SPECTRAL SHIFT FUNCTION OF THE SCHRÖDINGER OPERATOR IN THE LARGE COUPLING CONSTANT LIMIT

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Abstract

We consider the spectral shift function $\xi(\lambda; H_0 - \alpha V, H_0)$, where H_0 is a Schrödinger operator with a variable Riemannian metric and an electromagnetic field and V is a perturbation by a multiplication operator. We prove the Weyl type asymptotic formula for $\xi(\lambda; H_0 - \alpha V; H_0)$ in the large coupling constant limit $\alpha \to \infty$.

0 Introduction

0.1 Let $H_0 = -\triangle$ in $L_2(\mathbb{R}^d)$. Below in the introduction for the sake of simplicity of exposition we assume $d \geq 3$; in the main text of the paper we treat the general case $d \geq 1$. Let $V = V(x)$, $x \in \mathbb{R}^d$, be a perturbation potential which decays rapidly enough as $|x| \to \infty$. For a coupling constant $\alpha > 0$ and a spectral parameter $\lambda \leq 0$ denote

$$N(\lambda, \alpha) := \#\{n \in \mathbb{N} \mid \lambda_n(H_0 - \alpha V) < \lambda\}, \tag{0.1}$$

where $\lambda_n(H_0 - \alpha V)$ are the eigenvalues of $H_0 - \alpha V$, enumerated with the multiplicities taken into account. The following Weyl type asymptotic formula holds:

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N(\lambda, \alpha) = (2\pi)^{-d}\omega_d \int V_+^{d/2}(x)dx, \quad \lambda \le 0, \tag{0.2}$$

where ω_d is the volume of a unit ball in $\mathbb{R}^d$ and $V_+(x) = \max\{0, V(x)\}$. The following condition is necessary and sufficient (see, e.g., [9, 5]) for the validity of (0.2):

$$V \in L_{d/2}(\mathbb{R}^d). \tag{0.3}$$

0.2 Next, let H_0 be the Schrödinger operator of a more general form:

$$H_0 = H_0(g, \mathbf{A}, U) = \sum_{k,j=1}^{d} \left(-i\frac{\partial}{\partial x_k} - A_k(x)\right) g_{kj}(x) \left(-i\frac{\partial}{\partial x_j} - A_j(x)\right) + U(x), \tag{0.4}$$

where $g = \{g_{kj}\}_{k,j=1}^d$ is a symmetric $d \times d$-matrix-valued function with real entries (metric of the space),

$$g_-\mathbf{1} \le g(x) \le g_+\mathbf{1}, \quad 0 < g_- \le g_+ < \infty, \tag{0.5}$$

$\mathbf{A} = \{A_j\}_{j=1}^d$ is a real magnetic vector potential,

$$\mathbf{A} \in L_{2,\mathrm{loc}}(\mathbb{R}^d), \tag{0.6}$$

U is a scalar real electric potential,

$$U \in L_{1,\mathrm{loc}}(\mathbb{R}^d), \quad U \ge 0. \tag{0.7}$$

Under the above assumptions the operator H_0 can be correctly defined via the corresponding quadratic form (see §1.2 below). Next, let the potential V satisfy (0.3); denote

$$C_{0.8}(V, g) := (2\pi)^{-d}\omega_d \int V_+^{d/2}(x)(\det g(x))^{-1/2}dx. \tag{0.8}$$

Clearly, $C_{0.8}(V, g)$ is finite under the assumptions (0.3), (0.5). Suppose in addition that

$$|\mathbf{A}|^2 + U \in L_{d/2,\mathrm{loc}}(\mathbb{R}^d). \tag{0.9}$$

Then, for any $\lambda \le 0$, the function (0.1) obeys the following asymptotic formula, which extends (0.2):

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N(\lambda, \alpha) = C_{0.8}(V, g). \tag{0.10}$$

In §1.2 we give a precise statement of this result and references to the literature.

The spectrum $\sigma(H_0)$ of the operator (0.4) may have gaps (apart from the semi–infinite gap $(-\infty, \inf \sigma(H_0))$). One can study the behaviour of the discrete spectrum of $H_0 - \alpha V$ in these gaps for $\alpha \to \infty$. Below we discuss two consecutive generalisations of (0.10) known in this situation.

0.3 Suppose that the perturbation potential V is nonnegative: $V \geq 0$. In order to keep our notation coherent with those of [4, 30], we write the perturbation in a factorised form: $V = G^*G$; for example, one can take $G := \sqrt{V}$. The eigenvalues of $H_0 \mp \alpha V$ are monotone functions of α. For $\alpha > 0$ denote

$$N_\pm(\lambda; H_0, G; \alpha) := \#\{t \in (0, \alpha) \mid \lambda_n(H_0 \mp tV) = \lambda\}, \quad \lambda \in \mathbb{R} \setminus \sigma(H_0). \tag{0.11}$$

In other words, $N_\pm(\lambda; H_0, G; \alpha)$ is the number of eigenvalues (counting multiplicities) of $H_0 \mp tV$, which pass the point λ as t grows monotonically from 0 to α. Clearly,

$$N(\lambda, \alpha) = N_+(\lambda; H_0, G; \alpha), \quad \lambda < \inf \sigma(H_0) \tag{0.12}$$

(remind that $N(\lambda, \alpha)$ is defined by (0.1)). Obviously, the definition (0.11) is of an abstract character. In [4] there has been proved an abstract theorem on the stability (i.e., independence of λ and of some weak perturbations of H_0) of the leading term of the asymptotics of $N_+(\lambda; H_0, G; \alpha)$ as $\alpha \to \infty$. As an application of this theorem to the Schrödinger operator, it has been proved (see [4, §2] and [8]) that the limits

$$\limsup_{\alpha \to \infty} \alpha^{-d/2} N_+(\lambda; H_0, G; \alpha), \quad \liminf_{\alpha \to \infty} \alpha^{-d/2} N_+(\lambda; H_0, G; \alpha)$$

do not depend on $\lambda \in \mathbb{R} \setminus \sigma(H_0)$. This fact, together with (0.10), results in the formula

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N_+(\lambda; H_0, G; \alpha) = C_{0.8}(V, g), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0), \tag{0.13}$$

which is valid under the assumptions (0.3), (0.5), (0.7), (0.9) on V, g, U, $\mathbf{A}$. Under more restrictive assumptions on V, g, U, $\mathbf{A}$, the formula (0.13) has been first proved (by using a method different from the one described above) in [15, 16] (see also the survey [5] and references therein). Note also that in the papers [4, 8], cited above, V, g, U, $\mathbf{A}$ were subject to slightly more restrictive conditions than (0.3), (0.5), (0.7), (0.9); nevertheless, the general case can be considered in the same way. However, in this paper there is no need to justify formula (0.13) in its full generality ((0.10) is sufficient for our purposes).

If H_0 is a periodic operator, then, under some additional assumptions on V and on the structure of the spectrum of H_0, formula (0.13) is valid also for the points λ on the boundary of the continuous spectrum of H_0 — see, e.g., the survey [5].

Note that the right hand side of (0.13) does not depend on $\lambda \in \mathbb{R} \setminus \sigma(H_0)$. Allowing for a somewhat free interpretation of (0.13), one can consider this formula as an expression of a certain asymptotic “conservation law” for the eigenvalues of $H_0 - \alpha V$: as α grows, the eigenvalues which “disappear” at the left edge of a gap, eventually reappear at the right edge of the next gap (located to the left from the first one). Thus, we deal with a “flow of eigenvalues” leftwards, which “seeps” through the continuous spectrum.

0.4 Now suppose that the potential V is not necessarily nonnegative: $V = V_+ - V_-$, $V_\pm = G_\pm^* G_\pm \geq 0$. The eigenvalues of $H_0 - tV$ are real analytic functions of t. As t grows, these eigenvalues can cross the point $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ leftwards or rightwards or “turn” at the point λ. For $\alpha > 0$ denote by $N^{(+)}(\lambda, \alpha)$ and $N^{(-)}(\lambda, \alpha)$ the numbers of eigenvalues (counting multiplicities) of $H_0 - tV$, which cross λ respectively leftwards and rightwards as t grows from 0 to α. Following [30], we define

$$N(\lambda; H_0, G_+, G_-; \alpha) := N^{(+)}(\lambda, \alpha) - N^{(-)}(\lambda, \alpha), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0). \tag{0.14}$$

The eigenvalues which “turn” at λ, do not enter the expression (0.14) — see [30, 31]. Obviously, for the perturbations of a definite sign (0.14) coincides with $\pm N_\pm$:

$$\begin{aligned} N(\lambda; H_0, G_+, 0; \alpha) &= N_+(\lambda; H_0, G_+; \alpha), \qquad \lambda \in \mathbb{R} \setminus \sigma(H_0), \\ N(\lambda; H_0, 0, G_-; \alpha) &= -N_-(\lambda; H_0, G_-; \alpha), \qquad \lambda \in \mathbb{R} \setminus \sigma(H_0). \end{aligned} \tag{0.15}$$

Next, for $\lambda < \inf \sigma(H_0)$

$$N(\lambda; H_0, G_+, G_-; \alpha) = N(\lambda, \alpha), \quad \lambda < \inf \sigma(H_0), \tag{0.16}$$

where $N(\lambda, \alpha)$ is defined by (0.1). Note also that, as it has been found in [30], the function (0.14) admits a representation which corresponds to the consecutive “switching on” of the negative and positive parts of the perturbation:

$$N(\lambda; H_0, G_+, G_-; \alpha) = -N_-(\lambda; H_0, G_-; \alpha) + N_+(\lambda; H_0 + \alpha V_-, G_+; \alpha). \tag{0.17}$$

In [30] there has been proved an abstract theorem on the stability of the leading term of the asymptotics of $N(\lambda; H_0, G_+, G_-; \alpha)$ as $\alpha \to \infty$. Applying this theorem to the Schrödinger operator and taking into account (0.10), one obtains the following fact (see [30, §6]). Under the same conditions (0.3), (0.5), (0.7), (0.9) on V, g, U, $\mathbf{A}$ one has

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N(\lambda; H_0, G_+, G_-; \alpha) = C_{0.8}(V, g), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0). \tag{0.18}$$

Note that in [30] formula (0.18) was obtained under more restrictive conditions than those assumed above; nevertheless, the generalisation is almost obvious (though we do not need this). Concerning the behaviour of $N^{(\pm)}(\lambda, \alpha)$, see also [14, 31] and references therein.

0.5 The purpose of the present paper is to give a certain analogue of the relations (0.10), (0.13), (0.18) for the points λ lying *on the continuous spectrum* of the unperturbed operator H_0. In this case the natural generalisation of the counting functions (0.1), (0.11), (0.14) is given by the I. M. Lifshits–M. G. Krein *spectral shift function* (SSF). Below we remind some basic facts of the SSF theory and give references to the literature.

Let H_0 and H be selfadjoint operators in a Hilbert space $\mathcal{H}$ and let their difference V be a trace class operator:

$$V := H - H_0 \in \mathbf{S}_1(\mathcal{H}). \tag{0.19}$$

Then the following *Lifshits—Krein trace formula* holds [20, 18]:

$$\operatorname{Tr}(\phi(H) - \phi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda; H, H_0)\phi'(\lambda)d\lambda. \tag{0.20}$$

Here ϕ is any function of some appropriate functional class and $\xi(\lambda; H, H_0)$ is the SSF for the pair H_0, H. The SSF is given by the *Krein formula* [18]

$$\xi(\lambda; H, H_0) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \arg\det(I + V(H_0 - (\lambda + i\varepsilon)I)^{-1}), \text{ a. e. } \lambda \in \mathbb{R}. \tag{0.21}$$

The branch of the argument in (0.21) is fixed by the condition

$$\arg\det(I + V(H_0 - zI)^{-1}) \to 0 \quad \text{as} \quad \operatorname{Im} z \to +\infty.$$

A detailed exposition of the SSF theory can be found in [12, 38]. If operators H_0, H_1, H_2 in $\mathcal{H}$ are such that the differences $H_2 - H_1$ and $H_1 - H_0$ belong to the trace class, then, obviously,

$$\xi(\lambda; H_2, H_0) = \xi(\lambda; H_2, H_1) + \xi(\lambda; H_1, H_0). \tag{0.22}$$

The SSF is *monotone* in V [18]:

$$\pm V \geq 0 \quad \Rightarrow \quad \pm\xi(\lambda; H, H_0) \geq 0. \tag{0.23}$$

In applications, instead of (0.19), it is usually possible to check the inclusion

$$f(H) - f(H_0) \in \mathbf{S}_1(\mathcal{H}). \tag{0.24}$$

Here $f : (a,b) \to \mathbb{R}$ is some smooth enough, monotone function and $(a,b) \subseteq \mathbb{R}$ is an interval which contains $\sigma(H) \cup \sigma(H_0)$. Then the SSF for the pair H_0, H is defined by the natural formula

$$\xi(\lambda; H, H_0) := \operatorname{sign} f' \cdot \xi(f(\lambda); f(H), f(H_0)), \quad \text{a.e. } \lambda \in \mathbb{R}. \tag{0.25}$$

Obviously, in this case the trace formula (0.20) is still valid (only the class of admissible functions ϕ may have to be changed). The relation (0.23) for such a definition of SSF has been proved in [17] (see also [38, §8.10]) for $f(\lambda) = (\lambda - \lambda_0)^{-m}$ with any $m > 0$ and $\lambda_0 < \inf(\sigma(H) \cup \sigma(H_0))$. Finally, note that if the operators H_0, H_1, H_2 in $\mathcal{H}$ are such that the differences $f(H_2) - f(H_1)$ and $f(H_1) - f(H_0)$ are trace class operators, then (0.22) is still valid.

Let us discuss the behaviour of the SSF away from the continuous spectrum of H_0. First suppose that H_0 is semibounded from below. Then, taking an appropriate ϕ in (0.20), one finds:

$$\xi(\lambda; H_0 - \alpha V, H_0) = -N(\lambda, \alpha), \quad \lambda < \inf \sigma(H_0), \tag{0.26}$$

where $N(\lambda, \alpha)$ is defined by (0.1). Next, if H_0 is an arbitrary selfadjoint operator and $V = G^*G \geq 0$, then

$$\xi(\lambda; H_0 \mp \alpha V, H_0) = \mp N_\pm(\lambda; H_0, G; \alpha), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0), \tag{0.27}$$

where $N_\pm(\lambda; H_0, G; \alpha)$ is defined by (0.11). The relation (0.27) have been proved in [36]. Finally, if $V = G_+^* G_+ - G_-^* G_-$, then in concrete problems (see, e.g., Remark 5.4 below) one can often justify the formula

$$\xi(\lambda; H_0 - \alpha V, H_0) = -N(\lambda; H_0, G_+, G_-; \alpha), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0), \tag{0.28}$$

where $N(\lambda; H_0, G_+, G_-; \alpha)$ is defined by (0.14).

0.6 Let, as above, H_0 be the Schrödinger operator (0.4) with g, U, $\mathbf{A}$ satisfying the conditions (0.5), (0.7), (0.9). Next, let $V = V(x)$ be the perturbation potential which decays sufficiently fast as $|x| \to \infty$. We prove the following asymptotic formula for the SSF:

$$\lim_{\alpha \to \infty} \alpha^{-d/2} \xi(\lambda; H_0 - \alpha V, H_0) = -C_{0.8}(V, g), \quad \lambda \in \mathbb{R}, \tag{0.29}$$

(the precise result is formulated in §1.3).

From (0.26), (0.27), (0.28) it follows that (0.29) can be considered as a natural generalisation of (0.10), (0.13), (0.18). Note, however, that the fact of existence of the SSF imposes strong restrictions on the rate of decay of V at infinity. For this reason, (0.29) makes sense for a poorer class of potentials V, than the relations (0.10), (0.13), (0.18).

Formula (0.29) adds a new feature to our picture of the "flow of eigenvalues". It shows that this flow can be noticed and controlled not only in the gaps, but also on the spectrum of H_0, where it obeys the same asymptotic "conservation law". This point of view on the movement of eigenvalues of $H_0 - tV$ and the idea to interpret it in terms of the SSF is due to M. Sh. Birman.

The relation (0.29) shows that the functions (of λ) in the left hand side, which depend on a parameter α, converge to a constant $-C_{0.8}(V, g)$ as $\alpha \to \infty$. Let us specify the type of convergence. In fact, we prove two main results. Theorem 1.5 gives a pointwise convergence in (0.29) for a.e. $\lambda \in \mathbb{R}$ (see Remark 1.6). The hypothesis of this theorem includes the requirement $V \geq 0$. Next, Theorem 1.7 gives convergence in the weighted space $L_1(\mathbb{R}; d\mu(\lambda))$ with some power weight $d\mu(\lambda)$. We have managed to prove this theorem only for $d \geq 3$. Thus, neither of these two theorems is exhaustive, but together they provide a fairly complete picture. As a by-product of the proof of these theorems, we obtain some information on the asymptotic behaviour of $\xi(\lambda; H_0 + \alpha V, H_0)$ as $\alpha \to \infty$ (for $V \geq 0$) — see Theorem 1.8.

0.7 The paper is organised as follows. In §1 we give some necessary definitions and present the main results. In §2 we formulate some basic results of [24, 25] concerning a new representation for the SSF for the case of perturbations of a definite sign. This representation plays a crucial role in the proof of the main results of the paper. In §3 we formulate some known estimates for the Schrödinger operator and discuss the existence of the SSF. Theorem 1.5 (on the pointwise convergence in (0.29)) is proved in §4 and Theorem 1.7 (on the weighted integral convergence in (0.29)) — in §5. In §6 for the sake of completeness of the exposition we give the proof of formula (0.10), since earlier it was proved under somewhat more restrictive assumptions.

1 Main results

1.1 Notation. 1) The standard inner product in $\mathbb{C}^d$ is denoted by $\langle \cdot, \cdot \rangle$; $\mathbf{1}$ is a unit $d \times d$-matrix. Integral without the domain of integration explicitly specified implies integration over $\mathbb{R}^d$. By $\operatorname{meas} \delta$ we denote the Lebesgue measure of a Borel set $\delta \subset \mathbb{R}$. By ω_d we denote the volume of a unit ball in $\mathbb{R}^d$. Formulae and statements with double indices ($\pm$ and $\mp$) should be read as pairs of statements, in one of which all the indices take upper values and in another — the lower ones. A constant which first appears in formula $(i.j)$ is denoted by $C_{i.j}$.

2) *Functions.* The spaces $L_p(\mathbb{R}^d)$ and $L_{p,\,\mathrm{loc}}(\mathbb{R}^d)$ are defined in a usual way. The space $l_\tau(\mathbb{Z}^d; L_\sigma(\mathbf{Q}^d))$, $\tau > 0$, $\sigma \geq 1$, consists of the functions $u \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^d)$

such that the following functional is finite:

$$\|u\|^\tau_{l_\tau(L_\sigma)} := \sum_{j\in\mathbb{Z}^d}\left(\int_{\mathbf{Q}^d+j}|u|^\sigma dx\right)^{\tau/\sigma}, \quad \mathbf{Q}^d=(0,1)^d\subset\mathbb{R}^d. \qquad (1.1)$$

For a real-valued function F we put $F_\pm := (|F|\pm F)/2$. For an open set $\Omega\subset\mathbb{R}^d$, $H^1(\Omega)$ is a usual Sobolev space with the norm $\|u\|^2_{H^1}=\int_\Omega(|\nabla u|^2+|u|^2)dx$ and $H^1_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

3) *Operators.* Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. For a linear operator A, the notations $\operatorname{Dom} A$, $\operatorname{Ran} A$, $\operatorname{Ker} A$, A^*, $\sigma(A)$, $\rho(A)$ are standard; $\overline{A}$ is the closure of A, I is the identity operator. For a selfadjoint operator A the symbol $E_A(\delta)$ denotes the spectral measure of a Borel set $\delta\subset\mathbb{R}$; $A_\pm := (|A|\pm A)/2$. Resolvent of an "unperturbed" selfadjoint operator H_0 is denoted by $R_0(z)=(H_0-zI)^{-1}$. By $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ and $\mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ we denote respectively the spaces of bounded and compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$; $\mathbf{B}(\mathcal{H}) := \mathbf{B}(\mathcal{H},\mathcal{H})$, $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H},\mathcal{H})$. For $T=T^*\in\mathbf{S}_\infty(\mathcal{H})$ and $s>0$ we denote $n_\pm(s,T) := \operatorname{rank} E_{T_\pm}((s,+\infty))$, and for $T\in\mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ let $n(s,T) := n_+(s^2,T^*T)$.

4) *Classes of compact operators.* For $0<p<\infty$ the Neumann-Schatten class $\mathbf{S}_p(\mathcal{H}_1,\mathcal{H}_2)\subset\mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ is a set of operators T such that the following functional is finite:

$$\|T\|^p_{\mathbf{S}_p} := \sum_j s_j^p(T) = p\int_0^\infty s^{p-1}n(s,T)ds,$$

where $s_j(T)$ are the singular numbers of T. The functional $\|\cdot\|_{\mathbf{S}_p}$ is a norm for $p\ge 1$ and a quasinorm for $p<1$. For $0<p<\infty$ the class $\mathbf{\Sigma}_p(\mathcal{H}_1,\mathcal{H}_2)\subset\mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ is a set of all compact operators T such that the following functional is finite:

$$\|T\|_{\mathbf{\Sigma}_p} := \left(\sup_{s>0} s^p n(s,T)\right)^{1/p}.$$

The functional $\|\cdot\|_{\mathbf{\Sigma}_p}$ is a quasinorm. The classes $\mathbf{\Sigma}_p(\mathcal{H}_1,\mathcal{H}_2)$ are not separable (if $\dim\mathcal{H}_1=\dim\mathcal{H}_2=\infty$); a separable subspace $\mathbf{\Sigma}^0_p\subset\mathbf{\Sigma}_p$ is defined by

$$\mathbf{\Sigma}^0_p := \{T\in\mathbf{\Sigma}_p \mid \lim_{s\to+0} s^p n(s,T)=0\}.$$

Note that $\mathbf{S}_p\subset\mathbf{\Sigma}^0_p$. For $T=T^*\in\mathbf{S}_\infty$ the following functionals are introduced:

$$\Delta^{(\pm)}_p(T) := \limsup_{s\to\infty} s^p n_\pm(s,T), \qquad (1.2)$$

$$\delta^{(\pm)}_p(T) := \liminf_{s\to\infty} s^p n_\pm(s,T), \qquad (1.3)$$

so that $0 \le \delta_p^{(\pm)}(T) \le \Delta_p^{(\pm)}(T) \le \infty$. The functionals $\Delta_p^{(\pm)}$, $\delta_p^{(\pm)}$ are continuous in $\mathbf{\Sigma}_p$ and

$$\Delta_p^{(\pm)}(T_1) = \Delta_p^{(\pm)}(T_2) \quad \text{if} \quad T_1 - T_2 \in \mathbf{\Sigma}_p^0. \tag{1.4}$$

The last statement is essentially due to H. Weyl. More information on the classes $\mathbf{\Sigma}_p$ can be found in [10].

1.2 Schrödinger operator. In $\mathbb{R}^d$, $d \ge 1$, we fix a real-valued function U, satisfying (0.7) and measurable functions $g = \{g_{kj}\}_{k,j=1}^d$, $\mathbf{A} = \{A_j\}_{j=1}^d$ with real entries, satisfying (0.5), (0.6). In $\mathcal{H} = L_2(\mathbb{R}^d)$ define a quadratic form

$$h_{g,\mathbf{A},U}^{(0)}[u,u] = \sum_{k,j=1}^d \int g_{kj}(x)\left(-i\frac{\partial u}{\partial x_k} - A_k u\right)\overline{\left(-i\frac{\partial u}{\partial x_j} - A_j u\right)}dx + \int U|u|^2 dx \tag{1.5}$$

on the domain

$$d[h_{g,\mathbf{A},U}^{(0)}] = \{u \in L_2(\mathbb{R}^d) \mid i\nabla u + \mathbf{A}u \in L_2(\mathbb{R}^d), \quad U^{1/2}u \in L_2(\mathbb{R}^2)\}. \tag{1.6}$$

It is easy to see that the form $h_{g,\mathbf{A},U}^{(0)}$ is closed.

Proposition 1.1 *Under the assumptions* (0.5)–(0.7) *the set* $C_0^\infty(\mathbb{R}^d)$ *is dense in* $d[h_{g,\mathbf{A},U}^{(0)}]$ *with the metric* $h_{g,\mathbf{A},U}^{(0)}[u,u] + \|u\|^2$.

Proposition 1.1 follows from the estimate

$$g_- h_{\mathbf{1},\mathbf{A},U}^{(0)}[u,u] \le h_{g,\mathbf{A},U}^{(0)}[u,u] \le g_+ h_{\mathbf{1},\mathbf{A},U}^{(0)}[u,u] \tag{1.7}$$

and the fact that $C_0^\infty(\mathbb{R}^d)$ is dense in $d[h_{\mathbf{1},\mathbf{A},U}^{(0)}] = d[h_{g,\mathbf{A},U}^{(0)}]$ with the metric $h_{\mathbf{1},\mathbf{A},U}^{(0)}[u,u] + \|u\|^2$ — see [32].

Denote by $H_0 = H_0(g,\mathbf{A},U)$ the selfadjoint operator in $\mathcal{H}$ generated by the form $h_{g,\mathbf{A},U}^{(0)}$. By (1.7),

$$g_- H_0(\mathbf{1},\mathbf{A},U) \le H_0(g,\mathbf{A},U). \tag{1.8}$$

The last relation implies

Proposition 1.2 *Let V be an operator of multiplication by a function $V = V(x) \ge 0$, $x \in \mathbb{R}^d$. If V is $(-\triangle)$–form compact, then V is $H_0(g,\mathbf{A},U)$–form compact.*

Proof By (1.8), the problem reduces to the case of a flat metric $g \equiv \mathbf{1}$. Now, following the proof of Theorems 2.5 and 2.6 of [3], one gets the required statement. ∎

Let V be an operator of multiplication by a function $V = V(x)$, $x \in \mathbb{R}^d$, such that $|V|$ is $(-\triangle)$–form compact. Due to Proposition 1.2, the operators

$$H_+(\alpha) = H_0 + \alpha V_-, \quad H_-(\alpha) = H_0 - \alpha V_+, \quad H(\alpha) = H_0 - \alpha V, \quad \alpha > 0, \tag{1.9}$$

are well-defined via the sums of quadratic forms. Let (as in Introduction)

$$N(\lambda, \alpha) := \operatorname{rank} E_{H(\alpha)}((-\infty, \lambda)), \quad \lambda < 0, \quad \alpha > 0. \tag{1.10}$$

Proposition 1.3 *Assume the conditions* (0.5), (0.7) *and let*

$$\left.\begin{array}{lll} V \in L_{d/2}(\mathbb{R}^d) & \textit{for} & d \geq 3; \\ V \in l_1(\mathbb{Z}^2; L_\sigma(\mathbf{Q}^2)),\ \sigma > 1 & \textit{for} & d = 2; \\ V \in l_{1/2}(\mathbb{Z}^1; L_1(\mathbf{Q}^1)) & \textit{for} & d = 1. \end{array}\right\} \tag{1.11}$$

$$\left.\begin{array}{lll} |\mathbf{A}|^2 + U \in L_{d/2,\mathrm{loc}}(\mathbb{R}^d) & \textit{for} & d \geq 3; \\ |\mathbf{A}|^2 \log(1 + |\mathbf{A}|) + U \log(1 + U) \in L_{1,\mathrm{loc}}(\mathbb{R}^2) & \textit{for} & d = 2; \\ \mathbf{A} \equiv 0,\ U \in L_{1,\mathrm{loc}}(\mathbb{R}) & \textit{for} & d = 1. \end{array}\right\} \tag{1.12}$$

Then, for any $\lambda < 0$ the asymptotic formula (0.10) *holds.*

Proposition 1.3 was proved in [11] for $\mathbf{A} = 0$ and under somewhat more restrictive conditions on U, in [8] for $g \equiv \mathbf{1}$, $d \geq 3$, in [29] for $g \equiv \mathbf{1}$, $d = 2$, and also in many other particular cases. The precise in the order of α bounds of $N(\lambda, \alpha)$ are well-known — see (3.1)–(3.3) below. Once these bounds are established, the proof of the asymptotics (0.10) by means of the local compactness technique of [8] becomes a routine procedure. Nevertheless, to the best of the author's knowledge, the proof of (0.10) under the assumptions (1.11), (0.5), (0.7), (1.12) has not been published. In order to make the exposition complete, we give this proof in Appendix (§6).

Note that the condition $\mathbf{A} \equiv 0$ for $d = 1$ is motivated by the fact that in the one dimensional case one can always gauge away the magnetic field. For $d \geq 3$ one can take $\lambda = 0$ in (0.10); for $d = 1, 2$ formula (0.10) is valid for $\lambda = 0$ only under some additional assumptions on V — see, e.g., [9, 5].

The conditions imposed by the hypothesis of Proposition 1.3 on the negative part of V can be considerably relaxed. It is sufficient to assume

$$V_- \in L_{d/2,\mathrm{loc}}(\mathbb{R}^d) \text{ for } d \geq 3,$$
$$V_- \log(1 + V_-) \in L_{1,\mathrm{loc}}(\mathbb{R}^2) \text{ for } d = 2,$$
$$V_- \in L_{1,\mathrm{loc}}(\mathbb{R}) \text{ for } d = 1.$$

Moreover, if there exists an open set $\Omega \subset \mathbb{R}^d$ such that $V(x) \geq 0$ for $x \in \Omega$ and $V(x) \leq 0$ for $x \in \mathbb{R} \setminus \Omega$, then it is sufficient to assume that $V_- \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ for all dimensions $d \geq 1$ (see, e.g., [11, Appendix 6]). If U has a negative part which is $(-\triangle)$–form bounded with the relative bound zero, then (0.10) is still valid (for $\lambda < \inf \sigma(H_0)$) — see [4].

1.3 SSF of the Schrödinger operator. Main results. Below $H_0 = H_0(g, \mathbf{A}, U)$ is the Schrödinger operator, defined in the previous subsection, and $H(\alpha)$, $H_\pm(\alpha)$ are the operators (1.9). Let us first discuss the existence of the SSF.

Proposition 1.4 *Assume* (0.5)–(0.7) *and let the perturbation potential* $V = V(x)$, $x \in \mathbb{R}^d$, *be such that* $|V|$ *is* $(-\triangle)$*–form compact and*

$$V \in l_1(\mathbb{Z}^d; L_2(\mathbf{Q}^d)) \text{ for } d \geq 4, \tag{1.13}$$

$$V \in L_1(\mathbb{R}^d) \text{ for } d \leq 3. \tag{1.14}$$

Then for $d \geq 4$ *the inclusion*

$$(H(\alpha) - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1 \tag{1.15}$$

holds with any integer $k > (d-1)/2$ *and any* $\lambda_0 < 0$ *with large enough absolute value, and for* $d \leq 3$ *— with* $k = 1$ *and any* $\lambda_0 \in \rho(H_0) \cap \rho(H(\alpha))$.

The proof (given in §3.3) follows the pattern of [3, Theorem 2.11, Corollary 2.13], where the case $g \equiv \mathbf{1}$ has been considered. The inclusion (1.15) allows one to define the SSF for the pair H_0, $H(\alpha)$ according to (0.25) with $f(\lambda) = (\lambda - \lambda_0)^{-k}$.

Theorem 1.5 *Assume the conditions* (0.5), (0.7), (1.12) *and let the perturbation potential* $V = V(x) \geq 0$, $x \in \mathbb{R}^d$, *satisfy the conditions*

$$\left.\begin{array}{ll} V \in L_{d/2}(\mathbb{R}^d) \cap l_1(\mathbb{Z}^d; L_2(\mathbf{Q}^d)) & \text{for } d \geq 4; \\ V \in L_{3/2}(\mathbb{R}^3) \cap L_1(\mathbb{R}^3) & \text{for } d = 3; \end{array}\right\} \tag{1.16}$$

$$\left.\begin{array}{ll} V \in l_\tau(\mathbb{Z}^2; L_\sigma(\mathbf{Q}^2)),\ \tau < 1,\ \sigma > 1 & \text{for } d = 2; \\ V \in l_{1/2}(\mathbb{Z}^1; L_1(\mathbf{Q}^1)) & \text{for } d = 1. \end{array}\right\} \tag{1.17}$$

Then for a.e. $\lambda \in \mathbb{R}$:

$$\lim_{\alpha \to \infty} \alpha^{-d/2} \xi(\lambda; H(\alpha), H_0) = -C_{0.8}(V, g). \tag{1.18}$$

Remark 1.6 One might ask, what do we mean by a.e. $\lambda \in \mathbb{R}$ in Theorem 1.5: the set of λ's, where $\xi(\lambda; H(\alpha), H_0)$ is defined, may depend on α! Let us explain this. For every $\alpha > 0$ formulae (0.21), (0.25) define the SSF $\xi(\lambda; H(\alpha), H_0)$ as an equivalence class of functions which differ from each other on a set of a zero measure. In the proof of Theorem 1.5 we shall specify a set $M \subset \mathbb{R}$ of a full measure and construct functions $\mathcal{N}_+(\lambda, \alpha) > 0$, $\lambda \in M$, $\alpha > 0$, such that

(i) for any $\alpha > 0$ the function $-\mathcal{N}_+(\cdot, \alpha)$ belongs to the equivalence class of $\xi(\cdot\,; H(\alpha), H_0)$;

(ii) for all $\lambda \in M$:

$$\lim_{\alpha\to\infty} \alpha^{-d/2}\mathcal{N}_+(\lambda, \alpha) = C_{0.8}(V, g).$$

In the following theorem the potential V is not supposed to be of a definite sign.

Theorem 1.7 *Let $d \geq 3$; assume the conditions* (0.5), (0.7), (1.12), *and suppose that the perturbation potential V satisfies the inclusion* (1.16). *Then, for all $E > 0$ and $p > d/2$:*

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} |\alpha^{-d/2}\xi(\lambda; H(\alpha), H_0) + C_{0.8}(V, g)|(\lambda + \tilde{E})^{-p} d\lambda = 0, \quad \tilde{E} > E. \quad (1.19)$$

1.4 Comments. 1) The natural question, which arises in connection with Theorem 1.5, is that about the behaviour of $\xi(\lambda; H_+(\alpha), H_0)$ for $\alpha \to \infty$. The asymptotic behaviour of $N_-(\lambda; H_0, G_-; \alpha)$ has been studied in [14, 2]. The approach of the present paper does not allow us to obtain asymptotics of $\xi(\lambda; H_+(\alpha), H_0)$. However, some bounds for this function can be proved. As a by-product of the proof of Theorem 1.5, we obtain the following proposition.

Theorem 1.8 *Let $d \geq 1$; assume the conditions* (0.5)–(0.7). *Next, let the perturbation potential $V = V(x) \leq 0$, $x \in \mathbb{R}^d$, be $(-\triangle)$-form compact, satisfy the inclusion*

$$V \in l_\tau(\mathbb{Z}^d; L_1(\mathbf{Q}^d)), \quad 1/2 \leq \tau \leq 1, \quad (1.20)$$

and, if $d \geq 4$, satisfy an additional assumption (1.13). *Then for a.e. $\lambda \in \mathbb{R}$*

$$\lim_{\alpha\to\infty} \alpha^{-\rho}\xi(\lambda; H_+(\alpha), H_0) = 0, \quad (1.21)$$

where $\rho = \tau$, if $\tau < 1$ and $\rho > 1$ — any number, if $\tau = 1$.

Formula (1.21) should be understood in the same sense as (1.18) — see Remark 1.6. The proof of Theorem 1.8 is given in §4.

Besides, in [25] there have been obtained integral estimates for the SSF $\xi(\lambda; H_+(\alpha), H_0)$ (see also the inequality (2.24) below).

2) Note that the spectrum of the operator $H_0(g, \mathbf{A}, U)$ may have extremely "unregular" structure; for example, the absolutely continuous spectrum may be empty. At the same time, our results are formulated in the terms which are "not sensitive" to the quality of the spectrum of H_0.

On the other hand, one can consider the operators H_0 with absolutely continuous spectrum and operators V which are smooth with respect to H_0 in some proper sense. In the framework of this approach it is possible to obtain the estimates of the SSF which are locally uniform in λ; see [36].

3) There is extensive literature devoted to the large energy and semiclassical asymptotics of the SSF; we do not touch this subject here.

4) Theorems 1.5, 1.7, 1.8 can be generalised to the case of some differential operators H_0 of the order $l \neq 2$ — see Remarks 4.8, 5.5 below.

2 Representation for the SSF

In this section we collect necessary statements from [24, 25] concerning some new representation for the SSF.

2.1 Definition of $\mathcal{N}_\pm$. Let $\mathcal{H}$ be a basic and $\mathcal{K}$ — an auxiliary Hilbert space, H_0 be a selfadjoint operator in $\mathcal{H}$ and $G : \mathcal{H} \to \mathcal{K}$ be a closed linear operator. For simplicity we assume from the very beginning that

$$H_0 \geq 0, \tag{2.1}$$

since further on the assumption that H_0 be semibounded from below will become essential. Next, suppose that

$$GR_0^{1/2}(-1) \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}). \tag{2.2}$$

For $z \in \rho(H_0)$ define the operators

$$T(z; H_0, G) := \overline{GR_0(z)G^*} = (GR_0^{1/2}(-1))(H_0 + I)R_0(z)(GR_0^{1/2}(-1))^*, \tag{2.3}$$

$$A(z; H_0, G) := \operatorname{Re} T(z; H_0, G), \quad K(z; H_0, G) := \operatorname{Im} T(z; H_0, G), \tag{2.4}$$

which are compact in $\mathcal{K}$. We shall write $T(z)$ instead of $T(z; H_0, G)$, etc., if the choice of H_0, G is clear from the context. Suppose that for some $\lambda \in \mathbb{R}$ the pair H_0, G satisfies the following condition.

Condition 2.1 *The limit*

$$\lim_{\varepsilon \to +0} T(\lambda + i\varepsilon; H_0, G) =: T(\lambda + i0; H_0, G), \tag{2.5}$$

exists in the operator norm and

$$K(\lambda + i0; H_0, G) \in \mathbf{S}_1(\mathcal{K}). \tag{2.6}$$

Then define

$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\pm(\alpha^{-1}; A(\lambda + i0) + tK(\lambda + i0)), \quad \alpha > 0. \tag{2.7}$$

It is easy to see that the conditions (2.2), (2.5), (2.6) imply the convergence of the integral (2.7).

Proposition 2.2 [25, Corollary 3.8] *Assume that for some open interval* $\delta \subset \mathbb{R}$ *the following inclusion holds:*

$$GE_{H_0}(\delta) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}). \tag{2.8}$$

Then for a.e. $\lambda \in \delta$ *the pair* H_0, G *satisfies the Condition 2.1 and for any* $\alpha > 0$

$$\mathcal{N}_\pm(\cdot; H_0, G; \alpha) \in L_{1,\mathrm{loc}}(\delta).$$

In particular, if (2.8) *holds for any bounded interval* $\delta \subset \mathbb{R}$*, then*

$$\mathcal{N}_\pm(\cdot; H_0, G; \alpha) \in L_{1,\mathrm{loc}}(\mathbb{R}).$$

Define the operators

$$H_\pm(\alpha) = H_0 \pm \alpha V, \quad V = \overline{G^*G}, \quad \alpha > 0, \tag{2.9}$$

via the form sums and let the counting function $N_\pm(\lambda; H_0, G; \alpha)$ be as defined by (0.11). The following relation is known as the *Birman–Schwinger principle:*

$$N_\pm(\lambda; H_0, G; \alpha) = n_\pm(\alpha^{-1}, T(\lambda; H_0, G)), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0). \tag{2.10}$$

Comparing (2.10) and (2.7) and observing that for $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ one has $K(\lambda + i0) = 0$ and $A(\lambda + i0) = T(\lambda + i0) = T(\lambda)$, we arrive at the formula

$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) = N_\pm(\lambda; H_0, G; \alpha), \quad \lambda \in \mathbb{R} \setminus \sigma(H_0). \tag{2.11}$$

In particular, by (0.12),

$$\mathcal{N}_+(\lambda; H_0, G; \alpha) = N(\lambda, \alpha), \quad \lambda < 0. \tag{2.12}$$

2.2 Connection between $\mathcal{N}_\pm$ and SSF.

1) Trace class perturbations. Let H_0 be a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. It is well-known that under these conditions[1] for a.e. $\lambda \in \mathbb{R}$ the limits (2.5) exist in the Hilbert-Schmidt norm and (2.6) holds — see [6] or [38]. More precise results (on the limit values in $\mathbf{S}_p$ for $p > 1$) can be found in [22, 23] — see Proposition 4.3 below.

Proposition 2.3 [24, Theorem 1.1] *Let* H_0 *be a selfadjoint operator in* $\mathcal{H}$ *and* $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$*; then for any* $\alpha > 0$

$$\xi(\lambda; H_0 \pm \alpha G^*G, H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G; \alpha), \quad a.e.\ \lambda \in \mathbb{R}. \tag{2.13}$$

[1] The condition (2.1) in this case can be omitted

Note that, by (2.11), for $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ the relation (2.13) turns into the formula (0.27), which has been proved earlier in [36].

2) Relatively trace class perturbations. Let H_0, G satisfy the conditions (2.1), (2.2) and for some $m > 0$:

$$GR_0^m(-1) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}). \tag{2.14}$$

Then, by Proposition 2.2, for a.e. $\lambda \in \mathbb{R}$ the pair H_0, G satisfies Condition 2.1. Let us fix some $\alpha > 0$ and define the operators (2.9) via the form sums. In order to define the SSF for the pair H_0, $H_\pm(\alpha)$, assume that for some $k > 0$ and $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm(\alpha)))$ the following inclusion holds:

$$(H_\pm(\alpha) - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1(\mathcal{H}). \tag{2.15}$$

The inclusion (2.15) enables one to define the SSF by (0.25) with $f(\lambda) = (\lambda - \lambda_0)^{-k}$.

Proposition 2.4 [24, Theorem 1.2] *Assume the conditions* (2.1)*,* (2.2)*,* (2.14)*,* (2.15)*. Then,*

$$\xi(\lambda; H_\pm(\alpha), H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G; \alpha), \qquad a.e.\ \lambda \in \mathbb{R}. \tag{2.16}$$

2.3 Pointwise estimates for $\mathcal{N}_\pm$. For the operators

$$A = A^* \in \mathbf{S}_\infty(\mathcal{K}), \quad K = K^* \in \mathbf{S}_q(\mathcal{K}), \quad q \le 1, \tag{2.17}$$

introduce the functions which "simulate" the integral (2.7):

$$\eta_\pm(s; A, K) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\pm(s, A + tK), \quad s > 0. \tag{2.18}$$

One easily checks that the conditions (2.17) (for $q = 1$) are sufficient for the convergence of the integral (2.18).

Proposition 2.5 [24, Lemma 2.1] *For the functions* (2.18) *the following estimates hold:*

$$\eta_\pm(s; A, K) \le \inf_{0<\tau<1} (n_\pm(s(1-\tau), A) + (s\tau)^{-q} \mu_q(s\tau, K) \|K\|^q_{\mathbf{S}_q}), \tag{2.19}$$

$$\eta_\pm(s; A, K) \ge \sup_{\tau>0} (n_\pm(s(1+\tau), A) - (s\tau)^{-q} \mu_q(s\tau, K) \|K\|^q_{\mathbf{S}_q}), \tag{2.20}$$

where $\mu_q(\tau, K) \ge 0$ are some functions satisfying

$$\mu_q(\tau, K) \to 0 \text{ as } \tau \to +0. \tag{2.21}$$

2.4 Integral estimates for $\mathcal{N}_\pm$.

Proposition 2.6 [25] *Let the operators* H_0*,* G *satisfy the conditions* (2.1), (2.2) *and* (2.14) *for some* $m > 1/2$*. Then, for any* $E > 0$ *and* $\alpha > 0$ *the following estimates hold:*

$$\int_{-E}^{\infty} (\mathcal{N}_+(\lambda; H_0, G; \alpha) - N_+(-E; H_0, G; \alpha(1-\theta)^{-2}))_+ (\lambda + 2E)^{-2m} d\lambda \leq \alpha\theta^{-1-m} \|GR_0^m(-E)\|_{\mathbf{S}_2}^2, \quad \forall\theta \in (0,1); \tag{2.22}$$

$$\int_{-E}^{\infty} (\mathcal{N}_+(\lambda; H_0, G; \alpha) - N_+(-2E; H_0, G; \alpha(1+\theta)^{-1}))_- (\lambda + 2E)^{-2m} d\lambda \leq \alpha\theta^{-1}(4 + 2/(2m-1)) \|GR_0^m(-2E)\|_{\mathbf{S}_2}^2, \quad \forall\theta > 0; \tag{2.23}$$

$$\int_0^{\infty} \mathcal{N}_-(\lambda; H_0, G; \alpha)(\lambda + 2E)^{-2m} d\lambda \leq \alpha \|GR_0^m(-2E)\|_{\mathbf{S}_2}^2. \tag{2.24}$$

The estimates (2.22)–(2.24) are immediate consequences of Lemmas 4.3, 4.4 and 3.3 respectively of [25].

3 Schrödinger operator: auxiliary statements

In this section we collect some auxiliary statements concerning the Schrödinger operator $H_0 = H_0(g, \mathbf{A}, U)$ and prove Proposition 1.4. In §3.2-3.3 our reasoning follows almost literally the pattern of [3, §2]. Below we always suppose that (0.5)–(0.7) hold.

3.1 Spectral bounds.

Proposition 3.1 *Assume the conditions* (0.5)–(0.7) *and* (1.11)*. Then the function* $N(\lambda, \alpha)$*, defined by* (1.10)*, obeys the following estimates:*

$$N(\lambda, \alpha) \leq C_{3.1}(d)\alpha^{d/2} g_-^{-d/2} \int V_+^{d/2}(x)dx, \quad \lambda < 0, \quad d \geq 3, \tag{3.1}$$

$$N(\lambda, \alpha) \leq C_{3.2}(\lambda, \sigma)\alpha g_-^{-1} \|V_+\|_{l_1(L_\sigma)}, \quad \sigma > 1, \quad \lambda < 0, \quad d = 2, \tag{3.2}$$

$$N(\lambda, \alpha) \leq C_{3.3}(\lambda)\alpha^{1/2} g_-^{-1/2} \|V_+\|_{l_{1/2}(L_1)}^{1/2}, \quad \lambda < 0, \quad d = 1. \tag{3.3}$$

The estimate (3.1) follows from (1.8) and the magnetic variant of the Cwikel–Lieb–Rozenblum bound — see [19, 33] and also [21]. The estimate (3.2) follows from (1.8) and the corresponding estimate for $H_0(\mathbf{1}, \mathbf{A}, U)$, which has been recently proved in [29]. For $d = 1$, using (1.8) and gauging away the magnetic field, the problem is reduced to the case $H_0 = -\triangle$; in this case, the bound (3.3) is well known (see, e.g., [9]).

Obviously, the estimates (3.1)–(3.3) do not, in fact, require any assumptions on V_-, apart from the inclusion $V_- \in L_{1,\text{loc}}(\mathbb{R}^d)$; the latter is assumed in order to ensure that $C_0^\infty(\mathbb{R}^d)$ is a form core for $H(\alpha)$.

Remark 3.2 Denote

$$Z_\pm(\lambda) := R_0^{1/2}(\lambda) V_\pm R_0^{1/2}(\lambda), \quad \lambda < 0. \tag{3.4}$$

Let us write the Birman–Schwinger principle as (compare with (2.10))

$$N(\lambda, \alpha) = n_+(\alpha^{-1}; Z_+(\lambda) - Z_-(\lambda)), \quad \lambda < 0,\ \alpha > 0. \tag{3.5}$$

For $V \geq 0$ the relation (3.5) takes the form

$$N(\lambda, \alpha) = n_+(\alpha^{-1}; Z_+(\lambda)), \quad \lambda < 0,\ \alpha > 0.$$

It follows from the last relation that the estimates (3.1)–(3.3) can be written as

$$\|Z_+(\lambda)\|_{\mathbf{\Sigma}_{d/2}}^{d/2} \leq C_{3.1}(d) g_-^{-d/2} \int V_+^{d/2}(x) dx, \quad d \geq 3; \tag{3.6}$$

$$\|Z_+(\lambda)\|_{\mathbf{\Sigma}_1} \leq C_{3.2}(\lambda, \sigma) g_-^{-1} \|V_+\|_{l_1(L_\sigma)}, \quad d = 2; \tag{3.7}$$

$$\|Z_+(\lambda)\|_{\mathbf{\Sigma}_{1/2}}^{1/2} \leq C_{3.3}(\lambda) g_-^{-1/2} \|V_+\|_{l_{1/2}(L_1)}^{1/2}, \quad d = 1. \tag{3.8}$$

Obviously, in these estimates one can replace $Z_+ \mapsto Z_-$, $V_+ \mapsto V_-$.

3.2 Trace class estimates.

Proposition 3.3 *For some constants $M > 0$ and $\beta > 0$, which depend only on g, the following estimate holds:*

$$|e^{-tH_0}\psi| \leq M e^{t\beta\triangle} |\psi|, \quad \forall t > 0,\ \psi \in L_2(\mathbb{R}^d). \tag{3.9}$$

For a flat metric, Proposition 3.3 is a well known diamagnetic inequality (see [32] and references therein); in this case $M = \beta = 1$. With an additional assumption $U \in L_\infty(\mathbb{R}^d)$, Proposition 3.3 for the case of a general metric is contained in [13]. Similarly to [32], it is easy to get rid of this assumption. This is explained in detail in [25, §7]; an alternative scheme of proof of Proposition 3.3, which does not use the results of [13], is also presented there.

Proposition 3.4 *Let F be the operator of multiplication by the function $F \in l_{2q}(\mathbb{Z}^d; L_2(\mathbf{Q}^d))$, $1/2 \leq q \leq 1$. Then*

$$\|F R_0^m(-1)\|_{\mathbf{S}_{2q}} \leq C_{3.10}(q, m, d, g) \|F\|_{l_{2q}(L_2)}, \quad m > d/(4q). \tag{3.10}$$

Proof The desired result has been proved for $H_0 = H_0(\mathbf{1}, \mathbf{A}, 0)$ in [3, Theorems 2.11, 2.12] and for $H_0 = H_0(\mathbf{1}, 0, U)$ in [34, Theorem B.9.2] (in [34] the background potential U was allowed to have some nonzero negative part). Our proof follows the same pattern.

1. Let $q = 1$. Due to the representation

$$(H_0 + E)^{-m} = (\Gamma(m))^{-1} \int_0^\infty e^{-H_0 t} e^{-Et} t^{m-1} dt,$$

(3.9) implies

$$|R_0^m(-1)\psi| \le M(\beta(-\triangle) + I)^{-m}|\psi|, \quad \forall m > 0,\ \psi \in L_2(\mathbb{R}^d). \tag{3.11}$$

From here one obtains

$$\|FR_0^m(-1)\|_{\mathbf{S}_2} \le M\|F(\beta(-\triangle) + I)^{-m}\|_{\mathbf{S}_2} \le C(m, d, g)\|F\|_{L_2}, \quad m > d/4.$$

2. Let $q = 1/2$. Similarly to [3, Theorem 2.12], take $F \in L_2(\mathbb{R}^d)$ with a support in a cube $\mathbf{Q}^d + j$, $\mathbf{Q}^d = (0, 1)^d$, $j \in \mathbb{Z}^d$, and write

$$FR_0^m(-1) = [FR_0^{m/2}(-1)(1 + |x|^2)^{m/2}][(1 + |x|^2)^{-m/2} R_0^{m/2}(-1)], \quad m > d/2. \tag{3.12}$$

By (3.11), each of the two factors in the right hand side of (3.12) is pointwise dominated by the same object with $M(\beta(-\triangle) + I)^{-m}$ instead of $R_0^m(-1)$, in which case they are Hilbert–Schmidt operators (see, e.g, [34] or [35] for the first factor). Summing over $j \in \mathbb{Z}^d$, we get (3.10) with $q = 1/2$.

3. Interpolating between the cases $q = 1/2$ and $q = 1$, we obtain the desired result. ∎

3.3 Proof of Proposition 1.4. 1. Let $d \le 3$. Then, by (1.14) and Proposition 3.4 (with $q = 1$), for any $\lambda_0 \in \rho(H_0)$ one has $|V|^{1/2} R_0(\lambda_0) \in \mathbf{S}_2$. Thus, in the identity (see, e.g., [38, §1.9])

$$\begin{aligned}(H(\alpha) - \lambda_0 I)^{-1} - R_0(\lambda_0) = -(\operatorname{sign} V|V|^{1/2}R_0(\lambda_0))^* \\ \times(I + |V|^{1/2}R_0(\lambda_0)|V|^{1/2}\operatorname{sign} V)^{-1}(|V|^{1/2}R_0(\lambda_0)),\end{aligned} \tag{3.13}$$

which is valid for all $\lambda_0 \in \rho(H_0) \cap \rho(H(\alpha))$, the right hand side is a trace class operator. This proves (1.15) for $k = 1$.

2. Let $d \ge 4$. By (1.13) and Proposition 3.4 (with $q = 1/2$), one has $VR_0^m(\lambda_0) \in \mathbf{S}_1$ for all $\lambda_0 \in \rho(H_0)$ and $m > d/2$. From here, by [26, Theorem XI.12], (1.15) follows for any integer $k > m - \frac{1}{2}$ and any $\lambda_0 < 0$ with large enough absolute value. ∎

4 Pointwise asymptotics

The aim of this section is to prove Theorems 1.5 and 1.8. First we state and prove two abstract results (Theorems 4.1 and 4.2) on the asymptotics of the functions $\mathcal{N}_+(\lambda; H_0, G; \alpha)$ and $\mathcal{N}_-(\lambda; H_0, G; \alpha)$ for $\alpha \to \infty$. Next, applying Proposition 2.4, we obtain the desired results for the SSF of the Schrödinger operator.

4.1 Statement of abstract results

Let $\mathcal{H}$ be a basic and $\mathcal{K}$ — an auxiliary Hilbert space, $H_0 \geq 0$ be a selfadjoint operator in $\mathcal{H}$ and $G : \mathcal{H} \to \mathcal{K}$ be a closed operator such that

$$GR_0^{1/2}(-1) \in \mathbf{\Sigma}_{2\varkappa}(\mathcal{H}, \mathcal{K}), \quad \varkappa > 0, \tag{4.1}$$

$$GE_{H_0}([0, R)) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K}), \quad q \leq 1, \quad \forall R > 0. \tag{4.2}$$

By Proposition 2.2, for a.e. $\lambda \in \mathbb{R}$ the pair H_0, G satisfies Condition 2.1 and thus the functions $\mathcal{N}_\pm(\lambda; H_0, G; \alpha)$ are well-defined.

Theorem 4.1 *Let the conditions* (2.1), (4.1), (4.2) *be satisfied with* $\varkappa \geq q$ *if* $q < 1$ *and with* $\varkappa > 1$ *if* $q = 1$. *Then for a.e.* $\lambda \in \mathbb{R}$ *the limits*

$$\limsup_{\alpha\to\infty} \alpha^{-\varkappa} \mathcal{N}_+(\lambda; H_0, G; \alpha), \quad \liminf_{\alpha\to\infty} \alpha^{-\varkappa} \mathcal{N}_+(\lambda; H_0, G; \alpha) \tag{4.3}$$

are finite and do not depend on λ.

Note that in [4] in the framework of a similar abstract scheme, but under broader conditions on G, H_0 (without the trace class type hypotheses) it was proved that the functions (4.3) do not depend on $\lambda \in \mathbb{R} \setminus \sigma(H_0)$ (and also on some weak perturbations of H_0). Our construction uses some ideas of [4].

Passing to the discussion of $\mathcal{N}_-(\lambda; H_0, G; \alpha)$, we relax the condition (4.1); it is sufficient to assume (2.2). For the exponent $0 < q \leq 1$ from (4.2), we introduce q_* by

$$\begin{array}{ll} q_* = q & \text{if } q < 1; \\ q_* > 1 \text{ — any number} & \text{if } q = 1. \end{array}$$

Theorem 4.2 *Assume the conditions* (2.1), (2.2), (4.2). *Then for a.e.* $\lambda \in \mathbb{R}$:

$$\lim_{\alpha\to\infty} \alpha^{-q_*} \mathcal{N}_-(\lambda; H_0, G; \alpha) = 0. \tag{4.4}$$

4.2 Preliminary results

Below we use the notations (2.3), (2.4).

Proposition 4.3 [6, 22] *Let* H_0 *be a selfadjoint operator in* $\mathcal{H}$ *and* $G \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K})$, $q \leq 1$. *Then for a.e.* $\lambda \in \mathbb{R}$ *the operator* $T(\lambda + i\varepsilon; H_0, G)$ *has limit values for* $\varepsilon \to +0$ *in* $\mathbf{S}_{q_*}(\mathcal{K})$ *and* $K(\lambda + i0; H_0, G) \in \mathbf{S}_q(\mathcal{K})$.

Proposition 4.3 for $q = 1$, $q_* = 2$ has been proved in [6]; the general case was considered in [22]. Further results in this direction can be found in [23].

Lemma 4.4 *Let* A, K *be as specified in* (2.17). *Then for any* $p \geq q$ *the functions* (2.18) *obey*

$$\limsup_{s\to+0} s^p \eta_\pm(s; A, K) = \Delta_p^{(\pm)}(A), \tag{4.5}$$

$$\liminf_{s\to+0} s^p \eta_\pm(s; A, K) = \delta_p^{(\pm)}(A). \tag{4.6}$$

Proof Let us fix $0 < \tau < 1$ and use (2.19):

$$s^p \eta_\pm(s; A, K) \leq s^p n_\pm(s(1-\tau), A) + s^{p-q}\tau^{-q}\mu_q(\tau s, K)\|K\|^q_{\mathbf{S}_q}.$$

Passing to the upper limits as $s \to +0$ and taking into account (2.21), one finds

$$\limsup_{s\to+0} s^p \eta_\pm(s; A, K) \leq (1-\tau)^{-p}\Delta_p^{(\pm)}(A).$$

Since τ is arbitrary, the last estimate implies

$$\limsup_{s\to+0} s^p \eta_\pm(s; A, K) \leq \Delta_p^{(\pm)}(A).$$

Similarly, using (2.20) instead of (2.19), one obtains

$$\limsup_{s\to+0} s^p \eta_\pm(s; A, K) \geq \Delta_p^{(\pm)}(A),$$

which gives (4.5). In a similar way one proves (4.6). ∎

Lemma 4.5 *Conditions* (4.1) *and* (4.2) *with* $q \leq \varkappa$ *imply the inclusion*

$$GR_0(-1) \in \mathbf{\Sigma}^0_{2\varkappa}(\mathcal{H}, \mathcal{K}). \tag{4.7}$$

Proof By (4.2), for any $R > 0$

$$GR_0(-1)E_{H_0}([0,R)) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K}) \subset \mathbf{\Sigma}^0_{2\varkappa}(\mathcal{H}, \mathcal{K}).$$

Next,

$$\begin{aligned}\|GR_0(-1) - GR_0(-1)E_{H_0}([0,R))\|_{\mathbf{\Sigma}_{2\varkappa}} &= \|GR_0(-1)E_{H_0}([R,\infty))\|_{\mathbf{\Sigma}_{2\varkappa}} \\ &\leq \|GR_0^{1/2}(-1)\|_{\mathbf{\Sigma}_{2\varkappa}}(R+1)^{-1/2} \to 0, \quad R \to \infty.\end{aligned}$$

Since $\mathbf{\Sigma}^0_{2\varkappa}$ is closed in $\mathbf{\Sigma}_{2\varkappa}$, this implies (4.7). ∎

Lemma 4.6 *Assume the hypothesis of Theorem 4.1. For a.e.* $\lambda \in \mathbb{R}$*:*
(i) *the operator* $T(\lambda + i\varepsilon; H_0, G)$ *has limit values as* $\varepsilon \to +0$ *in* $\mathbf{\Sigma}_\varkappa(\mathcal{K})$*;*
(ii) $K(\lambda + i0; H_0, G) \in \mathbf{S}_q(\mathcal{K})$.
For a.e. λ, $\mu \in \mathbb{R}$*:*
(iii) $A(\lambda + i0; H_0, G) - A(\mu + i0; H_0, G) \in \mathbf{\Sigma}^0_\varkappa(\mathcal{K})$.

Proof Fix an open bounded interval $\delta \subset \mathbb{R}$. Below we check (i), (ii) for a.e. $\lambda \in \delta$ and (iii) for a.e. $\lambda, \mu \in \delta$.

1. One has:

$$T(z; H_0, G) = T(z; H_0, GE_{H_0}(\delta)) + T(z; H_0, GE_{H_0}(\mathbb{R} \setminus \delta)). \tag{4.8}$$

By (4.2), $GE_{H_0}(\delta) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K})$. Thus, by Proposition 4.3, the operator $T(\lambda + i\varepsilon; H_0, GE_{H_0}(\delta))$ for a.e. $\lambda \in \mathbb{R}$ has limit values as $\varepsilon \to +0$ in $\mathbf{S}_{q_*}(\mathcal{K}) \subset \mathbf{\Sigma}_\varkappa(\mathcal{K})$ and $K(\lambda + i0; H_0, GE_{H_0}(\delta)) \in \mathbf{S}_q(\mathcal{K})$. On the other hand, the limit $T(\lambda + i0; H_0, GE_{H_0}(\mathbb{R} \setminus \delta))$ exists in $\mathbf{\Sigma}_\varkappa(\mathcal{K})$ for all $\lambda \in \delta$ and $K(\lambda + i0; H_0, GE_{H_0}(\mathbb{R} \setminus \delta)) = 0$. From here follow (i), (ii).

2. Let us check (iii). For λ, $\mu \in \delta$ and $\varepsilon > 0$ we have

$$\begin{aligned} &A(\lambda + i\varepsilon; H_0, G) - A(\mu + i\varepsilon; H_0, G) \\ &= A(\lambda + i\varepsilon; H_0, GE_{H_0}(\delta)) - A(\mu + i\varepsilon; H_0, GE_{H_0}(\delta)) \\ &+(\lambda - \mu)\mathrm{Re}\,(GR_0(\lambda + i\varepsilon)E_{H_0}(\mathbb{R} \setminus \delta)(GR_0(\mu - i\varepsilon))^*). \end{aligned} \tag{4.9}$$

The first two summands in the right hand side of (4.9) for a.e. λ, $\mu \in \delta$ have limit values in $\mathbf{S}_{q_*}(\mathcal{K}) \subset \mathbf{\Sigma}^0_\varkappa(\mathcal{K})$ as $\varepsilon \to +0$ (see Proposition 4.3). The third summand for all λ, $\mu \in \delta$ has a limit as $\varepsilon \to +0$, which can be written as $(GR_0(-1))M(GR_0(-1))^*$ with some $M \in \mathbf{B}(\mathcal{H})$. From here, taking into account (4.7), we see that the limit of the third summand belongs to $\mathbf{\Sigma}^0_\varkappa$, and we get the required statement (iii). ∎

Lemma 4.7 *Assume the hypothesis of Theorem 4.2. Then, for a.e.* $\lambda \in \mathbb{R}$*:*
(i) *The operator* $T(\lambda + i\varepsilon; H_0, G)$ *has a limit*[2] *as* $\varepsilon \to +0$ *in* $\mathbf{S}_\infty(\mathcal{K})$*;*
(ii) $K(\lambda + i0; H_0, G) \in \mathbf{S}_q(\mathcal{K})$*;*
(iii) $(A(\lambda + i0; H_0, G))_- \in \mathbf{S}_{q_*}(\mathcal{K})$.

Proof Fix arbitrary $R > 0$ and an interval $\delta = [0, R)$. It is sufficient to check (i)–(iii) for a.e. $\lambda \in \delta$. As in the proof of Lemma 4.6, we write the representation (4.8) and see that for a.e. $\lambda \in \mathbb{R}$ the first summand in the right hand side has a limit in $\mathbf{S}_{q_*}(\mathcal{K})$ and for all $\lambda \in \delta$ the second summand has a limit in $\mathbf{S}_\infty(\mathcal{K})$, which is a nonnegative selfadjoint operator. This proves (i), (iii).

[2] In fact, this statement follows already from Proposition 2.2

The assertion (ii) follows from the inclusion $K(\lambda + i0; H_0, GE_{H_0}(\delta)) \in \mathbf{S}_q(\mathcal{K})$, which holds for a.e. $\lambda \in \mathbb{R}$. ∎

4.3 Proof of Theorems 4.1, 4.2.

Proof of Theorem 4.1 By Lemmas 4.4, 4.6(ii), for a.e. $\lambda \in \mathbb{R}$

$$\limsup_{\alpha\to\infty} \alpha^{-\varkappa}\mathcal{N}_+(\lambda; H_0, G; \alpha) = \Delta_\varkappa^{(+)}(A(\lambda + i0)),$$
$$\liminf_{\alpha\to\infty} \alpha^{-\varkappa}\mathcal{N}_+(\lambda; H_0, G; \alpha) = \delta_\varkappa^{(+)}(A(\lambda + i0)).$$

By Lemma 4.6(i), the right hand sides are finite. By Lemma 4.6(iii) and (1.4), they do not depend on λ. ∎

Proof of Theorem 4.2 By Lemmas 4.4, 4.7(ii), for a.e. $\lambda \in \mathbb{R}$:

$$\limsup_{\alpha\to\infty} \alpha^{-q_*}\mathcal{N}_-(\lambda; H_0, G; \alpha) = \Delta_{q_*}^{(-)}(A(\lambda + i0)).$$

By Lemma 4.7(iii), $(A(\lambda+i0))_- \in \mathbf{S}_{q_*}(\mathcal{K}) \subset \mathbf{\Sigma}^0_{q_*}(\mathcal{K})$ and thus $\Delta_{q_*}^{(-)}(A(\lambda+i0)) = 0$. ∎

4.4 Proof of Theorems 1.5, 1.8.

Proof of Theorem 1.5 1. Let us check the hypothesis of Theorem 4.1 for $\mathcal{H} = \mathcal{K} = L_2(\mathbb{R}^d)$, $H_0 = H_0(g, \mathbf{A}, U)$, $G = \sqrt{V}$, $\varkappa = d/2$. We choose q in (4.2) as follows: $q = 1/2$ if $d = 1$; $q = \max\{1/2, \tau\}$ if $d = 2$ (remind that τ is an exponent in (1.17)); $q = 1$ if $d \geq 3$. Next, by Proposition 3.4,

$$GR_0^m(-1) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K}) \tag{4.10}$$

for large enough m, where q is as specified above. Obviously, (4.10) implies (4.2). Finally, (4.1) holds by (3.6)–(3.8). Thus, the hypothesis of Theorem 4.1 holds.

2. For $\lambda < 0$, by Proposition 1.3 and (2.12), the limits (4.3) coincide and are equal to $C_{0.8}(V, g)$. From here and Theorem 4.1 it follows that there exists a set $M \subset \mathbb{R}$, meas $(\mathbb{R} \setminus M) = 0$, such that

$$\lim_{\alpha\to\infty} \alpha^{-d/2}\mathcal{N}_+(\lambda; H_0, G; \alpha) = C_{0.8}(V, g), \quad \forall\lambda \in M. \tag{4.11}$$

3. By (4.1), (4.10) and Proposition 1.4, for any $\alpha > 0$ the hypothesis of the Proposition 2.4 holds for the operators H_0, G. Thus, the equality (2.16) holds. More precisely, the function $-\mathcal{N}_+(\cdot\,; H_0, G; \alpha)$ belongs to the equivalence class defined by the function $\xi(\cdot\,; H(\alpha), H_0)$ (see Remark 1.6). From here and (4.11) follows the desired statement. ∎

Proof of Theorem 1.8 1. Let us check the hypothesis of Theorem 4.2 for $\mathcal{H} = \mathcal{K} = L_2(\mathbb{R}^d)$, $H_0 = H_0(g, \mathbf{A}, U)$, $G = \sqrt{|V|}$ and $q = \tau$. Indeed, (2.1) and (2.2) hold by hypothesis. By Proposition 3.4,

$$GR_0^m(-1) \in \mathbf{S}_{2\tau}(\mathcal{H}, \mathcal{K}) \tag{4.12}$$

for large enough $m > 0$; (4.12) implies (4.2).

2. Theorem 4.2 asserts that there exists a set $M \subset \mathbb{R}$ of a full measure such that

$$\lim_{\alpha\to\infty} \alpha^{-\rho}\mathcal{N}_-(\lambda; H_0, G; \alpha) = 0, \quad \forall\lambda \in M,$$

where the exponent ρ is as specified in the statement of Theorem 1.8.

3. Finally, as in the part 3 of the proof of Theorem 1.5, taking into account Proposition 2.4, we arrive at (1.21). ∎

Remark 4.8 Theorems 4.1 and 4.2 allow one to consider other types of operators H_0 as long as it is possible to establish some analogues of Propositions 3.1 and 3.4. In particular, one can take $H_0 = (-\triangle)^l$, $l > 0$. In this case the estimates of the type (3.1)–(3.3) and (3.10) are well known — see, e.g., [9, 11, 35]. Here one can also consider perturbations V by differential operators of an order lower than l with decaying coefficients.

Another possibility is to consider the pseudo-relativistic magnetic Schrödinger operator

$$\tilde{H}_0 = (H_0(\mathbf{1}, \mathbf{A}, 0) + I)^{1/2} + U(x). \tag{4.13}$$

In this case the bound similar to (3.1)–(3.3) can be obtained by using the Heinz inequality. In order to obtain the bounds similar to (3.10), one can exploit the pointwise inequality

$$|e^{-t\tilde{H}_0}\psi| \le e^{-t(-\triangle+I)^{1/2}}|\psi|, \quad t > 0,\ \psi \in L_2(\mathbb{R}^d),$$

in the spirit of §3.2. In connection with this, see also [28, 7].

5 Integral asymptotics

The aim of this section is to prove Theorem 1.7. First we prove an abstract result (Theorem 5.2) and then apply it to the Schrödinger operator.

5.1 Statement of an abstract result. Let $H_0 \ge 0$ be a selfadjoint operator in the basic Hilbert space $\mathcal{H}$. Next, let $\mathcal{K}_+$ and $\mathcal{K}_-$ be auxiliary Hilbert spaces, $G_+ : \mathcal{H} \to \mathcal{K}_+$ and $G_- : \mathcal{H} \to \mathcal{K}_-$ be closed operators and

$$G_-R_0^{1/2}(-1) \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}_-), \tag{5.1}$$

$$G_+R_0^{1/2}(-1) \in \mathbf{\Sigma}_{2\varkappa}(\mathcal{H}, \mathcal{K}_+), \quad \varkappa > 0. \tag{5.2}$$

Let $V_+ = G_+^*G_+$, $V_- = G_-^*G_-$ and define the operators (1.9) via the form sums; denote

$$R_+(\lambda; \alpha) = (H_+(\alpha) - \lambda I)^{-1}; \quad R(\lambda; \alpha) = (H(\alpha) - \lambda I)^{-1}.$$

Before stating the result on the SSF, we discuss the asymptotics of the spectrum below the point $\lambda = 0$. We introduce the notation (1.10) and for $\lambda < 0$ denote

$$\Omega_\varkappa(\lambda) := \limsup_{\alpha\to\infty} \alpha^{-\varkappa} N(\lambda;\alpha), \quad \omega_\varkappa(\lambda) := \liminf_{\alpha\to\infty} \alpha^{-\varkappa} N(\lambda;\alpha). \tag{5.3}$$

Proposition 5.1 *Assume the conditions* (2.1)*,* (5.1)*,* (5.2) *and suppose that for any* $R > 0$

$$G_+ E_{H_0}([0,R)) \in \mathbf{\Sigma}^0_{2\varkappa}(\mathcal{H},\mathcal{K}_+). \tag{5.4}$$

Then the limits $\Omega_\varkappa$*,* $\omega_\varkappa$ *do not depend on* $\lambda < 0$*.*

The proof is given in the next subsection. Note that the condition (5.1) in the hypothesis of Proposition 5.1 can be relaxed. In [30], in the framework of a similar abstract scheme but under slightly different assumptions on H_0, G_+, G_-, the stability of the leading term of the asymptotics of $N(\lambda; H_0, G_+, G_-;\alpha)$ (see (0.14)) has been established for $\lambda \in \mathbb{R}\setminus\sigma(H_0)$. In contrast to Proposition 5.1, the proof of the main result of [30] requires the use of a fairly complicated technique.

The main result of this section (Theorem 5.2) deals with the SSF for the pair H_0, $H(\alpha)$. Its proof uses also the SSF for the pairs H_0, $H_+(\alpha)$ and $H_+(\alpha)$, $H(\alpha)$. In order to define these SSF, suppose that for all $\alpha > 0$ and some $k > 0$, $\lambda_0 < \inf(\sigma(H_0)\cup\sigma(H(\alpha)))$, the following inclusions hold:

$$R_+^k(\lambda_0,\alpha) - R_0^k(\lambda_0) \in \mathbf{S}_1(\mathcal{H}), \tag{5.5}$$

$$R^k(\lambda_0,\alpha) - R_0^k(\lambda_0) \in \mathbf{S}_1(\mathcal{H}). \tag{5.6}$$

The numbers k, λ_0 may depend on α, but must be the same in (5.5) and (5.6). From (5.5) and (5.6) it follows that

$$R_+^k(\lambda_0,\alpha) - R^k(\lambda_0,\alpha) \in \mathbf{S}_1(\mathcal{H}). \tag{5.7}$$

Theorem 5.2 *Assume the conditions* (2.1)*,* (5.1)*,* (5.2) *for* $\varkappa > 1$*,* (5.5)*,* (5.6)*. Next, suppose that for some* $m > 1/2$*:*

$$G_- R_0^m(-1) \in \mathbf{S}_2(\mathcal{H},\mathcal{K}_-), \tag{5.8}$$

$$\sup_{\alpha>0} \|G_+ R_+^m(-1;\alpha)\|_{\mathbf{S}_2} =: C_{5.9} < \infty. \tag{5.9}$$

Then for any $E > 0$*:*

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} (\alpha^{-\varkappa}\xi(\lambda; H(\alpha), H_0) + \Omega_\varkappa)_-(\lambda+\tilde{E})^{-2m} d\lambda = 0, \quad \tilde{E} > E, \tag{5.10}$$

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} (\alpha^{-\varkappa}\xi(\lambda; H(\alpha), H_0) + \omega_\varkappa)_+(\lambda+\tilde{E})^{-2m} d\lambda = 0, \quad \tilde{E} > E, \tag{5.11}$$

where $\Omega_\varkappa$, $\omega_\varkappa$ *are defined by* (5.3) *(and do not depend on* λ *by*[3] *Proposition 5.1). In particular, if* $\Omega_\varkappa = \omega_\varkappa$, *then*

$$\lim_{\alpha\to\infty}\int_{-E}^{\infty}|\alpha^{-\varkappa}\xi(\lambda;H(\alpha),H_0)+\Omega_\varkappa|(\lambda+\tilde E)^{-2m}d\lambda=0,\quad \tilde E>E. \tag{5.12}$$

5.2 Proof of Proposition 5.1. 1. Clearly, $N(\lambda_1,\alpha)\le N(\lambda_2,\alpha)$ if $\lambda_1\le\lambda_2<0$. It follows that

$$\Omega_\varkappa(\lambda_1)\le\Omega_\varkappa(\lambda_2),\quad \omega_\varkappa(\lambda_1)\le\omega_\varkappa(\lambda_2),\quad \lambda_1\le\lambda_2<0.$$

Thus, it suffices to check the opposite inequalities

$$\Omega_\varkappa(\lambda_1)\ge\Omega_\varkappa(\lambda_2),\quad \omega_\varkappa(\lambda_1)\ge\omega_\varkappa(\lambda_2),\quad \lambda_1\le\lambda_2<0.$$

2. Denote

$$X_\pm(\lambda):=G_\pm R_0^{1/2}(\lambda),\quad Z_\pm(\lambda):=X_\pm^*(\lambda)X_\pm(\lambda),\quad \lambda<0. \tag{5.13}$$

Clearly, $Z_\pm(\lambda_1)\le Z_\pm(\lambda_2)$ for $\lambda_1\le\lambda_2<0$. Besides, below we will show that

$$Z_+(\lambda_1)-Z_+(\lambda_2)=:M\in\mathbf{\Sigma}_\varkappa^0(\mathcal H). \tag{5.14}$$

From here by (3.5) it follows that

$$\begin{aligned}N(\lambda_1,\alpha)=n_+(\alpha^{-1},Z_+(\lambda_1)-Z_-(\lambda_1))&\ge n_+(\alpha^{-1},Z_+(\lambda_1)-Z_-(\lambda_2))\\ &=n_+(\alpha^{-1},Z_+(\lambda_2)-Z_-(\lambda_2)+M),\end{aligned}$$

and thus, using (1.4),

$$\begin{aligned}\Omega_\varkappa(\lambda_1)=\Delta_\varkappa^{(+)}(Z_+(\lambda_1)-Z_-(\lambda_1))&\ge\Delta_\varkappa^{(+)}(Z_+(\lambda_2)-Z_-(\lambda_2)+M)\\ &=\Delta_\varkappa^{(+)}(Z_+(\lambda_2)-Z_-(\lambda_2))=\Omega_\varkappa(\lambda_2).\end{aligned}$$

Similarly, it follows that $\omega_\varkappa(\lambda_1)\ge\omega_\varkappa(\lambda_2)$.

3. Now it remains to check (5.14). Clearly, (5.14) will be established if we prove

$$X_+(\lambda_1)-X_+(\lambda_2)\in\mathbf{\Sigma}_{2\varkappa}^0(\mathcal H,\mathcal K_+). \tag{5.15}$$

One can write the operator from (5.15) as

$$X_+(\lambda_1)-X_+(\lambda_2)=G_+R_0(-1)B,\quad B\in\mathbf B(\mathcal H). \tag{5.16}$$

As in Lemma 4.5, we find that (5.4) and (5.2) imply $G_+R_0(-1)\in\mathbf{\Sigma}_{2\varkappa}^0(\mathcal H,\mathcal K_+)$. From here and (5.16) follows (5.15). ∎

5.3 Proof of Theorem 5.2.

[3]Condition (5.4) of Proposition 5.1 follows from (5.9): (5.9) $\Rightarrow$ $G_+R_0^m(-1)\in\mathbf S_2\subset\mathbf{\Sigma}_{2\varkappa}^0$ $\Rightarrow$ (5.4).

Lemma 5.3 *Assume the hypothesis of Proposition 5.1. Denote*

$$L(\lambda, \alpha, a) := \alpha^{-\varkappa} N_+(\lambda; H_+(\alpha), G_+; a\alpha), \quad \lambda < 0,\ \alpha > 0,\ a > 0. \tag{5.17}$$

Then the limits

$$B(a) := \limsup_{\alpha\to\infty} L(\lambda, \alpha, a), \quad b(a) := \liminf_{\alpha\to\infty} L(\lambda, \alpha, a) \tag{5.18}$$

do not depend on $\lambda < 0$ and depend continuously on $a > 0$. In particular,

$$\lim_{a\to 1} B(a) = \Omega_\varkappa, \quad \lim_{a\to 1} b(a) = \omega_\varkappa. \tag{5.19}$$

Proof For $\alpha > 0$ and $\beta > 0$ define the operators $H(\alpha, \beta) = H_0 + \alpha V_- - \beta V_+$ via the corresponding form sums. Clearly,

$$N_+(\lambda; H_+(\alpha), G_+; \beta) = \operatorname{rank} E_{H(\alpha,\beta)}((-\infty, \lambda)), \quad \lambda < 0.$$

Introducing the notations (5.13) and using (3.5), we find

$$N_+(\lambda; H_+(\alpha), G_+; a\alpha) = n_+(\alpha^{-1}; aZ_+(\lambda) - Z_-(\lambda)),$$

and thus

$$B(a) = \Delta_\varkappa^{(+)}(aZ_+(\lambda) - Z_-(\lambda)), \quad b(a) = \delta_\varkappa^{(+)}(aZ_+(\lambda) - Z_-(\lambda)).$$

Now the continuity of $B(a)$, $b(a)$ follows from the continuity of $\Delta_\varkappa^{(+)}$, $\delta_\varkappa^{(+)}$ in $\mathbf{\Sigma}_\varkappa$. The fact that $B(a)$, $b(a)$ do not depend on λ, follows from Proposition 5.1 (after the substitution $G_+ \mapsto \sqrt{a} G_+$). ∎

Proof of Theorem 5.2 1. First note that, by (0.22) and (5.5)–(5.7),

$$\xi(\lambda; H(\alpha), H_0) = \xi(\lambda; H(\alpha), H_+(\alpha)) + \xi(\lambda; H_+(\alpha), H_0). \tag{5.20}$$

Next, by Proposition 2.4,

$$\xi(\lambda; H(\alpha), H_+(\alpha)) = -\mathcal{N}_+(\lambda; H_+(\alpha), G_+; \alpha), \tag{5.21}$$

$$\xi(\lambda; H_+(\alpha), H_0) = \mathcal{N}_-(\lambda; H_0, G_-; \alpha). \tag{5.22}$$

Denote

$$F(\lambda, \alpha) := -\alpha^{-\varkappa} \xi(\lambda; H(\alpha), H_0),$$
$$Q(\lambda, \alpha) := \alpha^{-\varkappa} \mathcal{N}_+(\lambda; H_+(\alpha), G_+; \alpha),$$
$$J(\lambda, \alpha) := \alpha^{-\varkappa} \mathcal{N}_-(\lambda; H_0, G_-; \alpha).$$

Write (5.20) as

$$F(\lambda, \alpha) = Q(\lambda, \alpha) - J(\lambda, \alpha). \tag{5.23}$$

Finally, denote for brevity $d\mu(\lambda) := (\lambda+2E)^{-2m}d\lambda$ for $\lambda \geq -E$ and $d\mu(\lambda) = 0$ for $\lambda < -E$. In these notations, we need to prove that

$$\lim_{\alpha\to\infty} \int (F(\lambda,\alpha) - \Omega_\varkappa)_+ d\mu(\lambda) = 0, \tag{5.24}$$

$$\lim_{\alpha\to\infty} \int (F(\lambda,\alpha) - \omega_\varkappa)_- d\mu(\lambda) = 0. \tag{5.25}$$

2. Let prove (5.24). In (2.22), let us take $H_+(\alpha)$ for H_0 and G_+ for G and multiply the whole inequality by $\alpha^{-\varkappa}$. Using the notation (5.17), we obtain:

$$\begin{aligned} \int (Q(\lambda,\alpha) - L(-E,\alpha,(1-\theta)^{-2}))_+ d\mu(\lambda) \leq \alpha^{1-\varkappa}\theta^{-1-m} \\ \times\|G_+R_+^m(-E,\alpha)\|^2_{\mathbf{S}_2} \leq \alpha^{1-\varkappa}\theta^{-1-m}\max\{1,E^{-2m}\}\|G_+R_+^m(-1,\alpha)\|^2_{\mathbf{S}_2} \\ \leq \alpha^{1-\varkappa}\theta^{-1-m}\max\{1,E^{-2m}\}C_{5.9}. \end{aligned}$$

Next, taking into account (5.23) and the inequality $J(\lambda,\alpha) \geq 0$,

$$\begin{aligned} (Q(\lambda,\alpha) - L(-E,\alpha,(1-\theta)^{-2}))_+ \geq (F(\lambda,\alpha) - L(-E,\alpha,(1-\theta)^{-2}))_+ \\ \geq (F(\lambda,\alpha) - \Omega_\varkappa)_+ - (L(-E,\alpha,(1-\theta)^{-2}) - \Omega_\varkappa)_+. \end{aligned}$$

From here we find

$$\begin{aligned} \int (F(\lambda,\alpha) - \Omega_\varkappa)_+ d\mu(\lambda) \leq \alpha^{1-\varkappa}\theta^{-1-m}\max\{1,E^{-2m}\}C_{5.9} \\ +(L(-E,\alpha,(1-\theta)^{-2}) - \Omega_\varkappa)_+ \int d\mu(\lambda). \end{aligned}$$

Passing to the limit as $\alpha \to \infty$, we find (using the notation (5.18)):

$$\limsup_{\alpha\to\infty} \int (F(\lambda,\alpha) - \Omega_\varkappa)_+ d\mu(\lambda) \leq (B((1-\theta)^{-2}) - \Omega_\varkappa)_+ \int d\mu(\lambda).$$

Note that at this point the condition $\varkappa > 1$ is crucial. By (5.19), the right hand side of the last inequality goes to zero as $\theta \to +0$, and we arrive at (5.24).

3. Let us prove (5.25). In (2.23), let us take $H_+(\alpha)$ for H_0 and G_+ for G and multiply the whole inequality by $\alpha^{-\varkappa}$. Using the notation (5.17), we find:

$$\begin{aligned} \int (Q(\lambda,\alpha) - L(-2E,\alpha,(1+\theta)^{-1}))_- d\mu(\lambda) \\ \leq \alpha^{1-\varkappa}\theta^{-1}(4+2/(2m-1))\|G_+R_+^m(-2E,\alpha)\|^2_{\mathbf{S}_2} \\ \leq \alpha^{1-\varkappa}\theta^{-1}(4+2/(2m-1))\max\{1,(2E)^{-2m}\}\|G_+R_+^m(-1,\alpha)\|^2_{\mathbf{S}_2} \\ \leq \alpha^{1-\varkappa}\theta^{-1}(4+2/(2m-1))\max\{1,(2E)^{-2m}\}C_{5.9}. \end{aligned}$$

Next, taking into account (5.23),

$$
\begin{aligned}
&(Q(\lambda,\alpha)-L(-2E,\alpha,(1+\theta)^{-1}))_- \\
&=(F(\lambda,\alpha)-\omega_\varkappa+J(\lambda,\alpha)+(\omega_\varkappa-L(-2E,\alpha,(1+\theta)^{-1})))_- \\
&\geq (F(\lambda,\alpha)-\omega_\varkappa)_- - J(\lambda,\alpha)-(\omega_\varkappa-L(-2E,\alpha,(1+\theta)^{-1}))_+.
\end{aligned}
$$

From here we find:

$$
\begin{aligned}
\int(F(\lambda,\alpha)-\omega_\varkappa)_- d\mu(\lambda) &\leq \alpha^{1-\varkappa}\theta^{-1}(4+2/(2m-1))\max\{1,(2E)^{-2m}\}C_{5.9} \\
&+\int J(\lambda,\alpha)d\mu(\lambda)+(\omega_\varkappa-L(-2E,\alpha,(1+\theta)^{-1}))_+\int d\mu(\lambda). \qquad (5.26)
\end{aligned}
$$

By (2.24),

$$
\limsup_{\alpha\to\infty}\int J(\lambda,\alpha)d\mu(\lambda)\leq \limsup_{\alpha\to\infty}\alpha^{1-\varkappa}\|G_-R_0^m(-2E)\|^2_{\mathbf{S}_2}=0.
$$

Passing to the limit in (5.26) as $\alpha\to\infty$ and taking into account the last relation, we find:

$$
\limsup_{\alpha\to\infty}\int(F(\lambda,\alpha)-\omega_\varkappa)_- d\mu(\lambda)\leq(\omega_\varkappa-b((1+\theta)^{-1}))_+\int d\mu(\lambda).
$$

By (5.19), the right hand side of the last inequality goes to zero as $\theta\to+0$ and we obtain (5.25). ∎

Remark 5.4 In the hypothesis of Theorem 5.2, the formula (0.28) holds true. Indeed, using (0.17), (5.20)–(5.22), for $\lambda\in\mathbb{R}\setminus\sigma(H_0)$ we obtain:

$$
\begin{aligned}
&\xi(\lambda;H(\alpha),H_0)=-\mathcal{N}_+(\lambda;H_+(\alpha),G_+;\alpha)+\mathcal{N}_-(\lambda;H_0,G_-;\alpha) \\
&=-N_+(\lambda;H_+(\alpha),G_+;\alpha)+N_-(\lambda;H_0,G_-;\alpha)=-N(\lambda;H_0,G_+,G_-;\alpha).
\end{aligned}
$$

5.4 Proof of Theorem 1.7. 1. Let us check the hypothesis of Theorem 5.2 for $H_0=H_0(g,\mathbf{A},U)$, $G_\pm=\sqrt{V_\pm}$, $\mathcal{H}=\mathcal{K}_+=\mathcal{K}_-=L_2(\mathbb{R}^d)$ and $\varkappa=d/2$. Inclusions (5.1), (5.2) follow from (3.6). Conditions (5.8), (5.9) follow from (3.10) (with $q=1$); while checking (5.9) we include the term $+\alpha V_-$ into the background potential U and use the fact that the constant $C_{3.10}$ does not depend on $U\geq 0$. Conditions (5.5), (5.6) follow from Proposition 1.4.

2. By Proposition 1.3, $\Omega_\varkappa=\omega_\varkappa=C_{0.8}(V,g)$. Thus, (5.12) becomes (1.19) with $p=2m$. ∎

Remark 5.5 Let us comment on the possibility of applying Theorem 5.2 to other differential operators H_0. Here the main difficulty is in the check of the condition (5.9), which in our case required the use of rather specific technique of pointwise estimates for the Gaussian kernels. This technique is applicable to the elliptic differential operators of the order $l\leq 2$. Thus, one can apply Theorem 5.2 to the pseudo-relativistic magnetic Schrödinger operator (4.13).

6 Appendix. Proof of Proposition 1.3

6.1 Preliminary statements.

Proposition 6.1 *Let Ω be an open ball in $\mathbb{R}^d$, $d \geq 1$, and let*

$$\begin{array}{ll} F \in L_{d/2}(\Omega), & d \geq 3; \\ F \log(1+|F|) \in L_1(\Omega), & d = 2; \\ F \in L_1(\Omega), & d = 1. \end{array} \tag{6.1}$$

Then the quadratic form $\int_\Omega F|u|^2 dx$, $u \in H^1(\Omega)$, is compact in $H^1(\Omega)$.

For $d = 1$ Proposition 6.1 follows immediately from the Hölder inequality and the compactness of the embedding $H^1(\Omega) \subset L_\infty(\Omega)$, $d = 1$.

For $d \geq 2$, let us first observe that:

(i) for a bounded F, the form $\int_\Omega F|u|^2$ is compact in $H^1(\Omega)$;

(ii) (6.1) implies boundedness of the form $\int_\Omega F|u|^2$ in $H^1(\Omega)$.

The statement (i) follows by Hölder inequality from the compactness of the embedding $H^1(\Omega) \subset L_2(\Omega)$. For (ii), see, e.g., [1]. Approximating F by a sequence of bounded functions F_n and taking into account (i), (ii), we get the required statement.

Lemma 6.2 *Let Ω be an open ball in $\mathbb{R}^d$, $d \geq 1$; suppose that* (0.5), (0.7), (1.12) *hold. Then the form*

$$h_{g,\mathbf{A},U}[u,u] - h_{g,0,0}[u,u], \quad u \in H^1(\Omega), \tag{6.2}$$

is compact in $H^1(\Omega)$.

Proof Expand formula (6.2):

$$h_{g,\mathbf{A},U}[u,u] - h_{g,0,0}[u,u] = \int_\Omega \langle g\mathbf{A}u, \mathbf{A}u\rangle dx + 2\mathrm{Re} \int_\Omega \langle g\mathbf{A}u, i\nabla u\rangle dx + \int_\Omega U|u|^2 dx.$$

By Proposition 6.1 and (0.5), (1.12), the first and the last summands in the right hand side are compact in $H^1(\Omega)$. The second summand is also compact due to the estimate

$$\left| \int_\Omega \langle g\mathbf{A}u, i\nabla u\rangle dx \right| \leq g_+ \|\mathbf{A}u\|_{L_2(\Omega)} \|u\|_{H^1(\Omega)}$$

and to the compactness of the form $\|\mathbf{A}u\|^2_{L_2(\Omega)}$ in $H^1(\Omega)$. ∎

Let Ω be an open ball in $\mathbb{R}^d$, $d \geq 1$, $V \in C_0^\infty(\Omega)$ and let g satisfy the condition (0.5). Consider the the quotient

$$\int_\Omega V|u|^2 dx / (h_{g,0,0}[u,u] - \lambda \|u\|^2_{L_2(\Omega)}), \quad u \in H^1(\Omega), \tag{6.3}$$

where $\lambda < 0$. Denote by $n_\pm(s,(6.3))$, $s>0$, the distribution function of the consecutive maxima (minima) of (6.3). In other words, $n_\pm(s,(6.3)) = n_\pm(s,X)$, where X is a compact operator generated by the form in the numerator of (6.3) in the Hilbert space $H^1(\Omega)$ with the metric $h_{g,0,0}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}$. In what follows, we use the notations $\Delta_p^{(\pm)}((6.3))$, $\delta_p^{(\pm)}((6.3))$ instead of $\Delta_p^{(\pm)}(X)$, $\delta_p^{(\pm)}(X)$. Along with (6.3), consider the same quotient in $H^1_0(\Omega)$:

$$\int_\Omega V|u|^2dx/(h_{g,0,0}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}), \quad u \in H^1_0(\Omega). \tag{6.4}$$

Proposition 6.3 (see, e.g., [11]) *For any $\lambda < 0$,*

$$\Delta^{(+)}_{d/2}((6.3)) = \delta^{(+)}_{d/2}((6.3)) = \Delta^{(+)}_{d/2}((6.4)) = \delta^{(+)}_{d/2}((6.4)) = C_{0.8}(V,g).$$

Lemma 6.4 *Let Ω be an open ball in $\mathbb{R}^d$, $d \ge 1$, $V \in C_0^\infty(\Omega)$ and $\lambda < 0$; assume* (0.5), (0.7), (1.12). *For the quotients*

$$\int_\Omega V|u|^2dx/(h_{g,\mathbf{A},U}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}), \quad u \in H^1(\Omega), \tag{6.5}$$

$$\int_\Omega V|u|^2dx/(h_{g,\mathbf{A},U}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}), \quad u \in H^1_0(\Omega), \tag{6.6}$$

one has

$$\Delta^{(+)}_{d/2}((6.5)) = \delta^{(+)}_{d/2}((6.5)) = \Delta^{(+)}_{d/2}((6.6)) = \delta^{(+)}_{d/2}((6.6)) = C_{0.8}(V,g).$$

Proof By Lemma 6.2, the form in the denominators of (6.5), (6.6) differs from that in the denominators of (6.3), (6.4) by a compact term. Relatively compact perturbations of the metric in the Hilbert space do not affect the leading term in asymptotics — see, e.g., [11, Lemma 1.16]. Thus, the statement follows from Proposition 6.3. ∎

6.2 Proof of Proposition 1.3 By (3.5), the desired formula (0.10) is equivalent to

$$\Delta^{(+)}_{d/2}(Z_+(\lambda) - Z_-(\lambda)) = \delta^{(+)}_{d/2}(Z_+(\lambda) - Z_-(\lambda)) = C_{0.8}(V,g). \tag{6.7}$$

By (3.6)–(3.8) and the continuity of $\Delta^{(\pm)}_{d/2}$, $\delta^{(\pm)}_{d/2}$ in $\mathbf{\Sigma}_{d/2}$, it suffices to prove (6.7) for $V \in C_0^\infty(\mathbb{R}^d)$. Let $\Omega \subset \mathbb{R}^d$ be an open ball and $\operatorname{supp} V \subset \Omega$. Due to the variational principle (Dirichlet-Neumann bracketing), the upper and lower bounds for the spectrum of $Z_+(\lambda) - Z_-(\lambda)$ are given by the spectra of (6.5), (6.6):

$$n_+(s,(6.6)) \le n_+(s, Z_+(\lambda) - Z_-(\lambda)) \le n_+(s,(6.5)), \quad s > 0.$$

From here and Lemma 6.4 follows (6.7). ∎

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