THE SCHMIDT SUBSPACES OF HANKEL OPERATORS

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Abstract. These are (slightly expanded) lecture notes from a mini-course given at CRM Barcelona during 22–25 October 2019. The content is based on the author's joint work with Patrick Gérard.

1. Introduction

1.1. **Motivation.** The motivation for this topic is the work of Patrick Gérard and Sandrine Grellier in 2010–2014. In [3], they have introduced the *cubic Szegő equation*

$$i\frac{\partial u}{\partial t} = P(|u|^2 u), \quad u = u(z;t), \quad z \in \mathbb{T}, \quad t \in \mathbb{R},$$

as a model for totally non-dispersive evolution equations. Here for each $t \in \mathbb{R}$, the function $u(\cdot, t)$ is an element of the Hardy class $H^2 = H^2(\mathbb{T})$ and P is the the Szegő projection, i.e. the orthogonal projection in $L^2(\mathbb{T})$ onto the Hardy class H^2 (precise definitions will be given below).

It turned out [3, 4] that this equation is completely integrable and possesses a Lax pair. Indeed, a function u is a solution to the cubic Szegő equation if and only if

$$\frac{d}{dt} H_u = [B_u, H_u],$$

where H_u is a Hankel operator with the symbol u (see Section 1.7 for the definition), and B_u is a certain auxiliarly skew-selfadjoint operator. In particular, it follows that if the operator H_u is compact, then its singular values are integrals of motion for the cubic Szegő equation.

In order to solve the Cauchy problem for the cubic Szegő equation, one must therefore develop a version of direct and inverse spectral theory for H_u. The spectral data in this problem involves the sequence of singular values of H_u and the sequence of inner functions, parameterising the Schmidt subspaces of H_u (i.e. the eigenspaces of $|H_u|$). This was achieved in [5, 6] for $u \in \mathrm{VMOA}(\mathbb{T})$, which corresponds to *compact* Hankel operators H_u.

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1.2. **Summary.** The important ingredient of the work by Gérard-Grellier was the description of the structure of the Schmidt subspaces of H_u. This description was later made both more precise and more general in [7, 9]. The purpose of this mini-course is to describe this structure, its consequences and some related ideas.

To put it briefly, the aim of this mini-course is to state precisely and to prove

Main theorem. *Every Schmidt subspace of a Hankel operator has the form pK_θ, where θ is an inner function, K_θ is a model space and p is an isometric multiplier on K_θ.*

All the underlined terms will be defined and discussed; some ideas of the proof will be given, and some consequences will be mentioned.

1.3. **Schmidt subspaces.** Let A be a bounded operator in a Hilbert space. We will say that $s > 0$ is a *singular value* of A, if the *Schmidt subspace*

$$E_A(s) := \operatorname{Ker}(A^*A - s^2 I)$$

is non-trivial: $E_A(s) \neq \{0\}$. In other words, this means that there exists a non-zero pair (ξ, η) (called the *Schmidt pair*) of elements in our Hilbert space such that $A\xi = s\eta$ and $A^*\eta = s\xi$.

It is straightforward to see that A maps the Schmidt subspace $E_A(s)$ onto $E_{A^*}(s)$. Suppose for simplicity of discussion that A is compact. Observe that our Hilbert space can be represented as

$$\operatorname{Ker} A \oplus \Big(\bigoplus_s E_A(s)\Big),$$

where the orthogonal sum is taken over all singular values of A.

Observe that if we know all singular values of A, all Schmidt subspaces $E_A(s)$ and we know how A acts from $E_A(s)$ to $E_{A^*}(s)$, then we can reconstruct the operator A from this information.

1.4. **Hankel and Toeplitz matrices.** The most elementary way of approaching the definition of Hankel and Toeplitz operators is to consider them as infinite matrices in $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, with the following structure:

$$\Gamma = \{a(n+m)\}_{n,m=0}^\infty, \qquad T = \{a(n-m)\}_{n,m=0}^\infty,$$

where $\{a(n)\}_{n=-\infty}^\infty$ is a sequence of complex numbers. Our main focus will be on Hankel operators.

We will discuss the boundedness and other analytic properties very soon, but at present let us focus on the following "algebraic" aspect. Observe that the matrix Γ is symmetric, i.e. $\Gamma^\top = \Gamma$. It follows that

$$\mathcal{C}\Gamma = \Gamma^*\mathcal{C},$$

where $\mathcal{C}$ is the (anti-linear) operator of complex conjugation in ℓ^2:

$$\mathcal{C}\{x_n\}_{n=0}^\infty = \{\overline{x_n}\}_{n=0}^\infty.$$

From here we easily see that $\mathcal{C}$ maps $E_{\Gamma^*}(s)$ onto $E_\Gamma(s)$ and vice versa. Thus, the *anti-linear* operator $\Gamma\mathcal{C}$ maps the subspace $E_{\Gamma^*}(s)$ onto itself; in fact, it is an anti-linear involution on this subspace. Observe also that

$$(\Gamma\mathcal{C})^2 = \Gamma\mathcal{C}\Gamma\mathcal{C} = \Gamma\Gamma^*,$$

and so

$$E_{\Gamma^*}(s) = \operatorname{Ker}((\Gamma\mathcal{C})^2 - s^2 I).$$

Thus, it will be convenient to deal with the anti-linear operator $\Gamma\mathcal{C}$; our aim will be to describe the Schmidt subspaces $\operatorname{Ker}((\Gamma\mathcal{C})^2 - s^2 I)$ and the action of $\Gamma\mathcal{C}$ on these subspaces.

1.5. **Hardy space.** We use the standard notation for the unit circle and the unit disk in $\mathbb{C}$:

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

We recall basic definitions related to the Hardy space H^2 in the unit disk. This space consists of all functions in $L^2(\mathbb{T})$ of the form

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, \quad |z| = 1, \tag{1.1}$$

where $\|f\|^2 = \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 < \infty$. We will denote by $\langle f, g\rangle$ the inner product of f and g in H^2, and we will denote by $\mathbb{1}$ the function in H^2 which is identically equal to 1.

The Szegő projection P is the orthogonal projection in $L^2(\mathbb{T})$ onto H^2, given by

$$P : \sum_{n=-\infty}^{\infty} \widehat{f}(n) z^n \mapsto \sum_{n=0}^{\infty} \widehat{f}(n) z^n.$$

If $f \in H^2$, sometimes it is convenient to consider it not as a function on the unit circle $\mathbb{T}$, but as a holomorphic function in the open unit disk $\mathbb{D}$, given by the same formula (1.1). In this case, the function on the unit circle can be recovered, for example, as the radial (in fact, non-tangential) limit

$$f(e^{i\theta}) = \lim_{r \to 1_-} f(re^{i\theta}), \quad \text{a.e. } \theta \in (0, 2\pi).$$

We refer e.g. to [12, 14] for the theory of boundary behaviour of holomorphic functions in the unit disk.

The shift operator S in H^2 is defined by $Sf(z) = zf(z)$, and its adjoint S^* in H^2 is given by

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

We will also need the Hardy space H^∞, which can be defined as $H^2 \cap L^\infty$ (or as the space of all bounded analytic functions in the unit disk).

1.6. **Toeplitz operators in Hardy space.** Let $a \in L^\infty(\mathbb{T})$; consider the operator T_a in H^2, given by

$$T_a f = P(a \cdot f), \quad f \in H^2.$$

The function a is called the symbol of T_φ. It is straightforward to see that T_a is bounded and that the matrix of T_a in the standard basis $\{z^n\}_{n=0}^\infty$ is $\{\widehat{a}(n-m)\}_{n,m=0}^\infty$, i.e. it is a Toeplitz matrix.

Conversely, it is not difficult to prove that any bounded Toeplitz matrix

$$T = \{t(n-m)\}_{n,m=0}^\infty \quad \text{on } \ell^2(\mathbb{Z}_+)$$

is unitarily equivalent to T_a with some $a \in L^\infty$ (we will say that T is *realised* as T_a in the Hardy space).

Toeplitz operators satisfy the commutation relation

$$S^* T_a S = T_a;$$

the proof of this is a simple exercise. In fact, if a bounded operator T_a satisfies this commutation relation, then it is necessarily a Toeplitz operator (this is called the Brown-Halmos theorem).

We refer, e.g. to [13] for background information on Toeplitz operators.

1.7. **Hankel operators in Hardy space.** Consider the symbol $u \in H^\infty$ (this is the class of all bounded analytic functions on the open unit disk). We define the anti-linear Hankel operator in H^2 by

$$H_u f = P(u \cdot \overline{f}), \quad f \in H^2.$$

It is straightforward to see that the matrix of H_u in the standard basis is $\Gamma = \{\widehat{u}(n+m)\}_{n,m=0}^\infty$. Remembering the complex conjugation over f, we see that H_u is unitarily equivalent to the operator $\Gamma\mathcal{C}$ in ℓ^2.

We note that instead of $u \in H^\infty$, we could have taken $u \in L^\infty$. However, H_u depends only on the analytic part of u, and requiring that u is analytic ensures that the symbol u is uniquely defined by the operator H_u.

It is an easy exercise to check the commutation relation

$$S^* H_u = H_u S.$$

In fact, it is not difficult to show that any bounded anti-linear operator that satisfies this relation, is a Hankel operator.

Although this is not a focus of our mini-course, we briefly mention the following facts:

(1) Condition $u \in H^\infty$ is sufficient, but not necessary for the boundedness of H_u. In fact, H_u is bounded if and only if $u \in \mathrm{BMOA}$, i.e. u is an analytic BMO function; this fact is often called the Nehari-Fefferman theorem. The class BMOA satisfies

$$H^\infty \subset \mathrm{BMOA} \subset H^p, \quad \forall p < \infty.$$

In particular, BMOA $\subset H^2$, and so the symbol u of a bounded Hankel operator H_u is in H^2. This can also be seen directly, because

$$H_u \mathbb{1} = P(u\mathbb{1}) = u.$$

(2) Kronecker's theorem asserts that H_u is a finite rank operator if and only if u is a rational function with no poles in the closed unit disk.

Example. Let $u_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$, where $|\alpha| < 1$. Recall that u_α is the reproducing kernel of H^2, i.e.

$$\langle f, u_\alpha \rangle = f(\alpha), \quad f \in H^2.$$

It is a simple exercise to check that

$$H_{u_\alpha} f = \langle u_\alpha, f \rangle u_\alpha,$$

i.e. H_{u_α} is a rank one Hankel operator.

Using this example, it is not difficult to prove one part of Kronecker's theorem. If v is a rational function, we can represent it as a sum of elementary fractions u_α and their derivatives. Each of them gives rise to a finite rank Hankel operator.

We refer, e.g. to [13] and [15] for background information on Hankel operators.

2. Inner functions, model spaces and isometric multipliers

2.1. **Inner functions.** A non-constant function $\theta \in H^\infty$ is called *inner*, if $|\theta(z)| = 1$ for almost all z in the unit circle.

Example. Let $N \in \mathbb{N}$ and let $\{z_n\}_{n=1}^N$ be points in the open unit disk. Define

$$\theta(z) = \prod_{n=1}^{N} \frac{z_n - z}{1 - \overline{z_n} z}$$

for $|z| < 1$. Then θ is inner; it is a *Blaschke product of degree* N with zeros $\{z_n\}_{n=1}^N$.

Example. The previous example can be modified to the case of infinitely many zeros. The only new aspect is that one has to take care about the convergence of the infinite product. Let $\{z_n\}_{n=1}^\infty$ be points in the open unit disk, satisfying the condition

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

Define

$$\theta(z) = \prod_{n=1}^{N} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}$$

for $|z| < 1$. (The terms $\overline{z_n}/|z_n|$ are inserted in order to make the infinite product converge.) Then θ is inner; it is an infinite Blaschke product. We define the degree of θ to be infinity.

Example. Let $\mu \geq 0$ be a finite singular measure on the unit circle $\mathbb{T}$; define

$$\theta(z) = \exp\Big(-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t) \Big)$$

for $|z|<1$. Then θ is inner; it is a *singular inner function*; by definition, the degree of θ is infinity. For example, if μ is a point mass at 1 with $\mu(\{1\}) = c > 0$, we have

$$\theta(z) = \exp\Big(c\frac{z+1}{z-1} \Big).$$

In fact, every inner function can be represented as a product of a Blaschke product and a singular inner function.

2.2. **Model spaces.** The following theorem due to A.Beurling (1949) is fundamental to much of analysis.

Theorem 2.1. *Let $M \subset H^2$ be a closed subspace, $M \neq \{0\}$ and $M \neq H^2$. Suppose that M is invariant under the shift operator: $SM \subset M$. Then there exists an inner function θ such that*

$$M = \theta H^2 := \{\theta f : f \in H^2\}.$$

Of course, the converse is also true: every subspace θH^2 is invariant for S.
Observing that $SM \subset M$ if and only if $S^* M^\perp \subset M^\perp$, we obtain

Corollary 2.2. *Let $M \subset H^2$ be a closed subspace, $M \neq \{0\}$ and $M \neq H^2$. Suppose that M is invariant under the backwards shift operator: $S^* M \subset M$. Then there exists an inner function θ such that*

$$M = K_\theta := H^2 \cap (\theta H^2)^\perp.$$

The space K_θ is called a *model space.*
Let us rewrite the condition $f \in K_\theta$ in an equivalent way:
$f \perp \theta H^2 \Leftrightarrow \overline{\theta} f \perp H^2 \Leftrightarrow \overline{z}\theta\overline{f} \in H^2$.

Example. Let $\theta(z) = z^N$, $N \in \mathbb{N}$. Then

$$K_{z^N} = \{a_0 + \dots + a_{N-1} z^{N-1} : a_0, \dots, a_{N-1} \in \mathbb{C}\}$$

is simply the space of all polynomials of degree $\leq N-1$.

Example. Let $N \in \mathbb{N}$ and let θ be a Blaschke product of degree N with distinct zeros $\{z_n\}_{n=1}^N$. It is easy to see that

$$K_\theta = \operatorname{span}\{u_{z_n}\}_{n=1}^N, \quad u_\alpha(z) = \frac{1}{1-\overline{\alpha} z}.$$

Indeed, recalling that u_α is the reproducing kernel, we see that the orthogonal complement to $\operatorname{span}\{u_{z_n}\}_{n=1}^N$ is precisely the linear subspace of functions $f \in H^2$ that vanish at all points $\{z_n\}_{n=1}^N$. This subspace coincides with θH^2.

Remark. 1. It is easy to see that $\theta(0) = 0$ if and only if $\mathbb{1} \in K_\theta$.
2. It is easy to see that the map $f \mapsto \overline{z}\theta\overline{f}$ is an involution on K_θ.

2.3. **Isometric multipliers on model spaces.** Let M be a closed subspace in H^2, and let p be an analytic function in the open unit disk. One says that p is an *isometric multiplier* on M, if for every $f \in M$, we have $pf \in H^2$ and

$$\|pf\| = \|f\|.$$

In this case we will denote

$$pM = \{pf : f \in M\}.$$

Clearly, pM is a closed subspace in H^2.

Remark. Observe that if $\mathbb{1} \in M$, then (taking $f = \mathbb{1}$) we have $p \in M$ and $\|p\| = 1$.

Exercise. Check that if $\mathbb{1} \in M$, then p is (up to a unimodular complex factor) the normalised projection of $\mathbb{1}$ onto the space pM.

The interest to isometric multipliers on model spaces arose due to the following result by E.Hayashi from 1986:

Theorem 2.3. [10] *Let T be a bounded Toeplitz operator in H^2 with a non-trivial kernel. Then there exists an inner function θ and an isometric multiplier p on K_θ such that*

$$\operatorname{Ker} T = pK_\theta.$$

D.Sarason has characterised all isometric multipliers on a given space K_θ.

Theorem 2.4. [17] *Let θ be an inner function with $\theta(0) = 0$, and let $p \in H^2$ be a function of norm one. Then p is an isometric multiplier on K_θ if and only if it can be represented as*

$$p(z) = \frac{a(z)}{1 - \theta(z)b(z)}, \quad |z| < 1,$$

where $a, b \in H^\infty$ is a pair of functions such that

$$|a|^2 + |b|^2 = 1$$

almost everywhere on the unit circle.

Some ideas of the proof. We will proof only the easy part of the theorem (the "if" part) in the easy case when $|b| \leq \text{const} < 1$. Let us write $|p|^2$ on the unit circle. By a simple algebra, we have

$$|p|^2 = \frac{|a|^2}{|1 - \theta b|^2} = \frac{1 - |b|^2}{|1 - \theta b|^2} = 1 + \frac{\theta b}{1 - \theta b} + \frac{\overline{\theta b}}{1 - \overline{\theta b}}.$$

Now let us multiply this by $|f|^2$ and write the result as

$$|p|^2|f|^2 = |f|^2 + \frac{bf}{1 - \theta b}\theta\overline{f} + \frac{\overline{bf}}{1 - \overline{\theta b}}\overline{\theta} f. \tag{2.1}$$

Consider the second term in the right hand side. By the assumption on b, the term

$$\frac{bf}{1-\theta b}$$

is an element in H^2. Further, since $f \in K_\theta$, we have $\overline{z}\theta\overline{f} \in H^2$, and so $\theta\overline{f}$ is an element in H^2 which vanishes at the origin. It follows that

$$\frac{bf}{1-\theta b}\theta\overline{f}$$

is a function in H^1 which vanishes at the origin. Thus, its integral over the unit circle vanishes. The same considerations apply to the last term in the right hand side of (2.1): its integral over the unit circle vanishes. So, integrating (2.1), we obtain

$$\int_{-\pi}^{\pi} |p(e^{it})|^2 |f(e^{it})|^2 \, dt = \int_{-\pi}^{\pi} |f(e^{it})|^2 \, dt,$$

which means precisely that $\|pf\| = \|f\|$. □

2.4. **Frostman shifts.** Here we address the following question. Let $M = pK_\theta$, where p is an isometric multiplier on K_θ. Are the parameters p, θ unique in the representation $M = pK_\theta$?

It is clear that one can multiply both p and θ by unimodular complex numbers without changing the space pK_θ. It turns out that there is another natural family of transformations on p and θ that leaves the space pK_θ invariant. To begin, consider the example of the previous theorem with both a and b being constants. Changing notation slightly, we see that for every $|\alpha| < 1$, the function $\sqrt{1-|\alpha|^2}/(1-\overline{\alpha}\theta)$ is an isometric multiplier on K_θ.

Exercise. 1. Check that

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta} K_\theta \subset K_{\theta_\alpha}, \quad \theta_\alpha = \frac{\alpha-\theta}{1-\overline{\alpha}\theta}.$$

2. Check that in fact we have the equality

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta} K_\theta = K_{\theta_\alpha}.$$

Hint: use the fact that $(\theta_\alpha)_\alpha = \theta$ and

$$\frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha}\theta_\alpha} = \frac{1-\overline{\alpha}\theta}{\sqrt{1-|\alpha|^2}}.$$

We can rewrite the result of this exercise as follows:

$$K_\theta = g_\alpha K_{\theta_\alpha}, \quad g_\alpha = \frac{1 - \overline{\alpha}\theta}{\sqrt{1 - |\alpha|^2}},$$

and g_α is the isometric multiplier on K_{θ_α}. This transformation is called the *Frostman shift*. From here we see that if p is an isometric multiplier on K_θ, then the space pK_θ can be equivalently written as

$$pK_\theta = pg_\alpha K_{\theta_\alpha},$$

where pg_α is an isometric multiplier on K_{θ_α}.

In fact, the converse statement also holds, see [2]. If

$$pK_\theta = \widetilde{p}K_{\widetilde{\theta}},$$

where p is an isometric multiplier on K_θ and $\widetilde{p}$ is an isometric multiplier on $K_{\widetilde{\theta}}$, then for some constants $|\alpha| < 1$, $|c_1| = 1$, $|c_2| = 1$ we have

$$\widetilde{\theta} = c_1\theta_\alpha, \quad \widetilde{p} = c_2 p g_\alpha.$$

Suppose we have a subspace of the form pK_θ. It is often convenient to perform a Frostman shift with $\alpha = \theta(0)$. Then $\theta_\alpha(0) = 0$ and we write our subspace in an equivalent form $\widetilde{p}K_{\widetilde{\theta}}$ with $\widetilde{\theta}(0) = 0$.

Also, $\theta(0) = 0$ is a convenient normalisation which fixes the choices of θ and p up to unimodular constant factors.

2.5. **Nearly invariant subspaces.** The following definition was introduced by D.Hitt in 1988, see [11]. A closed subspace $M \subset H^2$ is called *nearly S^*-invariant*, if

$$S^*(M \cap \mathbb{1}^\perp) \subset M.$$

In other words, we require that if $f \in M$ and $f(0) = 0$, then $f(z)/z \in M$.

Observe that if $M \neq \{0\}$ is nearly S^*-invariant, then $M \not\perp \mathbb{1}$. Indeed, if $M \perp \mathbb{1}$ and if $f \in M$, then after dividing by z a finite number of times, we must arrive at a function which does not vanish at the origin, which contradicts the assumption $M \perp \mathbb{1}$. Because of this simple observation, the condition that $M \not\perp \mathbb{1}$ is often included in the definition of nearly S^*-invariance.

Theorem 2.5 (Hitt, [11])**.** *Every nearly S^*-invariant subspace M is of the form $M = pN$, where $S^*N \subset N$ and p is an isometric multiplier on N. Thus, we have two possibilities: (i) $M = pK_\theta$, where θ is inner and p is an isometric multiplier on K_θ; (ii) $M = pH^2$, where p is an inner function.*

Some ideas of the proof. 1) Let p be the normalised projection of $\mathbb{1}$ onto M. For $f \in M$, write

$$f = c_0 p + f_1, \quad f_1 \perp p.$$

Since both f and p are in M, we also have $f_1 \in M$. By orthogonality,

$$\|f\|^2 = |c_0|^2 + \|f_1\|^2.$$

Further, $f_1 \perp p$ means $f_1 \perp \mathbb{1}$ and so, by the nearly S^*-invariance, we have $S^* f_1 = f_1/z \in M$. For f_1/z we write again

$$f_1/z = c_1 p + f_2, \quad f_2 \perp p.$$

Then we get

$$\|f_1\|^2 = |c_1|^2 + \|f_2\|^2$$

and again $S^* f_2 \in M$. Continuing recursively, we get

$$f_n/z = c_n p + f_{n+1}, \quad f_{n+1} \perp p,$$

and

$$\|f_n\|^2 = |c_n|^2 + \|f_{n+1}\|^2.$$

Linking these equations together gives

$$f(z) = (c_0 + c_1 z + \cdots + c_n z^n) p(z) + z^n f_{n+1}(z)$$

and

$$\|f\|^2 = |c_0|^2 + \cdots + |c_n|^2 + \|f_{n+1}\|^2 \geq |c_0|^2 + \cdots + |c_n|^2.$$

Inspecting the Taylor series of f/p at zero, we find that

$$f(z)/p(z) = \sum_{n=0}^{\infty} c_n z^n$$

and

$$\|f/p\|^2 = \sum_{n=0}^{\infty} |c_n|^2 \leq \|f\|^2.$$

So the operator $T_{1/p} : M \to H^2$ is a contraction.

2) Consider the set

$$M_0 = \{f \in M : \|f/p\| = \|f\|\}.$$

Exercise. 1. Using the previous step of the proof, prove that M_0 is a linear (linearity is non-trivial!) closed subspace of M.

2. Prove that $T_{1/p}(M_0)$ is S^*-invariant.

3) Using a separate clever calculation with reproducing kernels, Hitt shows that actually $M_0 = M$. So now we have $T_{1/p} M = N$, or $M = pN$, where p is an isometric multiplier on N. □

2.6. **Toeplitz kernels.** As already mentioned, Toeplitz operators satisfy the key commutation relation

$$S^*T_aS = T_a. \tag{2.2}$$

Here we determine the structure of Toeplitz kernels. First we make two remarks:

1) Since I is a Toeplitz operator with the symbol $\mathbb{1}$, we have

$$\operatorname{Ker}(T_a - \lambda I) = \operatorname{Ker} T_{a-\lambda\mathbb{1}}.$$

Thus, describing the structure of Toeplitz kernels is equivalent to describing the structure of all Toeplitz eigenspaces.

2) A deep theorem by M.Rosenblum [16] says that if T_a is a bounded self-adjoint Toeplitz operator with a non-constant symbol a, then the spectrum of T_a is purely absolutely continuous. In particular, T_a has no eigenvalues. This shows that the study of kernels of Toeplitz operators is a specifically non-selfadjoint problem.

Now let us prove Hayashi's theorem on the structure of Toeplitz kernels by using Hitt's theorem. Let T_a be a bounded Toeplitz operator with $\operatorname{Ker} T_a \neq \{0\}$.

1) Let us check that $\operatorname{Ker} T_a$ is nearly S^*-invariant. Suppose $T_a f = 0$ and $f \perp \mathbb{1}$; then $f = SS^*f$. Let us apply the commutation relation (2.2) to S^*f: we get

$$S^*T_aSS^*f = T_aS^*f.$$

The left hand side is $S^*T_af = 0$, hence $S^*f \in \operatorname{Ker} T_a$, as claimed.

2) Since $\operatorname{Ker} T_a$ is a nearly invariant subspace, by Hitt's theorem there are two possibilities: (i) $\operatorname{Ker} T_a = pK_\theta$ where p is an isometric multiplier on K_θ, and (ii) $\operatorname{Ker} T_a = pH^2$, where p is inner. Let us show that the second possibility implies $T_a = 0$. Let $f \in H^2$, and $T_a(pf) = 0$, where p is inner. This means $P(apf) = 0$, i.e.

$$apf \in \overline{z}\overline{H^2}.$$

Since p is inner, this can be equivalently rewritten as

$$af \in \overline{pz}\overline{H^2}.$$

Since $\overline{p}\overline{H^2} \subset \overline{H^2}$, we obtain $f \in \operatorname{Ker} T_a$. Recall that f was an arbitrary element in H^2; this, we get $T_a = 0$.

Example. Let θ be an inner function; consider the Toeplitz operator $T_{\overline{\theta}}$. It is easy to see that in this case

$$\operatorname{Ker} T_{\overline{\theta}} = K_\theta.$$

3. Schmidt subspaces of Hankel operators

3.1. **Preliminaries.** Recall that Hankel operators satisfy the commutation relation

$$S^*H_u = H_uS. \tag{3.1}$$

First we discuss Hankel kernels. If $H_uf = 0$, then by (3.1) we also have

$$0 = S^*H_uf = H_uSf.$$

Thus, Hankel kernels are invariant under the shift operator S, and so by Beurling's theorem they have the form ψH^2 for some inner function ψ. Taking orthogonal complements, we obtain

$$\overline{\operatorname{Ran} H_u} = K_\psi. \tag{3.2}$$

Next, we want to discuss one particular example: Hankel operators with inner symbols. Let θ be inner; consider the Hankel operator H_θ,

$$H_\theta f = P(\theta \overline{f}).$$

It is straightforward to see that in this case we have

$$\operatorname{Ker} H_\theta = z\theta H^2, \quad \operatorname{Ran} H_\theta = K_{z\theta},$$

and H_θ is an anti-linear involution on $K_{z\theta}$,

$$H_\theta f = \theta \overline{f}, \quad f \in K_{z\theta}.$$

It follows that H_θ^2 is the orthogonal projection onto $K_{z\theta}$. In other words, in this case we have only one singular value $s = 1$, and the corresponding Schmidt subspace $E_{H_\theta}(1) = K_{z\theta}$.

3.2. **Main result.**

Theorem 3.1. *Let H_u be a bounded Hankel operator on H^2, and let $s > 0$ be a singular value of H_u:*

$$E_{H_u}(s) = \operatorname{Ker}(H_u^2 - s^2 I) \neq \{0\}.$$

Then there exists an inner function θ and an isometric multiplier p on K_θ such that

$$E_{H_u}(s) = pK_\theta.$$

Before embarking on the proof, we note that $E_{H_u}(s)$ may or may not be nearly S^*-invariant. Indeed, it may happen that $E_{H_u}(s)$ is orthogonal to $\mathbb{1}$.

Example. Let $0 < \alpha < 1$, and let

$$u(z) = \frac{1 - \alpha^2}{1 - \alpha z^2}.$$

Exercise. Check that $H_u u = u$.

Thus, 1 is a singular value, and $u \in E_{H_u}(1)$.

Exercise. Applying S^* to the identity $H_u u = u$, check that $H_u(zu) = \alpha(zu)$.

Thus, α is a singular value, and $zu \in E_{H_u}(\alpha)$.

Exercise. Check that $\operatorname{rank} H_u = 2$. Deduce that

$$E_{H_u}(1) = \operatorname{span}\{u\}, \qquad E_{H_u}(\alpha) = \operatorname{span}\{zu\}.$$

Summarising, we see that $E_{H_u}(\alpha) \perp \mathbb{1}$.

Proof of Theorem 3.1 in the case $E_{H_u}(s) \not\perp \mathbb{1}$.

1) Let us establish some identities. Besides (3.1), we need the rank one identity

$$SS^* = I - \langle \cdot, \mathbb{1}\rangle,$$

and the obvious identity $H_u\mathbb{1} = u$. Using (3.1), we have

$$S^*H_u^2S = H_uSS^*H_u = H_u^2 - \langle \cdot, H_u\mathbb{1}\rangle H_u\mathbb{1} = H_u^2 - \langle \cdot, u\rangle u.$$

(Compare this with the identity $S^*TS = T$ for Toeplitz operators!) Let us multiply the last identity by S^* on the right. After rearranging, we obtain

$$S^*H_u^2 - H_u^2S^* = \langle \cdot, \mathbb{1}\rangle S^*H_uu - \langle \cdot, Su\rangle u. \tag{3.3}$$

2) We need to establish the existence of an element $h \in E_{H_u}(s)$ such that $\langle u, g\rangle \neq 0$. This follows from the assumption $E_{H_u}(s) \not\perp \mathbb{1}$. Indeed, let $h \in E_{H_u}(s)$ be such that $\langle h, \mathbb{1}\rangle \neq 0$; take $g = H_uh$. Then

$$\langle u, g\rangle = \langle u\overline{g}, \mathbb{1}\rangle = \langle H_ug, \mathbb{1}\rangle = \langle H_u^2h, \mathbb{1}\rangle = s^2\langle h, \mathbb{1}\rangle \neq 0.$$

3) Let us prove that $E_{H_u}(s)$ is nearly S^*-invariant. Let $f \in E_{H_u}(s) \cap \mathbb{1}^\perp$, and let g as above. Let us take the bilinear form of (3.3) on the elements f, g. For the left hand side, we have

$$\langle S^*H_u^2f, g\rangle - \langle H_u^2S^*f, g\rangle = s^2\langle S^*f, g\rangle - \langle S^*f, H_u^2g\rangle = s^2\langle S^*f, g\rangle - s^2\langle S^*f, g\rangle = 0.$$

For the right hand side, we have

$$\langle f, \mathbb{1}\rangle\langle S^*H_uu, g\rangle - \langle f, Su\rangle\langle u, g\rangle;$$

by assumption $f \perp \mathbb{1}$, and so we obtain

$$\langle f, Su\rangle\langle u, g\rangle = 0.$$

But $\langle u, g\rangle \neq 0$, and so we obtain that $\langle f, Su\rangle = 0$.

Now let us substitute f back into (3.3) and use the latter fact; the right hand side vanishes and we have

$$S^*H_u^2f - H_u^2S^*f = 0.$$

Since $H_u^2f = s^2f$, this can be rewritten as

$$(H_u^2 - s^2I)S^*f = 0,$$

and so $S^*f \in E_{H_u}(s)$, as claimed.

Thus, $E_{H_u}(s)$ is a nearly S^*-invariant subspace.

4) By Hitt's theorem, either $E_{H_u}(s) = pK_\theta$ (where p is an isometric multiplier on K_θ) or $E_{H_u}(s) = pH^2$ (where p is inner).

Exercise. Show that the second option is not possible. Use the fact that the calculation from the previous step of the proof shows that

$$S^*(E_{H_u}(s) \cap \mathbb{1}^\perp) \subset E_{H_u}(s) \cap u^\perp.$$

Compare this with

$$S^*(pH^2 \cap \mathbb{1}^\perp) = pH^2$$

if p is inner. Bring this to a contradiction.

This completes the proof. $\square$

Proof of Theorem 3.1 in case $E_{H_u}(s) \perp \mathbb{1}$. Let α, $|\alpha| < 1$, be such that α is not a common zero of all elements of $E_{H_u}(s)$. Let μ be the Moebius map

$$\mu(z) = \frac{\alpha - z}{1 - \overline{\alpha} z},$$

mapping the unit disk onto itself, and let U_μ be the corresponding unitary operator on H^2:

$$U_\mu f(z) = \frac{\sqrt{1 - |\alpha|^2}}{1 - \overline{\alpha} z} f(\mu(z)).$$

By a direct calculation, U_μ is a unitary involution on H^2.

Exercise. Check that $U_\mu H_u U_\mu = H_w$ with some symbol w.

Now consider $M = U_\mu E_{H_u}(s)$. Then $M = E_{H_w}(s)$ and $z = 0$ is not a common zero of all elements of M, i.e. $M \not\perp \mathbb{1}$. By the previous part of the proof, it follows that $M = pK_\theta$ with some θ, and p is an isometric multiplier on K_θ. Now

$$E_{H_u}(s) = U_\mu(pK_\theta).$$

Exercise. Check that

$$U_\mu(pK_\theta) = (p \circ \mu) K_{\theta \circ \mu},$$

and $p \circ \mu$ is an isometric multiplier on $K_{\theta \circ \mu}$.

This completes the proof. $\square$

In the remainder of this section, we will discuss (mostly without proofs) some consequences of Theorem 3.1 and some related statements.

3.3. **The action of H_u on $E_{H_u}(s)$.** In fact, by the same method we obtain not only the formula for the subspace $E_{H_u}(s)$ but also the formula for the action of H_u on this subspace.

Assume the hypothesis of Theorem 3.1; we have

$$E_{H_u}(s) = pK_\theta,$$

where p is an isometric multiplier on K_θ. By performing a Frostman shift, we can always make sure that $\theta(0) = 0$. In this case, we will write $z\theta$ instead of θ.

Theorem 3.2. *Assume the hypothesis of Theorem 3.1, and let*

$$E_{H_u}(s) = pK_{z\theta},$$

where θ is an inner function and p is an isometric multiplier on $K_{z\theta}$. Then, for some unimodular constant $e^{i\gamma}$, the action of H_u on $E_{H_u}(s)$ is given by the following formula:

$$H_u(pf) = se^{i\gamma} p H_\theta f = se^{i\gamma} \theta \overline{f}, \quad f \in K_{z\theta}. \tag{3.4}$$

The constant $e^{i\gamma}$ depends on the normalisation of p and θ; in particular, we can normalise p and θ so that $e^{i\gamma} = 1$, in which case formula (3.4) becomes particularly simple:

$$H_u(pf) = spH_\theta f, \quad f \in K_{z\theta}.$$

One can also write this formula in operator theoretic terms as

$$H_u T_p = sT_p H_\theta, \quad \text{ on } K_{z\theta}.$$

Thus, the operator T_p intertwines the action of H_u on $E_{H_u}(s)$ and the action of the standard involution $f \mapsto \theta\overline{f}$ on $K_{z\theta}$.

3.4. **Decompositions of model spaces.** Let H_u be a finite rank Hankel operator. Recall that by (3.2), we have

$$\operatorname{Ran} H_u = K_\psi$$

for some inner function ψ. Suppose that H_u has singular values $s_1, \dots, s_N$, and the corresponding Schmidt subspaces are represented as

$$E_{H_u}(s_n) = p_n K_{z\theta_n},$$

where p_n is an isometric multiplier on $K_{z\theta_n}$. Thus, we arrive at an interesting orthogonal decomposition of the model space K_ψ:

$$K_\psi = \bigoplus_{n=1}^{N} p_n K_{z\theta_n}. \tag{3.5}$$

3.5. **The Adamyan-Arov-Krein theorem.** Here we briefly discuss the Adamyan-Arov-Krein (AAK) theorem, which gives additional information about the inner factors of p_n in formula (3.5). The theorem below is essentially due to AAK in [1]; however it was expressed there in a different form, since Theorem 3.1 was not available to AAK at that point. The precise statement below is from [7].

Theorem 3.3. *Assume the hypothesis of Theorem 3.1, and let*

$$E_{H_u}(s) = pK_{z\theta},$$

where θ is an inner function and p is an isometric multiplier on $K_{z\theta}$. Then the degree of the inner factor of p equals the total multiplicity of the spectrum of H_u^2 in the open interval (s^2, ∞).

To illustrate this, let us come back to the decomposition (3.5) and write $p_n = q_n\varphi_n$, where q_n is outer and φ_n is inner:

$$K_\psi = \bigoplus_{n=1}^{N} q_n \varphi_n K_{z\theta_n}.$$

Assume that the singular values have been ordered as $s_1 > s_2 > \cdots > s_N > 0$, and let $q_n \varphi_n K_{z\theta_n}$ correspond to the singular value s_n. As the dimension of $K_{z\theta_n}$ is $\deg(z\theta_n)$, we have for every $n = 1, \dots, N$

$$\deg \varphi_n = \sum_{k=1}^{n-1} \deg(z\theta_k).$$

In particular, for $n = 1$ this formula says that p_1 is an outer function. Below we prove the theorem in this particular case (following [1]). The general case is much more difficult.

Proof of Theorem 3.3 for the top singular value. Assume $s = \|H_u\|$; let us prove that p is outer.

1) First we observe that in this case $f \in E_{H_u}(s)$ iff $\|H_u f\| \geq s\|f\|$.

2) Let $f = af_0 \in E_{H_u}(s)$, where a is inner and $f_0 \in H^2$. We have

$$s\|f_0\| = s\|f\| = \|H_u f\| = \|P(u\overline{a}\overline{f_0})\| = \|P(\overline{a}u\overline{f_0})\|.$$

Observe that $P\overline{a}(I - P) = 0$ and therefore

$$P(\overline{a}u\overline{f_0}) = P(\overline{a}P(u\overline{f_0})).$$

Thus,

$$\|P(\overline{a}u\overline{f_0})\| = \|P(\overline{a}P(u\overline{f_0}))\| = \|P(\overline{a}H_u f_0)\| \leq \|H_u f_0\|.$$

We conclude that $s\|f_0\| \leq \|H_u f_0\|$, and therefore $f_0 \in E_{H_u}(s)$.

3) Suppose $p = q\varphi$, where φ is inner and q is outer. As $p \in E_{H_u}(s)$, by the previous step we have $q \in E_{H_u}(s)$. Then $q = ph$, $h \in K_{z\theta}$; so $q = q\varphi h$, and we conclude that $\varphi = \text{const}$. □

3.6. **Inverse spectral problems.** Finally, without going into details, we would like to show how to the parameterisation of Schmidt subspaces in terms of model spaces can be used in inverse spectral problems. The following facts are borrowed from [9]. Let $N \in \mathbb{N}$, let

$$s_1 > \widetilde{s}_1 > s_2 > \widetilde{s}_2 > \cdots > s_N > \widetilde{s}_N \geq 0$$

be real numbers, and let $\psi_1, \dots, \psi_N$ be any inner functions. For $z \in \mathbb{D}$, consider the $N \times N$ matrix

$$C(z) = \left\{ \frac{s_j - z\widetilde{s}_k \psi_j(z)}{s_j^2 - \widetilde{s}_k^2} \right\}_{j,k=1}^{N}$$

and the vectors in $\mathbb{C}^N$,

$$\Psi = \begin{pmatrix} \psi_1(z) \\ \vdots \\ \psi_N(z) \end{pmatrix}, \qquad \mathbf{1} = \begin{pmatrix} \mathbb{1} \\ \vdots \\ \mathbb{1} \end{pmatrix}.$$

Then (this is a non-trivial fact!) the matrix $C(z)$ is invertible for all $z \in \mathbb{D}$ and the function

$$u(z) = \langle C(z)^{-1}\Psi, \mathbf{1}\rangle_{\mathbb{C}^N}$$

is in H^∞. Consider the Hankel operator H_u. Then H_u has the singular values $\{s_j\}_{j=1}^N$ (and no others), and each Schmidt subspace of H_u can be represented as

$$E_{H_u}(s_j) = p_j K_{z\psi_j} \tag{3.6}$$

with some isometric multipliers p_j (which can also be written explicitly in terms of s_j, ψ_j).

Remark. (1) The numbers $\widetilde{s}_j$ are arbitrary parameters in this construction. In fact, they coincide with the singular values of the "associated Hankel operator" H_{S^*u}.

(2) In this construction, all Schmidt subspaces $E_{H_u}(s_j)$ are nearly S^*-invariant, i.e. none of them is orthogonal to $\mathbb{1}$. It is however possible to modify this construction so that H_u has also some Schmidt subspaces orthogonal to $\mathbb{1}$.

(3) The above construction gives *all* possible Hankel operators with the singular values $\{s_j\}_{j=1}^N$ (and no others), satisfying (3.6) and satisfying the additional requirement that $E_{H_u}(s_j) \not\perp \mathbb{1}$ for all j. In other words,

$$\big(\{s_j\}_{j=1}^N, \{\widetilde{s}_k\}_{k=1}^N, \{\psi_j\}_{j=1}^N\big)$$

is a complete independent set of spectral data for this inverse problem.

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