, which acts according to the formula

$$J(t) : \quad f(x) \mapsto (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{it\langle x, \nu\rangle} W(x) f(x) dx, \quad \nu \in \mathbb{S}^{d-1}.$$

For $d = 1$ by $\mathbb{S}^{d-1}$ we mean the set $\{-1, 1\}$. Clearly, $J(t) \in \mathbf{S}_2$ and

$$\|J(t)\|^2_{\mathbf{S}_2} = d\omega_d (2\pi)^{-d} \|V\|_{L_1}. \tag{6.2}$$

A straightforward calculation shows that the operator valued function (6.1) is differentiable in λ in the trace class and

$$\mathcal{F}(\lambda) := \frac{d}{d\lambda}(WE((0, \lambda)))(WE((0, \lambda)))^* = (2l)^{-1}\lambda^{\varkappa - 1} J^*(\lambda^{1/(2l)}) J(\lambda^{1/(2l)}) \geq 0. \tag{6.3}$$

If for some $\lambda > 0$ the limit $T(\lambda + i0)$ exists, then, clearly,

$$K(\lambda + i0) = \pi \mathcal{F}(\lambda). \tag{6.4}$$

Thus,

$$\|K(\lambda + i0)\|_{\mathbf{S}_1} = \pi \operatorname{Tr} \mathcal{F}(\lambda) = \pi (2l)^{-1} \lambda^{\varkappa - 1} \|J(\lambda^{1/(2l)})\|^2_{\mathbf{S}_2} = \pi \varkappa \omega_d (2\pi)^{-d} \lambda^{\varkappa - 1} \|V\|_{L_1}. \tag{6.5}$$

VII. Estimate for A_δ

7.1 Preliminary estimates. Define the function $\varphi(t)$, $t>0$, by

$$\varphi(t)=\begin{cases}|e^{it}-1|^2, & t\le\pi,\\ 4, & t\ge\pi.\end{cases}\tag{7.1}$$

Proposition 7.1 *Assume* (2.4)*. For any $t_1>0$, $t_2>0$ the following estimate holds:*

$$\|J(t_1)-J(t_2)\|_{\mathbf{S}_2}^2\le d\omega_d(2\pi)^{-d}\int_{\mathbb{R}^d}V(x)\varphi(|x||t_1-t_2|)dx.\tag{7.2}$$

Proof:

$$\begin{aligned}\|J(t_1)-J(t_2)\|_{\mathbf{S}_2}^2 &\le (2\pi)^{-d}\int_{\mathbb{R}^d}V(x)\int_{\mathbb{S}^{d-1}}|e^{it_1\langle\nu,x\rangle}-e^{it_2\langle\nu,x\rangle}|^2d\nu dx\\ &\le (2\pi)^{-d}\int_{\mathbb{R}^d}V(x)\int_{\mathbb{S}^{d-1}}\varphi(|x||t_1-t_2|)d\nu dx\\ &= (2\pi)^{-d}d\omega_d\int_{\mathbb{R}^d}V(x)\varphi(|x||t_1-t_2|)dx. \quad\blacksquare\end{aligned}$$

Remind that $\mathcal{F}(\lambda)$ is defined by (6.3).

Proposition 7.2 *Assume* (2.4)*. For any $\lambda_1>0$, $\lambda_2>0$:*

$$\begin{aligned}\|\mathcal{F}(\lambda_1)-\mathcal{F}(\lambda_2)\|_{\mathbf{S}_1} &\le \varkappa\omega_d(2\pi)^{-d}|\lambda_1^{\varkappa-1}-\lambda_2^{\varkappa-1}|\|V\|_{L_1}+2\varkappa\omega_d(2\pi)^{-d}\lambda_2^{\varkappa-1}\|V\|_{L_1}^{1/2}\\ &\quad\times\left(\int V(x)\varphi(|x||\lambda_1^{1/(2l)}-\lambda_2^{1/(2l)}|)dx\right)^{1/2}.\end{aligned}\tag{7.3}$$

Proof: By (6.3),

$$\begin{aligned}\|\mathcal{F}(\lambda_1)-\mathcal{F}(\lambda_2)\|_{\mathbf{S}_1} &= (2l)^{-1}\|\lambda_1^{\varkappa-1}J^*(\lambda_1^{1/(2l)})J(\lambda_1^{1/(2l)})-\lambda_2^{\varkappa-1}J^*(\lambda_2^{1/(2l)})J(\lambda_2^{1/(2l)})\|_{\mathbf{S}_1}\\ &\le (2l)^{-1}|\lambda_1^{\varkappa-1}-\lambda_2^{\varkappa-1}|\|J^*(\lambda_1^{1/(2l)})J(\lambda_1^{1/(2l)})\|_{\mathbf{S}_1}\\ &\quad+(2l)^{-1}\lambda_2^{\varkappa-1}\|(J^*(\lambda_1^{1/(2l)})-J^*(\lambda_2^{1/(2l)}))J(\lambda_1^{1/(2l)})\|_{\mathbf{S}_1}\\ &\quad+(2l)^{-1}\lambda_2^{\varkappa-1}\|J^*(\lambda_2^{1/(2l)})(J(\lambda_1^{1/(2l)})-J(\lambda_2^{1/(2l)}))\|_{\mathbf{S}_1}.\end{aligned}$$

Substituting (6.2) and (7.2) into the r.h.s. of the last estimate, we arrive at (7.3). ∎

Proposition 7.3 *Assume* (2.6)*; then there exist such constants $0<C_{7.4}^{(1)}(\gamma)$ and $0<C_{7.4}^{(2)}(\gamma)<1$, that*

$$\int V(x)\varphi(|x|s)dx\le C_{7.4}^{(1)}(\gamma)|\log s|^{-\gamma}\|VF_\gamma\|_{L_1},\quad\forall\, s\in(0,C_{7.4}^{(2)}(\gamma)).\tag{7.4}$$

Proof: One has

$$\int V(x)\varphi(|x|s)dx \le \sup_{x\in\mathbb{R}^d} \frac{\varphi(|x|s)}{F_\gamma(x)} \int V(x)F_\gamma(x)dx.$$

It is a straightforward calculation to check that

$$\sup_{x\in\mathbb{R}^d} \frac{\varphi(|x|s)}{F_\gamma(x)} \le C^{(1)}_{7.4}(\gamma)|\log s|^{-\gamma}$$

for some constant $C^{(1)}_{7.4}(\gamma)$ and for all small enough $s > 0$. ∎

7.2 Existence of $A_\delta(\lambda + i0)$.

Proposition 7.4 *Assume* (2.6)*, fix* $\lambda > 0$ *and let* $\delta = (0, 2\lambda)$*. Then the limit* $A_\delta(\lambda + i0)$ *exists in the trace norm and*

$$A_\delta(\lambda + i0) = \int_0^{2\lambda} \frac{\mathcal{F}(t) - \mathcal{F}(\lambda)}{t - \lambda} dt; \tag{7.5}$$

the last integral is absolutely convergent in the trace norm.

Proof We start from the obvious formula which is a consequence of the spectral theorem:

$$A_\delta(\lambda + i\varepsilon) = \int_0^{2\lambda} \frac{t - \lambda}{(t - \lambda)^2 + \varepsilon^2} \mathcal{F}(t)dt, \quad \varepsilon > 0.$$

Clearly,

$$\int_0^{2\lambda} \frac{t - \lambda}{(t - \lambda)^2 + \varepsilon^2} \mathcal{F}(t)dt = \int_0^{2\lambda} \frac{t - \lambda}{(t - \lambda)^2 + \varepsilon^2} (\mathcal{F}(t) - \mathcal{F}(\lambda))dt. \tag{7.6}$$

From Propositions 7.2, 7.3 it follows that

$$\|\mathcal{F}(\lambda + s) - \mathcal{F}(\lambda)\|_{\mathbf{S}_1} \le C(\lambda, V)|\log|s||^{-\gamma/2} \tag{7.7}$$

for all small enough $s \in \mathbb{R}$. Thus, the integrands both in (7.6) and in (7.5) are dominated (in the trace norm) by an integrable function $C(\lambda, V)|t - \lambda|^{-1}|\log|t - \lambda||^{-\gamma/2}$ in the neighbourhood of $t = \lambda$. It follows that the r.h.s. of (7.6) converges to the r.h.s. of (7.5) as $\varepsilon \to +0$. ∎

Remark 7.5 Note that it follows from the estimate (7.7) that under the assumptions (2.3), (2.6) the condition (4.3) holds for all $\lambda > 0$ and the operator $A_\delta(\lambda + i0)$ is continuous in $\lambda > 0$ in the trace norm. Thus, $T(\lambda + i0)$ is continuous in $\lambda > 0$ in the operator norm. Next, if instead of (2.6) one assumes a stronger condition

$$\int V(x)(1 + |x|)^\gamma dx < \infty, \quad \gamma \in (0, 2),$$

then the same reasoning leads to the estimate

$$\|\mathcal{F}(\lambda + s) - \mathcal{F}(\lambda)\|_{\mathbf{S}_1} \le C(\lambda, V)|s|^{\gamma/2}$$

for all small enough $s \in \mathbb{R}$. This estimate implies Hölder continuity of $A_\delta(\lambda+i0)$ in the trace norm with the exponent $\gamma/2$ and thus the Hölder continuity of $T(\lambda + i0)$ in the operator norm. However, in what follows we shall not need these facts.

7.3 Estimates for $A_\delta(\lambda+i0)$. First we consider the case $\lambda=1$.

Proposition 7.6 *Assume* (2.6)*; let* $\delta=(0,2)$*. One has:*

$$\|A_\delta(1+i0)\|_{\mathbf{S}_1}\le C(d,l)\|V\|_{L_1}+C(d,l)\|V\|_{L_1}^{1/2}\int_0^1\frac{ds}{s}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}. \qquad (7.8)$$

Proof: Let us use (7.5) and (7.3):

$$\begin{aligned}\|A_\delta(1+i0)\|_{\mathbf{S}_1} &\le \int_0^2\frac{\|\mathcal{F}(\lambda)-\mathcal{F}(1)\|_{\mathbf{S}_1}}{|\lambda-1|}d\lambda\le\varkappa\omega_d(2\pi)^{-d}\|V\|_{L_1}\int_0^2\frac{|\lambda^{\varkappa-1}-1|}{|\lambda-1|}d\lambda\\ &+ 2\varkappa\omega_d(2\pi)^{-d}\|V\|_{L_1}^{1/2}\int_0^2\frac{d\lambda}{|\lambda-1|}\left(\int V(x)\varphi(|x||\lambda^{1/(2l)}-1|)dx\right)^{1/2} \qquad (7.9)\end{aligned}$$

Changing variable in the last integral, we obtain:

$$\begin{aligned}&\int_0^2\frac{1}{|\lambda-1|}\left(\int V(x)\varphi(|x||\lambda^{1/(2l)}-1|)dx\right)^{1/2}d\lambda\\ &= \int_{-1}^{2^{1/(2l)}-1}\frac{2l(s+1)^{2l-1}}{|(s+1)^{2l}-1|}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}ds\\ &\le C(l)\int_{-1}^1\frac{ds}{|s|}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}+\int_1^{2^{1/(2l)}-1}\frac{2l(s+1)^{2l-1}}{|(s+1)^{2l}-1|}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}ds\\ &\le C(l)\int_0^1\frac{ds}{s}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}+C(l)\|V\|_{L_1}^{1/2}. \qquad (7.10)\end{aligned}$$

Note that the integral over $(1,2^{1/(2l)}-1)$ enters the last calculation only if $2^{1/(2l)}-1>1$, i.e., if $l<1/2$. In order to estimate this integral, we use the bound $\varphi(t)\le 4$. The estimates (7.9) and (7.10) together give (7.8). ∎

Proposition 7.7 *Assume* (2.6)*, fix* $\lambda>0$ *and let* $\delta=(0,2\lambda)$*. The following estimate holds:*

$$\|A_\delta(\lambda+i0)\|_{\mathbf{S}_1}\le C(d,l,\gamma)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\|VF_\gamma\|_{L_1}^{1/2}+C(d,l)\lambda^{\varkappa-1}(\log_+\lambda)\|V\|_{L_1}. \qquad (7.11)$$

Proof: We start from the standard dilatation argument. Namely, let U_r, $r>0$, be the unitary dilatation operator in $L_2(\mathbb{R}^d)$: $(U_rf)(x)=r^{d/2}f(rx)$. Then

$$U_rA(z)U_r^*=r^{2l}A^{(r)}(r^{2l}z),$$

where $A^{(r)}$ corresponds to the perturbation potential $V^{(r)}(x)=V(xr)$. Thus, taking $r=\lambda^{-1/(2l)}$, using (7.8), and changing variable in the resulting integrals, we obtain:

$$\begin{aligned}\|A(\lambda+i0)\|_{\mathbf{S}_1} &= \|U_rA(\lambda+i0)U_r^*\|_{\mathbf{S}_1}=\lambda^{-1}\|A^{(r)}(1+i0)\|_{\mathbf{S}_1}\le\lambda^{\varkappa-1}C(d,l)\|V\|_{L_1}\\ &+C(d,l)\lambda^{\varkappa-1}\|V\|_{L_1}^{1/2}\int_0^{\lambda^{1/(2l)}}\frac{ds}{s}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2}.\end{aligned}$$

It remains to estimate the last integral. First let $\lambda \geq 1$. Using (7.4) and the bound $\varphi(t) \leq 4$, we get:

$$\begin{aligned}\int_0^{\lambda^{1/(2l)}} \frac{ds}{s}\left(\int V(x)\varphi(|x|s)dx\right)^{1/2} &\leq (C_{7.4}^{(1)}(\gamma))^{1/2}\|VF_\gamma\|_{L_1}^{1/2}\int_0^{C_{7.4}^{(2)}}\frac{ds}{s}|\log s|^{-\gamma/2} + 2\|V\|_{L_1}^{1/2}\int_{C_{7.4}^{(2)}}^{\lambda^{1/(2l)}}\frac{ds}{s} \\ &\leq C(\gamma)\|VF_\gamma\|_{L_1}^{1/2} + \|V\|_{L_1}^{1/2} l^{-1}\log_+\lambda - 2\|V\|_{L_1}^{1/2}\log C_{7.4}^{(2)} \\ &\leq (C(\gamma) - 2\log C_{7.4}^{(2)})\|VF_\gamma\|_{L_1}^{1/2} + \|V\|_{L_1}^{1/2} l^{-1}\log_+\lambda \\ &\leq C(\gamma)\|VF_\gamma\|_{L_1}^{1/2} + \|V\|_{L_1}^{1/2} l^{-1}\log_+\lambda.\end{aligned}$$

Finally, if $\lambda < 1$, we merely replace the integration interval $(0, \lambda^{1/(2l)})$ by $(0, 1)$, thus getting the upper bound. ∎

VIII. Proof of Theorem 2.2. Concluding remarks.

8.1 Proof of Theorem 2.2. Obviously,

$$n_+(1/4, A_\delta(\lambda + i0)) \leq 4\|A_\delta(\lambda + i0)\|_{\mathbf{S}_1}.$$

It remains to substitute the last estimate together with (5.1), (6.5), (7.11) into (4.11) and take into account Proposition 4.2. ∎

8.2 Remarks. 1. Our Proposition 7.4 is fairly close to the “limiting absorption principle in the trace class” of [3], though these two statements are proved by using very different technique.

2. One can exploit (4.11) by using some other (different from ours) estimates for $K(\lambda + i0)$, $A_\delta(\lambda + i0)$, $A_{\mathbb{R}\setminus\delta}(\lambda + i0)$, thus obtaining new estimates for the SSF. Let us give an example.

In [23], the following estimate has been proved for $l = 1$, $d \geq 2$:

$$\|A_\delta(\lambda + i0)\|_{\mathbf{S}_2}^2 \leq C(d)\lambda^{d-2-(q/2)}\int\int\frac{V(x)V(x')}{|x - x'|^q}dxdx', \quad q \in [0, d-1]. \tag{8.1}$$

For $d = 1$ this estimate is also true (which follows from the explicit formula for the integral kernel of the resolvent of H_0). Observing that

$$n_\pm(s, A_\delta(\lambda + i0)) \leq \frac{1}{2}s^{-2}\|A_\delta(\lambda + i0)\|_{\mathbf{S}_2}^2 \tag{8.2}$$

and substituting (8.1), (8.2) into (4.11), one obtains the following estimates for $l = 1$, $\lambda > 0$, $q \in [0, d-1]$:

$$\begin{aligned}\xi(\lambda; H_0 + V, H_0) &\leq C(d)\lambda^{d-2-(q/2)}\int\int\frac{V(x)V(x')}{|x - x'|^q}dxdx' + C(d)\lambda^{(d/2)-1}\|V\|_{L_1}, \\ |\xi(\lambda; H_0 - V, H_0)| &\leq C(d)\lambda^{d-2-(q/2)}\int\int\frac{V(x)V(x')}{|x - x'|^q}dxdx' + C(d)\lambda^{(d/2)-1}\|V\|_{L_1} \\ &\quad + |\xi(-\lambda; H_0 - 6V, H_0)|.\end{aligned}$$

One can combine the last estimate with (3.11), (3.13), (3.16), (3.18) or similar bounds in an obvious way. Note that the estimate (2.10) of [20] is contained in this series of estimates (for $d = 1$).

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