# Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds

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Brian Davies 65th birthday conference
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Dear Brian, many happy returns on the day!

1. A. Cianchi & V.M. Bounds for eigenfunctions of the Laplacian on noncompact Riemannian manifolds, preprint.
2. A.Cianchi & V.M. On the discreteness of the spectrum of complete Riemannian manifolds, in preparation

# 1

Let $M$ be an $n$-dimensional Riemannian manifold (of class $C^1$) such that

$$\mathcal{H}^n(M) < \infty.$$

Here, $\mathcal{H}^n$ is the $n$-dimensional Hausdorff measure on $M$, namely, the volume measure on $M$ induced by its Riemannian metric.

Problem: estimates for eigenfunctions of the Laplacian on $M$.
Weak formulation: a function $u \in W^{1,2}(M)$ is an eigenfunction of the Laplacian associated with the eigenvalue $\gamma$ if

$$\int_\Omega \nabla u \cdot \nabla \Phi \, d\mathcal{H}^n(x) = \gamma \int_\Omega u\Phi \, d\mathcal{H}^n(x) \qquad (1)$$

for every $\Phi \in W^{1,2}(M)$.

If $M$ is complete, then (1) is equivalent to

$$-\Delta u = \gamma u \qquad \text{on } M. \tag{2}$$

If $M$ is an open subset of a Riemannian manifold, in particular of $\mathbb{R}^n$, then (1) is the weak formulation of the Neumann problem

$$\begin{cases} -\Delta u = \gamma u & \text{on } M \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial M \end{cases} \tag{3}$$

Case $M$ compact.
The eigenvalue problem for the Laplacian has been extensively studied.

By the classical Rellich's Lemma , the compactness of the embedding

$$W^{1,2}(M) \to L^2(M)$$

is equivalent to the discreteness of the spectrum of the Laplacian on $M$.

Bounds for eigenfunctions in $L^q(M)$, $q > 2$, and $L^\infty(M)$ follow via local bounds, owing to the compactness of $M$.

Pb.: noncompact $M$.

Much less seems to be known.

Not even the existence of eigenfunctions is guaranteed.

Major problem: the embedding $W^{1,2}(M) \to L^2(M)$ need not be compact.

Example 1.

$$M = \Omega$$

an open subset of $\mathbb{R}^n$ endowed with the Eulcidean metric.
The eigenvalue problem (2) turns into the Neumann problem

$$\begin{cases} -\Delta u = \gamma u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \,. \end{cases}$$

The point here is that no regularity on $\partial\Omega$ is (a priori) assumed.

Example 2.
A noncompact manifold of revolution in $\mathbb{R}^n$,

$$M = \{(r, \omega) : r \in [0, \infty), \omega \in \mathbb{S}^{n-1}\},$$

with metric (in polar coordinates) given by

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 . \tag{4}$$

Here, $d\omega^2$ stands for the standard metric on $\mathbb{S}^{n-1}$, and $\varphi : [0, L) \to [0, \infty)$ is a smooth function such that $\varphi(r) > 0$ for $r \in (0, L)$, and

$$\varphi(0) = 0 , \qquad \text{and} \qquad \varphi'(0) = 1 .$$

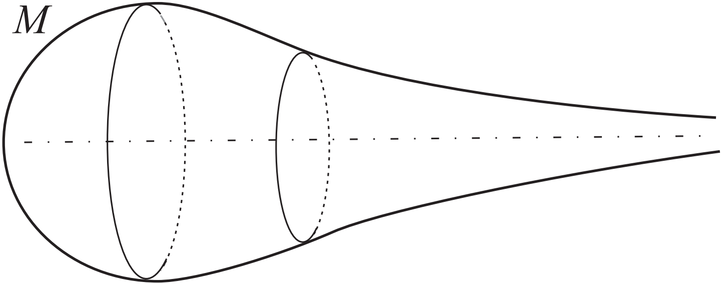


FIGURE: A manifold of revolution

Example 3.

Manifolds with a sequence of mushroom-shaped submanifolds .

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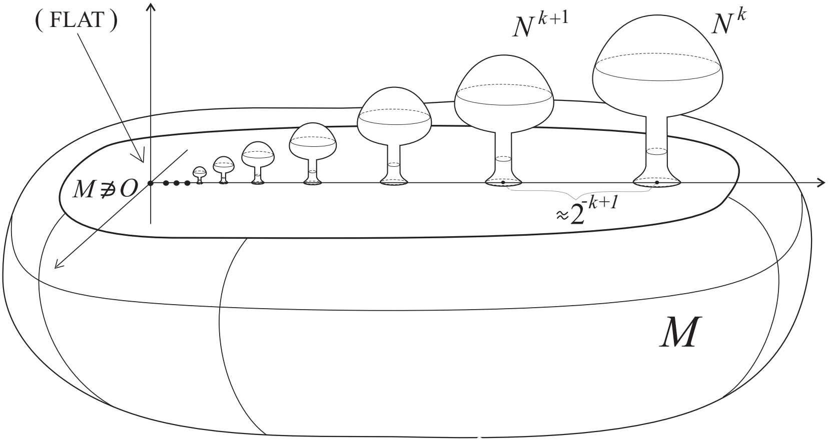
( FLAT )
$N^{k+1}$
$N^k$
$M \not\ni O$
$\approx 2^{-k+1}$
$M$

The integrability of eigenfunctions depends on the geometry of $M$.

The geometry of the manifold $M$ can be described through either the

isocapacitary function $\nu_M$ of $M$,

or the

isoperimetric function $\lambda_M$ of $M$.

Classical isoperimetric inequality [De Giorgi]

$$\mathcal{H}^{n-1}(\partial^* E) \geq n\omega_n^{1/n}|E|^{1/n'} \qquad \forall E \subset \mathbb{R}^n .$$

Here:

- $\partial^* E$ stands for the essential boundary of $E$,
- $|E| = \mathcal{H}^n(E)$, the Lebesgue measure of $E$,
- $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (the surface area).

In other words,

the ball has the least surface area among sets of fixed volume.

In general the isoperimetric function $\lambda_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty)$ of $M$ is defined as

$$\lambda_M(s) = \inf\{\mathcal{H}^{n-1}(\partial^* E) : s \le \mathcal{H}^n(E) \le \mathcal{H}^n(M)/2\}\,, \qquad (5)$$

for $s \in [0, \mathcal{H}^n(M)/2]$.

Isoperimetric inequality on $M$:

$$\mathcal{H}^{n-1}(\partial^* E) \ge \lambda_M(\mathcal{H}^n(E)) \quad \forall E \subset M, \mathcal{H}^n(E) \le \mathcal{H}^n(M)/2.$$

The geometry of $M$ is related to $\lambda_M$, and, in particular, to its asymptotic behavior at 0. For instance, if $M$ is compact, then

$$\lambda_M(s) \approx s^{1/n'} \qquad \text{as } s \to 0.$$

Here, $f \approx g$ means that $\exists\ c, k > 0$ such that

$$cg(cs) \leq f(s) \leq kg(ks).$$

Moreover, $n' = \dfrac{n}{n-1}$.

Approach by isocapacitary inequalities.
Standard capacity of $E \subset M$:

$$C(E) = \inf \Big\{ \int_M |\nabla u|^2\, dx : u \in W^{1,2}(M),$$
$$\text{"}u \geq 1\text{" in } E, \text{and } u \text{ has compact support} \Big\}.$$

Capacity of a condenser $(E; G)$, $E \subset G \subset M$:

$$C(E; G) = \inf \Big\{ \int_M |\nabla u|^2\, dx : u \in W^{1,2}(M),$$
$$\text{"}u \geq 1\text{" in } E \text{ "}u \leq 0\text{" in } M \setminus G \Big\}.$$

Isocapacitary function

$$\nu_M : [0, \mathcal{H}^n(M)/2] \to [0, \infty)$$

$$\nu_M(s) = \inf\{C(E,G) : E \subset G \subset M,\ s \leq \mathcal{H}^n(E) \text{ and } \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2\}$$

for $s \in [0, \mathcal{H}^n(M)/2]$.

Isocapacitary inequality:

$$C(E,G) \geq \nu_M(\mathcal{H}^n(E)) \quad \forall\ E \subset G \subset M,\ \mathcal{H}^n(G) \leq \mathcal{H}^n(M)/2.$$

If $M$ is compact and $n \geq 3$, then

$$\nu_M(s) \approx s^{\frac{n-2}{n}} \qquad \text{as } s \to 0.$$

The isoperimetric function and the isocapacitary function of a manifold $M$ are related by

$$\frac{1}{\nu_M(s)} \leq \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2). \tag{6}$$

A reverse estimate does not hold in general.
Roughly speaking, a reverse estimate only holds when the geometry of $M$ is sufficiently regular.

Both the conditions in terms of $\nu_M$, and those in terms of $\lambda_M$, for eigenfunction estimates in $L^q(M)$ or $L^\infty(M)$ to be presented are sharp in the class of manifolds $M$ with prescribed asymptotic behavior of $\nu_M$ and $\lambda_M$ at 0.

Each one of these approaches has its own advantages.

The isoperimetric function $\lambda_M$ has a transparent geometric character, and it is usually easier to investigate.

The isocapacitary function $\nu_M$ is in a sense more appropriate: it not only implies the results involving $\lambda_M$, but leads to finer conclusions in general. Typically, this is the case when manifolds with complicated geometric configurations are taken into account.

Estimates for eigenfunctions.
If $u$ is an eigenfunction of the Laplacian, then, by definition, $u \in W^{1,2}(M)$. Hence, trivially, $u \in L^2(M)$.

Problem: given $q \in (2, \infty]$, find conditions on $M$ ensuring that any eigenfunction $u$ of the Laplacian on $M$ belongs to $L^q(M)$.

**Theorem 1: $L^q$ bounds for eigenfunctions**

Assume that

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,. \tag{7}$$

Then for any $q \in (2, \infty)$ there exists a constant $C$ such that

$$\|u\|_{L^q(M)} \le C\|u\|_{L^2(M)} \tag{8}$$

for every eigenfunction $u$ of the Laplacian on $M$.

The assumption

$$\lim_{s \to 0} \frac{s}{\nu_M(s)} = 0 \tag{9}$$

is essentially minimal in Theorem 1.

**Theorem 2: Sharpness of condition (9)**

For any $n \geq 2$ and $q \in (2, \infty]$, there exists an $n$-dimensional Riemannian manifold $M$ such that

$$\nu_M(s) \approx s \qquad \text{near } 0, \tag{10}$$

and the Laplacian on $M$ has an eigenfunction $u \notin L^q(M)$.

Conditions in terms of the isoperimetric function for $L^q$ bounds for eigenfunctions can be derived via Theorem 2.

**Corollary 2**

Assume that

$$\lim_{s\to 0} \frac{s}{\lambda_M(s)} = 0\,. \tag{11}$$

Then for any $q \in (2,\infty)$ there exists a constant $C$ such that

$$\|u\|_{L^q(M)} \le C\|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$.

Assumption (12) is minimal in the same sense as the analogous assumption in terms of $\nu_M$.

Estimate for the growth of constant in the $L^q(M)$ bound for eigenfunctions in terms of the eigenvalue.

## Proposition

Assume that

$$\lim_{s\to 0}\frac{s}{\nu_M(s)} = 0\,. \tag{12}$$

Define

$$\Theta(s) = \sup_{r\in(0,s)}\frac{r}{\nu_M(r)} \qquad \text{for } s\in(0,\mathcal{H}^n(M)/2].$$

Then $\|u\|_{L^q(M)} \le C\|u\|_{L^2(M)}$ for any $q\in(2,\infty)$ and for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$, where

$$C(\nu_M, q, \gamma) = \frac{C_1}{\left(\Theta^{-1}(C_2/\gamma)\right)^{\frac{1}{2}-\frac{1}{q}}}\,,$$

and $C_1 = C_1(q,\mathcal{H}^n(M))$ and $C_2 = C_2(q,\mathcal{H}^n(M))$.

Example.

Assume that there exists $\beta \in [(n-2)/n, 1)$ such that

$$\nu_M(s) \geq C s^{\beta}.$$

Then there exists a constant $C = C(q, \mathcal{H}^n(M))$ such that

$$\|u\|_{L^q(M)} \leq C \gamma^{\frac{q-2}{2q(1-\beta)}} \|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$.

A digression on the discreteness of the spectrum of the Laplace operator.

Condition

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,,$$

which implies $L^q(M)$ bounds for eigenfunctions, can be shown to be equivalent to the compactness of the embedding

$$W^{1,2}(M) \to L^2.$$

When $M$ is complete, this condition is also equivalent to the discreteness of the spectrum of the Laplacian on $M$.

**Theorem 1: Discreteness of the spectrum**

Let $M$ be a complete Riemannian manifold. Then the spectrum of the Laplacian on $M$ is discrete if and only if

$$\lim_{s\to 0} \frac{s}{\nu_M(s)} = 0\,.$$

Back to bounds for eigenfunctions.
Consider now the case when $q = \infty$, namely the problem of the boundedness of the eigenfunctions.

**Theorem 3: boundedness of eigenfunctions**

Assume that

$$\int_0 \frac{ds}{\nu_M(s)} < \infty \,. \tag{13}$$

Then there exists a constant $C$ such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{14}$$

for every eigenfunction $u$ of the Laplacian on $M$.

The condition

$$\int_0 \frac{ds}{\nu_M(s)} < \infty \tag{15}$$

is essentially sharp in Theorem 4.
This is the content of the next result.
Recall that $f \in \Delta_2$ near 0 if there exist constants $c$ and $s_0$ such that

$$f(2s) \leq cf(s) \qquad \text{if } 0 < s \leq s_0. \tag{16}$$

**Theorem 4: sharpness of condition (15)**

Let $\nu$ be a non-decreasing function, vanishing only at 0, such that

$$\lim_{s\to 0} \frac{s}{\nu(s)} = 0\,, \qquad (17)$$

but

$$\int_0 \frac{ds}{\nu(s)} = \infty\,. \qquad (18)$$

Assume in addition that $\nu \in \Delta_2$ near 0 and

$$\frac{\nu(s)}{s^{\frac{n-2}{n}}} \quad \text{is equivalent to a non-decreasing function near 0,} \qquad (19)$$

for some $n \geq 3$. Then, there exists an $n$-dimensional Riemannian manifold $M$ fulfilling

$$\nu_M(s) \approx \nu(s) \qquad \text{as } s \to 0, \qquad (20)$$

and such that the Laplacian on $M$ has an unbounded eigenfunction.

Assumption (19) is consistent with the fact that $\nu_M(s) \approx s^{\frac{n-2}{n}}$ near 0 if the geometry of $M$ is nice (e.g. $M$ compact), and that $\nu_M(s) \to 0$ faster than $s^{\frac{n-2}{n}}$ otherwise.

Owing to the inequality

$$\frac{1}{\nu_M(s)} \le \int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2),$$

Theorem 4 has the following corollary in terms of isoperimetric inequalities.

**Corollary 3**

Assume that

$$\int_0 \frac{s}{\lambda_M(s)^2}\, ds < \infty\,. \tag{21}$$

Then there exists a constant $C$ such that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{L^2(M)} \tag{22}$$

for every eigenfunction $u$ of the Laplacian on $M$.

Assumption (21) is sharp in the same sense as the analogous assumption in terms of $\nu_M$.

Estimate for the growth of constant in the $L^\infty(M)$ bound for eigenfunctions in terms of the eigenvalue.

**Proposition**

Assume that

$$\int_0 \frac{ds}{\nu_M(s)} < \infty.$$

Define

$$\Xi(s) = \int_0^s \frac{dr}{\nu_M(r)} \qquad \text{for } s \in (0, \mathcal{H}^n(M)/2].$$

Then $\|u\|_{L^\infty(M)} \leq C\|u\|_{L^2(M)}$ for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$, where

$$C(\nu_M, \gamma) = \frac{C_1}{\left(\Xi^{-1}(C_2/\gamma)\right)^{\frac{1}{2}}},$$

and $C_1$ and $C_2$ are absolute constants.

Example.
Assume that there exists $\beta \in [(n-2)/n, 1)$ such that

$$\nu_M(s) \geq C s^{\beta}.$$

Then there exists an absolute constant $C$ such that

$$\|u\|_{L^\infty(M)} \leq C \gamma^{\frac{1}{2(1-\beta)}} \|u\|_{L^2(M)}$$

for every eigenfunction $u$ of the Laplacian on $M$ associated with the eigenvalue $\gamma$.

Example 4 Manifold of revolution, with metric

$$ds^2 = dr^2 + \varphi(r)^2 d\omega^2 \tag{23}$$

and $\varphi : [0, \infty) \to [0, \infty)$ such that

$$\varphi(r) = e^{-r^\alpha} \qquad \text{for large } r. \tag{24}$$

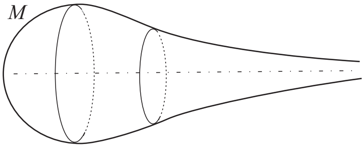


FIGURE: A manifold of revolution

The larger is $\alpha$, the better is $M$.
One can show that

$$\lambda_M(s) \approx s\big(\log(1/s))\big)^{1-1/\alpha} \qquad \text{near } 0,$$

and

$$\nu_M(s) \approx \left(\int_s^{\mathcal{H}^n(M)/2} \frac{dr}{\lambda_M(r)^2}\right)^{-1} \approx s\big(\log(1/s)\big)^{2-2/\alpha} \qquad \text{near } 0.$$

The criteria involving $\lambda_M$ tell us that all eigenfunctions of the Laplacian on $M$ belong to $L^q(M)$ for $q < \infty$ if

$$\alpha > 1, \tag{25}$$

and to $L^\infty(M)$ if

$$\alpha > 2. \tag{26}$$

The same conclusions follow via the criteria involving $\nu_M$.

Moreover, if $\alpha > 1$, then there exist constants $C_1 = C_1(q)$ and $C_2 = C_2(q)$ such that

$$\|u\|_{L^q(M)} \le C_1 e^{C_2 \gamma^{\frac{\alpha}{2\alpha-2}}} \|u\|_{L^2(M)}$$

for any eigenfunction $u$ of the Laplacian associated with the eigenvalue $\gamma$.

If $\alpha > 2$, then there exist absolute constants $C_1$ and $C_2$ such that

$$\|u\|_{L^\infty(M)} \le C_1 e^{C_2 \gamma^{\frac{\alpha}{\alpha-2}}} \|u\|_{L^2(M)}$$

for any eigenfunction $u$ associated with $\gamma$.

Example 5

Manifolds with clustering submanifolds.

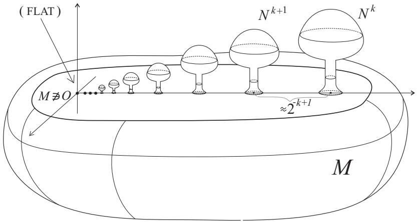


FIGURE: A manifold with a family of clustering submanifolds

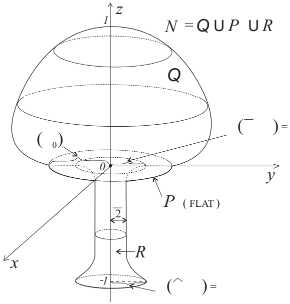


FIGURE: An auxiliary submanifold

In the sequence of mushrooms, the width of the heads and the length of the necks decay like $2^{-k}$, the width of the neck decays like $\sigma(2^{-k})$ as $k \to \infty$, where

$$\lim_{s \to 0} \frac{\sigma(s)}{s} = 0.$$

Assume, for instance, that $b > 1$ and

$$\sigma(s) = s^b \qquad \text{for } s > 0.$$

Then the criterion involving $\lambda_M$ ensures that all eigenfunctions of the Laplacian on $M$ are bounded provided that

$$b < 2.$$

The criterion involving $\nu_M$ yields the boundedness of eigenfunctions under the weaker assumption that

$$b < 3.$$

This shows that the use of the isocapacitary function can actually lead to sharper conclusions than those obtained via the isoperimetric function.

By the use of $\nu_M$ we also get that if $b < 3$, then there exists a constant $C = C(q)$ such that

$$\|u\|_{L^q(M)} \leq C\gamma^{\frac{q-2}{q(3-b)}} \|u\|_{L^2(M)}$$

for every $q \in (2, \infty]$ and for any eigenfunction $u$ of the Laplacian associated with the eigenvalue $\gamma$.

Thank you very much for your attention!