Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds

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Dear Brian, many happy returns on the day!

Let $M$ be an $n$-dimensional Riemannian manifold (of class $C^1$) such that

$$\mathcal{H}^n(M) < \infty.$$

Here, $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure on $M$, namely, the volume measure on $M$ induced by its Riemannian metric.

**Problem:** estimates for eigenfunctions of the Laplacian on $M$.

**Weak formulation:** a function $u \in W^{1,2}(M)$ is an eigenfunction of the Laplacian associated with the eigenvalue $\gamma$ if

$$\int_{\Omega} \nabla u \cdot \nabla \Phi \, d\mathcal{H}^n(x) = \gamma \int_{\Omega} u \Phi \, d\mathcal{H}^n(x)$$

(1)

for every $\Phi \in W^{1,2}(M)$. 
If $M$ is complete, then (1) is equivalent to

$$-\Delta u = \gamma u \quad \text{on } M. \quad (2)$$

If $M$ is an open subset of a Riemannian manifold, in particular of $\mathbb{R}^n$, then (1) is the weak formulation of the Neumann problem

$$\begin{cases} -\Delta u = \gamma u & \text{on } M \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial M \end{cases} \quad (3)$$
Case $M$ compact.
The eigenvalue problem for the Laplacian has been extensively studied.

By the classical Rellich’s Lemma, the compactness of the embedding

$$W^{1,2}(M) \to L^2(M)$$

is equivalent to the discreteness of the spectrum of the Laplacian on $M$.

Bounds for eigenfunctions in $L^q(M)$, $q > 2$, and $L^\infty(M)$ follow via local bounds, owing to the compactness of $M$. 
Pb.: noncompact $M$.

Much less seems to be known.

Not even the existence of eigenfunctions is guaranteed.

**Major problem**: the embedding $W^{1,2}(M) \rightarrow L^2(M)$ need not be compact.
Example 1.

\[ M = \Omega \]

an open subset of \( \mathbb{R}^n \) endowed with the Euclidean metric.

The eigenvalue problem (2) turns into the Neumann problem

\[
\begin{cases}
-\Delta u = \gamma u & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The point here is that no regularity on \( \partial \Omega \) is (a priori) assumed.
Example 2.
A noncompact manifold of revolution in $\mathbb{IR}^n$,

$$M = \{(r, \omega) : r \in [0, \infty), \omega \in \mathbb{S}^{n-1}\},$$

with metric (in polar coordinates) given by

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 .$$ (4)

Here, $d\omega^2$ stands for the standard metric on $\mathbb{S}^{n-1}$, and $\varphi : [0, L) \to [0, \infty)$ is a smooth function such that $\varphi(r) > 0$ for $r \in (0, L)$, and

$$\varphi(0) = 0 , \quad \text{and} \quad \varphi'(0) = 1 .$$
**Figure:** A manifold of revolution
Example 3.
Manifolds with a sequence of mushroom-shaped submanifolds.
A manifold with a family of clustering submanifolds

Figure:

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Eigenfunctions of the Laplacian

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\[ M \bowtie O \]

(FLAT)

\[ M \approx \bigotime_{k=0}^{k+1} N^k \]

\[ \approx 2^{k+1} \]

\[ N^k \]

\[ N^{k+1} \]
The integrability of eigenfunctions depends on the geometry of $M$.

The geometry of the manifold $M$ can be described through either the isocapacitary function $\nu_M$ of $M$,

or the isoperimetric function $\lambda_M$ of $M$. 
Classical isoperimetric inequality [De Giorgi]

\[ \mathcal{H}^{n-1}(\partial^* E) \geq n \omega_n^{1/n} |E|^{1/n'} \quad \forall E \subset \mathbb{R}^n. \]

Here:
- \( \partial^* E \) stands for the essential boundary of \( E \),
- \( |E| = \mathcal{H}^n(E) \), the Lebesgue measure of \( E \),
- \( \mathcal{H}^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure (the surface area).

In other words,

the ball has the least surface area among sets of fixed volume.
In general the isoperimetric function $\lambda_M : [0, \mathcal{H}^n(M)/2] \rightarrow [0, \infty)$ of $M$ is defined as

$$\lambda_M(s) = \inf \{ \mathcal{H}^{n-1}(\partial^* E) : s \leq \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2 \},$$

for $s \in [0, \mathcal{H}^n(M)/2]$.

**Isoperimetric inequality on $M$:**

$$\mathcal{H}^{n-1}(\partial^* E) \geq \lambda_M(\mathcal{H}^n(E)) \quad \forall E \subset M, \mathcal{H}^n(E) \leq \mathcal{H}^n(M)/2.$$
The geometry of $M$ is related to $\lambda_M$, and, in particular, to its asymptotic behavior at 0. For instance, if $M$ is compact, then

$$\lambda_M(s) \approx s^{1/n'} \quad \text{as } s \to 0.$$ 

Here, $f \approx g$ means that $\exists c, k > 0$ such that

$$cg(cs) \leq f(s) \leq kg(ks).$$

Moreover, $n' = \frac{n}{n - 1}$. 
Approach by isocapacitary inequalities.

Standard capacity of $E \subset M$:

$$C(E) = \inf \left\{ \int_M |\nabla u|^2 \, dx : u \in W^{1,2}(M), \quad "u \geq 1" \text{ in } E, \text{ and } u \text{ has compact support} \right\}.$$

Capacity of a condenser $(E; G)$, $E \subset G \subset M$:

$$C(E; G) = \inf \left\{ \int_M |\nabla u|^2 \, dx : u \in W^{1,2}(M), \quad "u \geq 1" \text{ in } E \quad "u \leq 0" \text{ in } M \setminus G \right\}.$$
Isocapacitary function

\[ \nu_M : [0, \mathcal{H}^n(M)/2] \rightarrow [0, \infty) \]

\[ \nu_M(s) = \inf \{ C(E, G) : E \subset G \subset M, s \leq \mathcal{H}^n(E) \text{ and } \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2 \} \]

for \( s \in [0, \mathcal{H}^n(M)/2] \).

Isocapacitary inequality:

\[ C(E, G) \geq \nu_M(\mathcal{H}^n(E)) \quad \forall \ E \subset G \subset M, \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2. \]

If \( M \) is compact and \( n \geq 3 \), then

\[ \nu_M(s) \approx s \frac{n-2}{n} \quad \text{as } s \rightarrow 0. \]
The isoperimetric function and the isocapacitary function of a manifold $M$ are related by

\[
\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \quad \text{for } s \in (0, \mathcal{H}^n(M)/2).
\]  

(6)

A reverse estimate does not hold in general. Roughly speaking, a reverse estimate only holds when the geometry of $M$ is sufficiently regular.
Both the conditions in terms of $\nu_M$, and those in terms of $\lambda_M$, for eigenfunction estimates in $L^q(M)$ or $L^\infty(M)$ to be presented are sharp in the class of manifolds $M$ with prescribed asymptotic behavior of $\nu_M$ and $\lambda_M$ at $0$.

Each one of these approaches has its own advantages.

The isoperimetric function $\lambda_M$ has a transparent geometric character, and it is usually easier to investigate.

The isocapacitary function $\nu_M$ is in a sense more appropriate: it not only implies the results involving $\lambda_M$, but leads to finer conclusions in general. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.
Estimates for eigenfunctions.
If $u$ is an eigenfunction of the Laplacian, then, by definition, $u \in W^{1,2}(M)$. Hence, trivially, $u \in L^2(M)$.

Problem: given $q \in (2, \infty]$, find conditions on $M$ ensuring that any eigenfunction $u$ of the Laplacian on $M$ belongs to $L^q(M)$. 
Theorem 1: $L^q$ bounds for eigenfunctions

Assume that

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0. \quad (7)$$

Then for any $q \in (2, \infty)$ there exists a constant $C$ such that

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)} \quad (8)$$

for every eigenfunction $u$ of the Laplacian on $M$. 
The assumption

\[
\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0 \tag{9}
\]

is essentially minimal in Theorem 1.

**Theorem 2: Sharpness of condition (9)**

For any \( n \geq 2 \) and \( q \in (2, \infty] \), there exists an \( n \)-dimensional Riemannian manifold \( M \) such that

\[
\nu_M(s) \approx s \quad \text{near } 0, \tag{10}
\]

and the Laplacian on \( M \) has an eigenfunction \( u \notin L^q(M) \).
Conditions in terms of the isoperimetric function for $L^q$ bounds for eigenfunctions can be derived via Theorem 2.

**Corollary 2**

Assume that

$$\lim_{s \to 0} \frac{s}{\lambda_M(s)} = 0. \quad (11)$$

Then for any $q \in (2, \infty)$ there exists a constant $C$ such that

$$\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$.

Assumption (12) is minimal in the same sense as the analogous assumption in terms of $\nu_M$. 
Estimate for the growth of constant in the $L^q(M)$ bound for eigenfunctions in terms of the eigenvalue.

**Proposition**

Assume that

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0. \quad (12)$$

Define

$$\Theta(s) = \sup_{r \in (0,s)} \frac{r}{\nu_M(r)} \quad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then $\|u\|_{L^q(M)} \leq C\|u\|_{L^2(M)}$ for any $q \in (2, \infty)$ and for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$, where

$$C(\nu_M, q, \gamma) = \frac{C_1}{(\Theta^{-1}(C_2/\gamma))^{\frac{1}{2} - \frac{1}{q}}},$$

and $C_1 = C_1(q, \mathcal{H}^n(M))$ and $C_2 = C_2(q, \mathcal{H}^n(M))$. 
Example.
Assume that there exists $\beta \in \left[\frac{(n-2)}{n}, 1\right)$ such that

$$\nu_M(s) \geq Cs^\beta.$$ 

Then there exists a constant $C = C(q, \mathcal{H}^n(M))$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{2q(1-\beta)}}\|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$. 

A digression on the **discreteness of the spectrum** of the Laplace operator.

Condition

\[
\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0,
\]

which implies \(L^q(M)\) bounds for eigenfunctions, can be shown to be equivalent to the **compactness of the embedding**

\[
W^{1,2}(M) \to L^2.
\]
When $\mathcal{M}$ is complete, this condition is also equivalent to the discreteness of the spectrum of the Laplacian on $\mathcal{M}$.

**Theorem 1: Discreteness of the spectrum**

Let $\mathcal{M}$ be a complete Riemannian manifold. Then the spectrum of the Laplacian on $\mathcal{M}$ is discrete if and only if

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0.$$
Back to bounds for eigenfunctions.
Consider now the case when $q = \infty$, namely the problem of the boundedness of the eigenfunctions.

**Theorem 3: boundedness of eigenfunctions**

Assume that

$$\int_0^\infty \frac{ds}{\nu_M(s)} < \infty. \quad (13)$$

Then there exists a constant $C$ such that

$$\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)} \quad (14)$$

for every eigenfunction $u$ of the Laplacian on $M$. 
The condition

\[ \int_{0} \frac{ds}{\nu_M(s)} < \infty \]  

(15)

is essentially sharp in Theorem 4.

This is the content of the next result.

Recall that \( f \in \Delta_2 \) near 0 if there exist constants \( c \) and \( s_0 \) such that

\[ f(2s) \leq cf(s) \quad \text{if } 0 < s \leq s_0. \]  

(16)
Theorem 4: sharpness of condition (15)

Let \( \nu \) be a non-decreasing function, vanishing only at 0, such that

\[
\lim_{s \to 0} \frac{s}{\nu(s)} = 0, \tag{17}
\]

but

\[
\int_0^s \frac{ds}{\nu(s)} = \infty. \tag{18}
\]
Assume in addition that \( \nu \in \Delta_2 \) near 0 and
\[
\frac{\nu(s)}{s^{n-2}} \quad \text{is equivalent to a non-decreasing function near 0,} \quad (19)
\]
for some \( n \geq 3 \). Then, there exists an \( n \)-dimensional Riemannian manifold \( M \) fulfilling
\[
\nu_M(s) \approx \nu(s) \quad \text{as } s \to 0, \quad (20)
\]
and such that the Laplacian on \( M \) has an unbounded eigenfunction.

Assumption (19) is consistent with the fact that \( \nu_M(s) \approx s^{\frac{n-2}{n}} \) near 0 if the geometry of \( M \) is nice (e.g. \( M \) compact), and that \( \nu_M(s) \to 0 \) faster than \( s^{\frac{n-2}{n}} \) otherwise.
Owing to the inequality

\[
\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}
\]

for \( s \in (0, \mathcal{H}^n(M)/2) \),

Theorem 4 has the following corollary in terms of isoperimetric inequalities.
Corollary 3

Assume that
\[ \int_0^s \frac{s}{\lambda_M(s)^2} \, ds < \infty. \] (21)

Then there exists a constant $C$ such that
\[ \|u\|_{L^\infty(M)} \leq C \|u\|_{L^2(M)} \] (22)

for every eigenfunction $u$ of the Laplacian on $M$.

Assumption (21) is sharp in the same sense as the analogous assumption in terms of $\nu_M$. 
Estimate for the growth of constant in the $L^\infty(M)$ bound for eigenfunctions in terms of the eigenvalue.

**Proposition**

Assume that

$$\int_0^\infty \frac{ds}{\nu_M(s)} < \infty.$$ 

Define

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)}$$ 

for $s \in (0, \mathcal{H}^n(M)/2]$.

Then $\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}$ for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$, where

$$C(\nu_M, \gamma) = \frac{C_1}{(\Xi^{-1}(C_2/\gamma))^{\frac{1}{2}}}$$

and $C_1$ and $C_2$ are absolute constants.
Example.
Assume that there exists $\beta \in [(n - 2)/n, 1)$ such that

$$\nu_M(s) \geq Cs^\beta.$$

Then there exists an absolute constant $C$ such that

$$\|u\|_{L^\infty(M)} \leq C\gamma^{\frac{1}{2(1-\beta)}}\|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$. 
Example 4 Manifold of revolution, with metric

\[ ds^2 = dr^2 + \varphi(r)^2 d\omega^2 \]  

and \( \varphi : [0, \infty) \to [0, \infty) \) such that

\[ \varphi(r) = e^{-r^\alpha} \quad \text{for large } r. \]
The larger is $\alpha$, the better is $M$.

One can show that

$$\lambda_M(s) \approx s \left( \log \left( \frac{1}{s} \right) \right)^{1-1/\alpha}$$

near 0,

and

$$\nu_M(s) \approx \left( \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \right)^{-1} \approx s \left( \log \left( \frac{1}{s} \right) \right)^{2-2/\alpha}$$

near 0.
The criteria involving $\lambda_M$ tell us that all eigenfunctions of the Laplacian on $M$ belong to $L^q(M)$ for $q < \infty$ if

$$\alpha > 1,$$  \hspace{1cm}  (25)

and to $L^\infty(M)$ if

$$\alpha > 2.$$  \hspace{1cm}  (26)

The same conclusions follow via the criteria involving $\nu_M$. 
Moreover, if \( \alpha > 1 \), then there exist constants \( C_1 = C_1(q) \) and \( C_2 = C_2(q) \) such that

\[
\| u \|_{L^q(M)} \leq C_1 e^{C_2 \gamma \frac{\alpha}{2\alpha - 2}} \| u \|_{L^2(M)}
\]

for any eigenfunction \( u \) of the Laplacian associated with the eigenvalue \( \gamma \).

If \( \alpha > 2 \), then there exist absolute constants \( C_1 \) and \( C_2 \) such that

\[
\| u \|_{L^\infty(M)} \leq C_1 e^{C_2 \gamma \frac{\alpha}{\alpha - 2}} \| u \|_{L^2(M)}
\]

for any eigenfunction \( u \) associated with \( \gamma \).
Example 5
Manifolds with clustering submanifolds.

\[ \text{Figure: A manifold with a family of clustering submanifolds} \]
In the sequence of mushrooms, the width of the heads and the length of the necks decay like $2^{-k}$, the width of the neck decays like $\sigma(2^{-k})$ as $k \to \infty$, where

$$\lim_{s \to 0} \frac{\sigma(s)}{s} = 0.$$
Assume, for instance, that $b > 1$ and

$$\sigma(s) = s^b \quad \text{for } s > 0.$$ 

Then the criterion involving $\lambda_M$ ensures that all eigenfunctions of the Laplacian on $M$ are bounded provided that

$$b < 2.$$ 

The criterion involving $\nu_M$ yields the boundedness of eigenfunctions under the weaker assumption that

$$b < 3.$$ 

This shows that the use of the isocapacitary function can actually lead to sharper conclusions than those obtained via the isoperimetric function.
By the use of $\nu_M$ we also get that if $b < 3$, then there exists a constant $C = C(q)$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{q(3-b)}} \|u\|_{L^2(M)}$$

for every $q \in (2, \infty]$ and for any eigenfunction $u$ of the Laplacian associated with the eigenvalue $\gamma$. 
Thank you very much for your attention!