Natural Boundaries and Spectral Theory

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In preparation

Definition of Natural Boundary

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One also says Ω is a domain of holomorphy for f.

Weierstrass' Example

We'll focus on a classical case where $\Omega = \mathbb{D} \equiv \{z \mid |z| < 1\}$ and f is described by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

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for which, when $\theta = 2\pi p/q$, p, q integral,

$$\lim_{r \uparrow 1} |f(re^{i\theta})| = \infty$$

Weierstrass' Example

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Kronecker found an example relevant to elliptic functions

$$f(z) = \sum_{n=0}^{\infty} z^{n^2}$$

has a natural boundary on $|z| = 1$.

Gap Theorems

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Theorem (Hadamard, 1892). *If*

$$f(z) = \sum_{j=0}^{\infty} a_j z^{n_j}$$

has a finite radius of convergence and

$$\inf_j \frac{n_{j+1}}{n_j} > 1$$

then the circle of convergence is a natural boundary.

Gap Theorems

Fabry (1896) proved the same result needing only

$$\lim_{j\to\infty} \frac{n_j}{j} = \infty$$

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Mordell found an especially simple proof of Hadamard's gap theorem.

A Theorem of Szegő

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Theorem (Szegő, 1922)**.** *If* $f(z) = \sum a_n z^n$ *and the set of values of* $\{a_n\}$ *is a finite set, then either* $|z| = 1$ *is a natural boundary, or else* a_n *is eventually periodic, in which case* f *is a rational function with poles on* $\partial\mathbb{D}$*.*

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When I saw this theorem, my mouth fell open.

Why my mouth fell open: Jacobi Matrices

For the past thirty years, a major focus of my research has been the spectral theory of Jacobi matrices and two-sided Jacobi matrices:

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Especially discrete Schrödinger operators where $a_n \equiv 1$.

One studies the relation of the a's and b's to properties of the spectral measure.

Why my mouth fell open: Kotani

In 1982, Kotani studied ergodic Jacobi matrices (actually, he studied ergodic Schrödinger ODEs; I'm describing my 1983 results on analogs of his work for the discrete case). That is, $Q \xrightarrow{T} Q$ is invariant and ergodic for a probability measure ω on Q. $A\colon Q \to (0,\infty)$, $B\colon Q \to \mathbb{R}$ bounded and measurable, and J_ω has parameters $a_n(\omega) = A(T^n\omega)$, $b_n(\omega) = B(T^n\omega)$. He proved results about a.c. spectrum (i.e., the spectral measures have a piece that is a.c. w.r.t. dx); I'll say more about this later.

Why my mouth fell open: Kotani

In 1989, using these ideas and some others, he proved:

Theorem (Kotani, 1989). *If J_ω is an ergodic Jacobi matrix so that a_n, b_n take only finitely many values, then either J_ω has no a.c. spectrum or it is periodic.*

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Theorem (Remling, 2007 (or 2011?)). *Let J be a* (*one-sided*) *Jacobi matrix where a_n and b_n take only finitely many values. Then either J has no a.c. spectrum or a_n and b_n are eventually periodic.*

Remling's Theory

Remling's paper not only had many really new results but essentially optimal ones for no a.c. spectrum in the class where $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are bounded and do not approach constants.

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$$\text{no a.c. spectrum} \sim \text{natural boundary}$$

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Thiele and Tao have emphasized that the spectral analysis of OPs is a kind of nonlinear Fourier transform so that Szegő's OP theorem (*not* the one above) is a nonlinear Plancherel, and Christ–Kiselev for L^2 would be a kind of nonlinear analog of Carleson's L^2 convergence theorem.

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Just so, our result is sort of Remling theory at infinitesimal coupling, so the proofs are much simpler.

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- Gap Theorems $\sim$ Sparse Potentials
- Szegő $\sim$ Kotani–Remling Finite Value
- Random Power Series $\sim$ Anderson Localization

Dense G_δ Result

Motivated by the Wonderland theorem of Simon from spectral theory, we found

Theorem. *Fix* $K \subset \mathbb{C}$ *compact with more than one point. Let* K^∞ *be* $\{a_n\}_{n=1}^\infty$, $a_n \in K$ *in the product topology which is a compact metric space. Then* $\{a \in K^\infty \mid \sum a_n z^n$ *has a natural boundary on* $\partial\mathbb{D}\}$ *is a dense* G_δ. *A similar result is true for strong natural boundaries.*

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The proof is only a few lines. The existence of a single natural boundary (e.g., Weierstrass) and the weak topology prove density, and the Vitali theorem implies the complement is a countable union of closed sets.

Reflectionless Jacobi Matrices

For a half-line Jacobi matrix J, its m-function is defined on $\mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$ by

$$m(z; J) = \langle \delta_1, (J - z)^{-1} \delta_1 \rangle$$

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By general principles for Lebesgue a.e. x, $m(x + i0) = \lim_{\varepsilon \downarrow 0} m(x + i\varepsilon)$ exists.

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The whole-line Jacobi matrix J is called *reflectionless* on $\mathfrak{e} \subset \mathbb{R}$ if and only if

$$m^+(x + i0) = \left(a_0^2\, \overline{m_0^-(x + i0)}\right)^{-1}$$

for a.e. $x \in \mathfrak{e}$.

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For here, the point is the object on the left and the object on the right have boundary values that determine each other.

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If J is a one-sided Jacobi matrix with parameters $\{a_n, b_n\}_{n=1}^{\infty}$, $J^{(\infty)}$ a two-sided Jacobi matrix is called a right limit for J if and only if, for some $n_k \to \infty$ and all m (but *not* uniformly in m),

$$a_m^{(\infty)} = \lim_{k\to\infty} a_{m+n_k} \qquad b_m^{(\infty)} = \lim_{k\to\infty} b_{m+n_k}$$

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By compactness, if the a's and b's are bounded, there are always right limits.

Remling's Key Tool

Theorem (Remling)**.** *Let J be a half-line Jacobi matrix. If J has a.c. spectrum on $\mathfrak{e} \subset \mathbb{R}$, then every right limit is reflectionless on $\mathfrak{e}$.*

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Earlier Kotani had proven in the ergodic case that for a.e. ω, $d\mu_\omega$ is reflectionless on $\mathfrak{e} =$ a.c. spectrum.

Right Limits of Power Series

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By compactness, right limits exist. Indeed, for any $m_j \to \infty$, there is a sub-subsequence with convergence.

Analytic Functions Associated to $\{b_n\}_{n=-\infty}^{\infty}$

We do not form the Laurent series $\sum_{n=-\infty}^{\infty} b_n z^n$ which, in typical examples (e.g., where both $\limsup_{n\to\infty}|b_n| \neq 0$ and $\limsup_{n\to-\infty}|b_n| \neq 0$), converges nowhere. Rather, we form two functions:

$$f_+(z) = \sum_{n=0}^{\infty} b_n z^n \qquad f_-(z) = \sum_{n=-\infty}^{-1} b_n z^n$$

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Example: $b_n \equiv 1 \Rightarrow f_+ = (1-z)^{-1} \quad f_- = -(1-z)^{-1}$

Notice for this case, f_+ has a continuation $\tilde{f}_+$ outside $\mathbb{D}$ so that $\tilde{f}_+ + f_- = 0$.

Reflectionless Double Series

We call $\{b_n\}_{n=-\infty}^{\infty}$ reflectionless on an open interval $I \subset \partial\mathbb{D}$ if and only if there is a function g analytic in a neighborhood, N, of I so that

$$g = f_+ \text{ on } N \cap \mathbb{D} \qquad g = -f_- \text{ on } N \cap (\mathbb{C} \setminus \mathbb{D})$$

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As above, $b_n \equiv 1$ is reflectionless on $\partial\mathbb{D} \setminus \{1\}$.

More generally, and if b_n is periodic, it is reflectionless and f_+ is rational with poles on $\partial\mathbb{D}$.

The Classical Case

Theorem (Breuer–Simon). *Let* $\sup_n |a_n| < \infty$ *and suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *has an analytic continuation to a neighborhood of an open interval,* $I \subset \partial\mathbb{D}$*. Then any right limit is reflectionless on* I*.*

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Corollary. *Let* $\sup_n |a_n| < \infty$*. If there is a right limit not reflectionless on any* $I \subset \partial\mathbb{D}$*, then* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *has a natural boundary on* $\partial\mathbb{D}$*.*

L^1 Natural Boundaries

Theorem (Breuer–Simon). *Let* $\sup_n |a_n| < \infty$ *and suppose for an open interval,* $I \subset \partial\mathbb{D}$*, and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$*, we have*

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Corollary. *Let* $\sup_n |a_n| < \infty$*. If there is a right limit not reflectionless on any* $I \subset \partial\mathbb{D}$*, then* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *has a strong natural boundary on* $\partial\mathbb{D}$ (*i.e., not* H^1 *in any sector and, in particular, not bounded there*)*.*

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Clearly, the L^1 theorem $\Rightarrow$ the classical theorem, but we use the classical theorem in the proof of the L^1 theorem.

Uniform Bounds

In both theorems, one proves something stronger than just the reflectionless property. For any right limit b, let $f_b(z)$ be the analytic function on $\mathbb{C} \cup \{\infty\} \setminus (\partial\mathbb{D} \setminus I) \equiv \Omega_I$. Let $\mathcal{R}$ be the set of right limits.

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Then for any compact $K \subset \Omega_I$,

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Then for any compact $K \subset \Omega_I$,

$$\sup_{b \in \mathcal{R}} \left[\sup_{z \in K} |f_n(z)| \right] < \infty$$

In particular, since $\mathcal{R}$ is closed under right limits, $\{f_b\}_{b \in \mathcal{R}}$ is compact in the topology of uniform convergence on compact sets of Ω_I.

Ultimate Gap Theorem

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Theorem (Ultimate Gap Theorem). *Let* $\sup_n |a_n| < \infty$*. Suppose there is* $n_j \to \infty$ *so that*

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Note, by compactness, we only need $\liminf|a_{n_j}| \neq 0$.

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2. This result is not new; it follows from a result of Agmon (1951).

Soft and Hard Edges

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Theorem (Polya (1942), Erdös (1945)). *Let n_k be such that for every b_k with*

$$0 < \limsup |b_k| \leq \sup |b_k| < \infty$$

$\sum_{k=1}^{\infty} b_k z^{n_k}$ has a natural boundary on $\partial\mathbb{D}$. Then $n_k/k \to \infty$.

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Or rather, since Erdös is earlier, they are reminiscent of Erdös.

Nondeterminism Implies NB

Theorem (Breuer–Simon). *If there exists $m_j \to \infty$ and $n_j \to \infty$ so that $a_{n_j} - a_{m_j} \to b \neq 0$, $a_{n_j+m} - a_{m_j+m} \to 0$, either all $m < 0$ or all $m > 0$, then $\sum_{n=0}^{\infty} a_n z^n$ has a strong natural boundary.*

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Proof. Two right limits; same f_-, different $f_+(0)$, or same f_+, different $f'_-(\infty)$. □

Rudin–Shapiro Series

Definition.

$$P_0(x) = Q_0(x) = 1; \qquad \deg(P_n) = \deg(Q_n) = 2^{n-1}$$

$$P_{n+1}(x) = P_n(x) + x^{2^n} Q_n(x) \quad Q_{n+1}(x) = P_n(x) - x^{2^n} q_n(x)$$

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Remark. Brillheart uses Szegő.

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Lemma. *If a_n is a sequence with finitely many values and which is not eventually periodic, then $\forall\, N\, \forall p$, $\exists n, m \geq N$ so that*

$$a_{j+n} = a_{j+m} \qquad j = 0, 1, 2, \dots, p-1$$

$$a_{p+n} \neq a_{p+m}$$

Proof of Szegő's Theorem

Proof of Lemma. Suppose not. Then, because there are only finitely many p blocks, $\exists N \exists P$ so that every p block determines the next 1, then by induction, $2, \dots$ full p block. That is, there is a function F on p blocks, so F (p block) $=$ next p block.

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For some k, $\operatorname{ran}(F^k) = \operatorname{ran}(F^{k+1}) = \dots$ since the set of p blocks is finite. So for some k, $\operatorname{ran}(F^k) = \operatorname{ran}(F^{k+1})$, so then all equal.

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Once F is a bijection, on a finite set, $F^\ell = 1$ for some ℓ, so eventually periodic. □

Proof of Szegő's Theorem

To get Szegő, if a_n is not eventually periodic, there are, by the lemma, two right limits, $\tilde{a}$ and $\tilde{b}$ so $\tilde{a}_j = \tilde{b}_j$ for $j \le 0$, but $a_1 \neq b_1$, so at most one is reflectionless, so not periodic $\Rightarrow$ natural boundary.

Classical Results

Theorem (Steinhaus, 1929)**.** *Suppose that* $\sum a_n z^n$ *has a finite radius of convergence.* $\{\omega_n\}_{n=0}^{\infty}$ *are iidrv with uniform distribution on* $\partial\mathbb{D}$*. For a.e. choice,* $\sum a_n \omega_n z^n$ *has a natural boundary on radius of convergence.*

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Various irv extensions by Kahane (1968 book)

Reflectionless View

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Theorem (Breuer–Simon)**.** *Let $a_n(\omega)$ be a stationary, ergodic, bounded, nondeterministic process. Then for a.e. ω, $\sum_{n=0}^{\infty} a_n(\omega)z^n$ has a strong natural boundary.*

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Theorem (Breuer–Simon)**.** *Let $a_n(\omega)$ be independent (but not necessarily identically distributed) uniformly bounded random variable so that for some $n_j \to \infty$, $\mathrm{Var}(a_{n_j}(\omega)) \to c \neq 0$. Then for a.e. ω, $\sum_{n=0}^{\infty} a_n(\omega) z^n$ has a strong natural boundary.*

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For example, for $n \neq n_j$, a_n can be nonrandom.

Pólya–Carleson Theorem

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These a_n are unbounded so, in general, we can't hope to prove this.

Hecke's Example

Determining which case one is in may not be easy.

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Theorem (Hecke, 1921)**.** *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$*. Let* $[\dots] =$ *integral part of* $\dots$*. Then* $\sum_{n=0}^{\infty} [\alpha n] z^n$ *has a natural boundary on the unit circle.*

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Since $\sum_{n=0}^{\infty} \alpha n z^n = \alpha/(1-z)^2$, this is equivalent to $\sum_{n=0}^{\infty} \{\alpha n\} z^n$ having a natural boundary, and that is a bounded sequence, so we can hope to prove this. Here $\{y\} = y - [y]$.

Damanik–Killip Theorem

Theorem (Damanik–Killip, 2005). *Consider a Jacobi matrix, $J(\theta)$, with $a_n \equiv 1$ and $b_n(\theta) = g(\alpha n + \theta)$ where α is irrational, g is bounded and periodic with period 1, and on $[0, 1]$, g is continuous except at finitely many points, at one of which it has different right and left limit. Then for a.e. θ, $J(\theta)$ has no a.c. spectrum.*

Our Result

Theorem (Breuer–Simon). *Let g be as in Damanik–Killip. Then for all* θ, $\sum_{n=0}^{\infty} g(\alpha n + \theta) z^n$ *has a strong natural boundary.*

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Corollary. *Hecke's result*

Sketch. Suppose θ_0 is a point of discontinuity. By finding n_j and m_j so $\{\alpha n_j\} \to \theta_0$ from below and $\{\alpha m_j\} \to \theta_0$ from above, we get two right limits, b and $\tilde{b}$, so that $b_0 \neq \tilde{b}_0$, but $b_n \neq \tilde{b}_n$ for finitely many n's. By shifting, we can suppose $b_n = \tilde{b}_n$ for $n \leq -1$ but $b_0 \neq \tilde{b}_0$. They cannot both be reflectionless across the same interval.

M. Riesz's Lemma

Lemma (M. Riesz, 1916). *Suppose* $\sup_n |a_n| = A < \infty$ *and* $f(z) = \sum_{n=0}^\infty a_n z^n$ *has an analytic continuation to* $\mathbb{D} \cup S$ *where* (*for* $R > 1$ *and* $\alpha < \beta$),

$$S = \{z \mid 0 < |z| < R, \, \alpha < \arg(z) < \beta\}$$

continuous on $\bar{S}$ *with* $M = \sup_{z \in \bar{S}} |f(z)|$. *Let* $z_1 = e^{i\alpha}$, $z_2 = e^{i\beta}$.

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Define, for $N = 0, 1, 2, \dots,$

$$S_+^N(z) = \sum_{n=0}^{\infty} a_{n+N} z^n \qquad S_-^N(z) = \sum_{n=-N}^{-1} a_{n+N} z^n$$

M. Riesz's Lemma

Then S_+^N has an analytic continuation to $\mathbb{D} \cup S$, continuous on $\bar{S}$, and for $N \geq 1$,

$$\sup_{\bar{S}} |(z - z_1)(z - z_2) S_+^N(z)| \leq (A + M)(1 + R)^2 (R - 1)^{-1}$$

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Note. For later purposes, we note S_+^N has a continuation to $\bar{S} \setminus \{0\}$ and that on $\bar{S} \setminus \{0\}$,

$$S_+^N + S_-^N = z^{-N} f(z) \qquad (*)$$

Riesz's Proof

By the maximum principle, we only need the bound on $\partial S \setminus \{z_1, z_2\}$. Let $g(z) = (z - z_1)(z - z_2)S_+^N(z)$.

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On $\{z \mid 0 \leq |z| = r < 1;\ \arg(z) = \alpha \text{ or } \beta\}$,

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On $|z| > 1$,

$$\begin{aligned} |S_-^N(z)| &\le A|z|^{-1}(1 - |z|^{-1})^{-1} \\ &= A(|z| - 1)^{-1} \end{aligned}$$

Riesz's Proof

On $\{z \mid 1 < |z| < R;\ \arg(z) = \alpha \text{ or } \beta\}$, by $(*)$,

$$
\begin{aligned}
|g(z)| &\leq [M|z|^{-N} + A(|z|-1)^{-1}][(|z|-1)(1+R)] \\
&\leq (A+M)(1+R) \leq (A+M)(1+R)^2(R-1)^{-1}
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since $|z|^{-N}(|z|-1) < 1$.

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By Riesz, if $N_j \to \infty$, $\{S_+^N\}$ are uniformly bounded on compact subsets of S. By right limit, S_+^N converges on $\mathbb{D}$. So, by Vitali, $S_+ \to f_+$ on $\mathbb{D} \cup S$.

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By $(*)$, on $S \cap \{|z| > 1\}$, $f_+ + f_- = 0$, so Q.E.D.

E^1 (Sector)

We need the following facts about a function f analytic on $\mathbb{D}$ obeying $\sup_{0<r<1} \int_\alpha^\beta |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty$:

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- $F(z) = \int_{\alpha+\varepsilon}^{\beta-\varepsilon} f(e^{i\theta})(e^{i\theta} - z)^{-1} \frac{d\theta}{2\pi}$ is analytic on $\mathbb{C} \setminus \{e^{i\theta} \mid \alpha + \varepsilon < \theta < \beta - \varepsilon\}$ with a jump discontinuity

$$F(e^{i\theta}(1-0)) - F(e^{i\theta}(1+0)) = f(e^{i\theta})$$

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These all follow from results in Chapter 10 of Duren's H^p-spaces book on E^1 (interior of a rectifiable Jordan curve)

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Theorem. *Let f_+ be analytic in $\mathbb{D}$, f_- in $\mathbb{C} \setminus \overline{\mathbb{D}}$ with* $\sup_{0<r<1} \int_\alpha^\beta |f_+(re^{i\theta})| \frac{d\theta}{2\pi} < \infty$ *and that* $\sup_{1<r<2} \int_\alpha^\beta |f_-(re^{i\theta})| \frac{d\theta}{2\pi} < \infty$. *Suppose $f_+(e^{i\theta}) = f_-(e^{i\theta})$ a.e. on (α, β). Then there is a function $F(z)$ analytic on $\mathbb{C} \setminus [\partial\mathbb{D} \setminus (\alpha, \beta)]$ equal to f_+ on $\mathbb{D}$.*

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This is an easy Morera-like argument.

Breuer–Simon Proof

Fix $\varepsilon > 0$ and let F be the function given by the Cauchy integral of $f(e^{i\theta}) \restriction (\alpha+\varepsilon, \beta-\varepsilon)$. Let

$$b_n = \int_{\alpha+\varepsilon}^{\beta-\varepsilon} e^{-in\theta} f(e^{i\theta}) \, \frac{d\theta}{2\pi}$$

Then F has Taylor expansions $\sum_{n=0}^{\infty} b_n z^n$ near 0 and $\sum_{n=-\infty}^{-1} b_n z^n$ near ∞.

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$$b_n = \int_{\alpha+\varepsilon}^{\beta-\varepsilon} e^{-in\theta} f(e^{i\theta}) \, \frac{d\theta}{2\pi}$$

Then F has Taylor expansions $\sum_{n=0}^{\infty} b_n z^n$ near 0 and $\sum_{n=-\infty}^{-1} b_n z^n$ near ∞.
Moreover, $b_n \to 0$ as $|n| \to \infty$ by the Riemann–Lebesgue lemma. In particular, $\sup_n |b_n| < \infty$.

Breuer–Simon Proof

Let $c_n = a_n - b_n$ and $f_+(z) = \sum_{n=0}^{\infty} c_n z^n$, so

$$\lim_{r \uparrow 1} f_+(re^{i\theta}) = f(e^{i\theta}) - F((1-0)e^{i\theta})$$

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Thus, by the discontinuity of F, Painlevé's theorem applies and $f_-(z)$ has an analytic continuation through $(\alpha + \varepsilon, \beta - \varepsilon)$. Since $b_n \to 0$ right limits of $\{a_n\}$ are the same as right limits of $\{c_n\}$ so the classical result implies the L^1 result.