INTRODUCTION TO SPECTRAL THEORY OF HANKEL AND TOEPLITZ OPERATORS

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ABSTRACT. These are preliminary notes of the lecture course to be given at LTCC in 2015. The plan for the course is to consider the following three classes of operators: Toeplitz and Hankel operators on the Hardy space on the unit circle and Toeplitz operators on the Bergman space on the unit disk. For each of these three classes of operators, we consider the following questions: boundedness and estimates or explicit expressions for the norm; compactness; essential spectrum; operators of the finite rank.

1. Introduction

1.1. $L^p(T)$ spaces. We denote by $T$ the unit circle on the complex plane, parameterised by $e^{i\theta}$, $\theta \in (-\pi, \pi]$ and equipped with the normalised Lebesgue measure $d\theta/2\pi$. Elements $f$ of $L^p(T)$ can be written either as $f(z)$, $|z| = 1$ or as $f(e^{i\theta})$, $|\theta| < \pi$. The norm of $f$ in $L^p(T)$ will be denoted by $\|f\|_p$,

$$\|f\|_p^p = \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi}. $$

We will be mostly interested in the space $L^2(T)$, which is a Hilbert space. In the case $p = 2$, we will drop the subscript of the norm: $\|f\| = \|f\|_2$. The set $\{z^n\}_{n \in \mathbb{Z}}$ is an orthonormal basis in $L^2$, so every $f$ can be represented as

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n, \quad z \in T,$$

where $\{\hat{f}(n)\}_{n \in \mathbb{Z}}$ is the (two-sided) sequence of the Fourier coefficients of $f$,

$$\hat{f}(n) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}. $$

We have the Parseval identity

$$\|\hat{f}\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \|f\|_{L^2(T)}^2.$$
1.2. **The harmonic extension.** The harmonic extension of \( f \in L^1(\mathbb{T}) \) is a function of \( z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), defined by

\[
 f(z) = \sum_{n \geq 0} \hat{f}(n) z^n + \sum_{n \geq 1} \hat{f}(-n) \overline{z}^n = \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|1 - \overline{z} e^{i\theta}|^2} f(e^{i\theta}) \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.
\]

The integral kernel above can be written as the series

\[
 \frac{1 - |z|^2}{|1 - \overline{z} e^{i\theta}|^2} = \sum_{n \geq 0} e^{-i\theta} z^n + \sum_{n \geq 1} e^{i\theta} \overline{z}^n.
\]

In particular, for \( 0 < r < 1 \), we will denote

\[
 f_r(e^{i\theta}) = f(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) \frac{d\theta}{2\pi},
\]

where \( P_r(\theta) \) is the Poisson kernel,

\[
 P_r(\theta) = \frac{1 - r^2}{|1 - re^{i\theta}|^2} = \sum_{n \geq 0} r^n e^{i\theta} + \sum_{n \geq 1} r^n e^{-i\theta}.
\]

The Poisson kernel satisfies the following properties:

(i) \( P_r(\theta) > 0 \) for all \( 0 < r < 1 \) and all \( |\theta| \leq \pi \);

(ii) \( \int_{-\pi}^{\pi} P_r(\theta) d\theta / 2\pi = 1 \) for all \( r \);

(iii) for any \( \delta > 0 \), \( \int_{|\theta|>\delta} P_r(\theta) d\theta \to 0 \) as \( r \to 1 \).

**Proposition 1.1.** The map \( f \mapsto f_r \) is a contraction (i.e. has norm \( \leq 1 \)) on \( L^p(\mathbb{T}) \) for all \( 1 \leq p \leq \infty \). Further, we have

\[
 \| f_r - f \|_p \to 0, \quad r \to 1, \quad \forall f \in L^p(\mathbb{T}), \quad p < \infty. \tag{1.1}
\]

For \( f \in C(\mathbb{T}) \), the convergence \( f_r \to f \) is uniform on \( \mathbb{T} \).

The proof is outlined in exercises.

There are also results about almost-everywhere pointwise convergence \( f_r \to f \), but they are more advanced and we will not need them.

1.3. **The Hardy classes.** For \( 1 \leq p \leq \infty \), the Hardy class \( H^p_+ = H^p_+(\mathbb{T}) \) is defined as

\[
 H^p_+(\mathbb{T}) = \{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \quad n < 0 \}. \tag{1.2}
\]

We will only need the cases \( p = 1, 2, \infty \). One also defines

\[
 H^p_+(\mathbb{T}) = \{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0, \quad n \geq 0 \}.
\]

There is a lack of complete symmetry between \( H^p_+ \) and \( H^p_-(\mathbb{T}) \), since the constant function belongs to \( H^p_+ \) but not to \( H^p_-(\mathbb{T}) \).

The functions \( f \in H^p_+ \) have a natural analytic extension into the unit disk,

\[
 f(z) = \sum_{n \geq 0} \hat{f}(n) z^n = \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1 - z e^{-i\theta}} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.
\]
This formula can be written as
\[ f(z) = (f, k_z), \quad k_z(e^{i\theta}) = \frac{1}{1 - \overline{z} e^{i\theta}}, \]
where \((\cdot, \cdot)\) is the inner product in \(L^2(T)\), and \(k_z\) is called the reproducing kernel.

We will often identify functions \(f(e^{i\theta})\) on the unit circle with their analytic extensions \(f(z)\) on the unit disk.

It is clear that for \(f \in H^p_+\), its harmonic extension coincides with the analytic extension. We will use the same notation \(f_+(e^{i\theta}) = f(re^{i\theta})\) for the analytic extension. As a corollary of (1.1), we have
\[ \|f_r - f\|_p \to 0, \quad r \to 0, \quad \forall f \in H^p_+, \quad p < \infty. \tag{1.3} \]

Consider the case \(p = 2\). The Hardy space \(H^2_+\) inherits the Hilbert space structure from \(L^2\). We will use the orthogonal projection
\[ P_+: L^2(T) \to H^2_+(T), \quad \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n \mapsto \sum_{n \geq 0} \hat{f}(n) z^n. \]

It is easy to see that for \(f \in H^p_+\), the norm \(\|f_r\|_p\) is monotone increasing in \(p\) (see exercises). Using this and (1.1), we obtain
\[ \|f\|_p = \lim_{r \to 1} \|f_r\|_p, \quad p < \infty. \tag{1.4} \]

In fact, this is also true for \(p = \infty\). The above relation is often used as an alternative definition of the \(H^p_+\) classes; a function \(f\) analytic in \(D\) is said to belong to \(H^p_+, \quad p < \infty\), if
\[ \sup_{r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty. \tag{1.5} \]

The space \(H^\infty(T)\) can be alternatively defined as the space of all bounded analytic functions on the unit disk. Relation (1.4) shows that any function that belongs to \(H^p_+\) in the sense of (1.2), also satisfies (1.5). The converse implication is not as straightforward, unless \(p = 2\) (see exercises).

1.4. The Bergman space. Let \(L^2(\mathbb{D})\) be the space of all square integrable functions on the unit disk, equipped with the standard norm,
\[ \|f\|_{L^2}^2 = \int_{\mathbb{D}} |f(z)|^2 d\mathcal{A}(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_0^1 |f(re^{i\theta})|^2 r dr d\theta; \]

here \(d\mathcal{A}(z)\) is the normalised 2-dimensional Lebesgue measure on the unit disk. The Bergman space \(A^2 = A^2(\mathbb{D})\) is the subspace of \(L^2(\mathbb{D})\) which consists of analytic functions. One can analogously define the spaces \(A^p(\mathbb{D})\) for all \(p\), but we will not need them. We will use the orthogonal projection
\[ \Pi_+: L^2(\mathbb{D}) \to A^2, \]
sometimes called the Bergman projection. The set \(\{(n+1)^{-1/2} z^n\}_{n=0}^\infty\) is an orthonormal basis in \(A^2\); we will call it the standard basis. Using the standard basis,
it is easy to see that the projection $\Pi_+$ can be expressed via the Bergman kernel $k_w(z) = (1 - z\overline{w})^{-2}$:

$$(\Pi_+ f)(w) = \sum_{n \geq 0} (n + 1)w^n \int_{\mathbb{D}} f(z)z^n dA(z) = \int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}w)^2} dA(z).$$

1.5. **The multiplication operators.** Let $a \in L^\infty(\mathbb{T})$. We denote by $M(a)$ the multiplication operator on $L^2(\mathbb{T})$:

$$(M(a)f)(z) = a(z)f(z), \quad z \in \mathbb{T}.$$  

We will call $a$ the symbol of $M(a)$. It is clear that $M(a)$ is bounded, and $\|M(a)\| \leq \|f\|_\infty$ (we will soon see that in fact $\|M(a)\| = \|f\|_\infty$). It is also obvious that $M(a)^* = M(\overline{a})$, and in particular $M(a)$ is self-adjoint for real-valued symbols.

The Fourier transform maps $M(a)$ into a discrete convolution operator in $\ell^2(\mathbb{Z})$:

$$(M(a)f)_n = \int_{-\pi}^{\pi} a(e^{i\theta})f(e^{i\theta})e^{-im\theta} \frac{d\theta}{2\pi} = \sum_{m \in \mathbb{Z}} \hat{f}(m) \int_{-\pi}^{\pi} a(e^{i\theta})e^{i(m-n)\theta} \frac{d\theta}{2\pi} = \sum_{m \in \mathbb{Z}} \hat{a}(n-m)\hat{f}(m). \quad (1.6)$$

Similarly, let $a \in L^\infty(\mathbb{D})$; one can define the multiplication operator $M(a)$ on $A^2$ by

$$(M(a)f)(z) = a(z)f(z), \quad z \in \mathbb{D}.$$  

Again, we have the obvious estimate $\|M(a)\| \leq \|f\|_\infty$ (in fact, $\|M(a)\| = \|f\|_\infty$). In contrast to (1.6), in general there is no simple matrix representation for multiplication operators on $L^2(\mathbb{D})$, because there is no simple basis on $L^2(\mathbb{D})$.

1.6. **Toeplitz operators on the Hardy space.** For a symbol $a \in L^\infty(\mathbb{T})$, the **Toeplitz operator** $T(a)$ on $H^2_+(\mathbb{T})$ is defined as

$$T(a)f = P_+ M(a)f = P_+(af), \quad f \in H^2_+(\mathbb{T}).$$

$T(a)$ is sometimes called Hardy-Toeplitz operator. It is clear that $T(a)$ is bounded and

$$\|T(a)\| \leq \|a\|_\infty.$$  

The Fourier transform maps $T(a)$ onto the class of matrix **Toeplitz operators** on $\ell^2(\mathbb{Z}_+)$; these are infinite matrices of the type $\{a_{n-m}\}_{n,m \geq 0}$. Indeed, as a consequence of (1.6), we have

$$(T(a)f)_n = \sum_{m \geq 0} \hat{a}(n-m)\hat{f}(m).$$

Clearly, $T(a)^* = T(\overline{a})$ and so, in particular, $T(a)$ is self-adjoint for real-valued symbols $a.$
1.7. Toeplitz operators on the Bergman space. For a symbol $a \in L^\infty(\mathbb{D})$, the Toeplitz operator $T(a)$ on $A^2(\mathbb{D})$ is defined as

$$T(a)f = \Pi_+ M(a)f = \Pi_+(af), \quad f \in A^2(\mathbb{D}).$$

$T(a)$ is sometimes called Bergman-Toeplitz operator. It is clear that $T(a)$ is bounded and

$$\|T(a)\| \leq \|a\|_\infty.$$

Bergman-Toeplitz operator can be represented by an infinite matrix in the standard basis $\{(n+1)^{1/2}z^n\}_{n \geq 0}$ of the Bergman space:

$$a_{nm} = (n+1)^{1/2}(m+1)^{1/2}(T(a)z^n, z^m)_{L^2(\mathbb{D})}$$

$$= (n+1)^{1/2}(m+1)^{1/2} \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{0}^{1} a(re^{i\theta})r^{n+m+1}e^{i(n-m)\theta} dr d\theta.$$

In contrast to the Hardy-Toeplitz case, the matrix $\{a_{nm}\}_{n,m \geq 0}$ in general does not have any simple structure. Important interesting particular cases when the structure of this matrix simplifies, are

- $a(z) = a(|z|)$ — radial symbols;
- $a(z) = a_1(r)a_2(e^{i\theta}), \quad z = re^{i\theta}$ — factorizable symbols.

1.8. Hankel operators on the Hardy space. Let $J$ on $L^2(\mathbb{T})$ be the involution:

$$(Jf)(e^{i\theta}) = f(e^{i\theta}).$$

If $f \in H^2_+$ and $\hat{f}(0) = 0$, then $Jf \in H^2_+$. For a symbol $a \in L^\infty(\mathbb{T})$, the Hankel operator $H(a)$ on $H^2_+$ is defined by

$$H(a)f = P_+ M(a)Jf = P_+(aJf), \quad f \in H^2_+(\mathbb{T}).$$

Clearly, we have the norm bound

$$\|H(a)\| \leq \|a\|_\infty.$$

The Fourier transform unitarily maps Hankel operators into the class of matrix Hankel operators on $\ell^2(\mathbb{Z}_+)$; these are matrices of the type $\{a_{n+m}\}_{n,m \geq 0}$. Indeed,

$$(H(a)z^n, z^m) = (P_+ aJz^n, z^m) = (az^{-n}, z^m) = \hat{a}(n + m).$$

It is easy to see that $H(a)^* = H(\hat{a})$, where $a_*(e^{i\theta}) = \overline{a(-e^{-i\theta})}$. In particular, symbols that satisfy the symmetry condition $a = a_*$ generate self-adjoint Hankel operators.

Remark. There are also two types of Hankel operators studied on Bergman space: “big Hankel” and “little Hankel”. We will not discuss these in this course.
1.9. **The aim of the course.** For each of the above four classes of operators (multiplication operators, Hardy-Toeplitz, Bergman-Toeplitz, Hankel), we will address the following questions.

- Is the symbol uniquely defined by the operator?
- What are the sufficient (necessary?) conditions in terms of the symbol for the boundedness of the operator? Is there a simple expression for the norm?
- What are the sufficient (necessary?) conditions for the operator to be compact? To be trace class? To be of the finite rank?
- Is there a simple description of the spectrum of the operator?
- For non-compact operators, is there a simple description of the essential spectrum?

The multiplication operators is the simplest class of all four; for multiplication operators, complete answers to all of the above questions are readily available. For the other three classes, some of the questions above turn out to be very non-trivial.

1.10. **Exercises.**

**Exercise 1.1.** Using the properties (i), (ii) of the Poisson kernel, prove that the map $f \mapsto f_r$ is a contraction in $L^p(\mathbb{T})$ for all $1 \leq p \leq \infty$. *Hint:* for $p = 1$ and $p = \infty$ this is a straightforward calculation. For $1 < p < \infty$, let $q$ be the dual exponent, $1/p + 1/q = 1$; using the Hölder inequality, prove the estimate

$$\left| \int_{-\pi}^{\pi} f_r(e^{i\theta}) g(e^{i\theta}) \frac{d\theta}{2\pi} \right| \leq \|f\|_p \|g\|_q,$$

and use the fact that $L^q$ is the dual space to $L^p$.

**Exercise 1.2.** Using the properties (i)–(iii) of the Poisson kernel, prove that for any $f \in C(\mathbb{T})$, we have $\|f_r - f\|_\infty \to 0$ as $r \to 1$. *Hint:* write

$$f_r(e^{i\theta}) - f(e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)(f(e^{it}) - f(e^{i\theta})) \frac{dt}{2\pi}.$$  

Split the interval of integration into the one where $|\theta - t| < \delta$ and its complement. Use the uniform continuity of $f$ and the property (iii) to estimate each of the two integrals resulting from this split.

**Exercise 1.3.** Using the above two exercises, prove (1.1). (*Hint:* $C$ is dense in $L^p$ for $p < \infty$.) Show that (1.1) is false for $p = \infty$. (*Hint:* $f_r$ is continuous for all $r < 1$.)

**Exercise 1.4.** Let $f \in L^p(\mathbb{T})$. Prove that $\|f_r\|_p$ is monotone increasing in $r$. *Hint:* $(f_{r_1})_{r_2} = f_{r_1 \cdot r_2}$.
Exercise 1.5. Let $f$ be a function analytic in the unit disk, satisfying (1.5) with $p = 2$. Prove that $f \in H^2_+$. Hint: write the Taylor series of $f$,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

and using (1.5), prove that the sequence $\{c_n\}_{n=0}^{\infty}$ is in $\ell^2$.

2. Multiplication operators

Here we consider the multiplication operators $M(a)$ on $L^2(\mathbb{T})$ and on $L^2(\mathbb{D})$.

2.1. Uniqueness of the symbol. Let us prove that the symbol $a$ is uniquely determined by the operator $M(a)$, i.e. $M(a_1) = M(a_2)$ implies $a_1 = a_2$ almost everywhere. It suffices to prove that $M(a) = 0$ implies $a = 0$ almost everywhere. Suppose to the contrary that $a \neq 0$ on some set $E \subset \mathbb{T}$ (resp. $E \subset \mathbb{D}$) of a positive Lebesgue measure. Let $\chi_E$ be the characteristic function of $E$. Then $\|M(a)\chi_E\| \neq 0$, contrary to the assumption $M(a) = 0$. This contradiction proves the claim.

2.2. The norm. The norm bound

$$\|M(a)\| \leq \|a\|_{\infty},$$

is obvious. Let us prove the opposite inequality:

$$\|a\|_{\infty} \leq \|M(a)\|.$$ 

Let’s assume $\|a\|_{\infty} > 0$ (otherwise there is nothing to prove). For any sufficiently small $\varepsilon > 0$, there exists a set $E_\varepsilon \subset \mathbb{T}$ (resp. $E_\varepsilon \subset \mathbb{D}$) of a positive Lebesgue measure such that $|a(z)| > \|a\|_{\infty} - \varepsilon$ for $z \in E_\varepsilon$. Now take $f = \chi_{E_\varepsilon}$; then

$$|(M(a)f)(z)| \geq (\|a\|_{\infty} - \varepsilon)|f(z)|,$$

and so $\|M(a)f\| \geq (\|a\|_{\infty} - \varepsilon)\|f\|$. It follows that $\|M(a)\| \geq \|a\|_{\infty} - \varepsilon$; since $\varepsilon$ can be taken arbitrarily small, we obtain $\|a\|_{\infty} \leq \|M(a)\|$.

By a similar argument, it is easy to see that the inclusion $a \in L^\infty$ is not only sufficient, but also necessary condition for the boundedness of $M(a)$.

2.3. Non-compactness. Let us prove that the operator $M(a)$ is compact only if $a = 0$ almost everywhere.

The proof is very easy in the case of the operator of multiplication in $L^2(\mathbb{T})$. If $M(a)$ is compact, then it maps weakly convergent sequences to strongly convergent ones. Consider the sequence $f_n(z) = z^n$, $n \in \mathbb{N}$ in $L^2(\mathbb{T})$; this sequence converges weakly to zero, and $\|M(a)f_n\|_2 = \|a\|_2 \neq 0$ unless $a = 0$.

Consider the case of the operator in $L^2(\mathbb{D})$. Suppose $a \neq 0$; then there exists $\varepsilon > 0$ and a set $E \subset \mathbb{D}$ of a positive Lebesgue measure such that $|a(z)| > \varepsilon$ for all $z \in E$. Let us choose a disjoint sequence of sets $E_n \subset E$ of a positive Lebesgue
measure. Consider \( f_n = \chi_{E_n}/\|\chi_{E_n}\| \). Then \( f_n \) converges weakly to zero (exercise). But we have
\[
\|M(a)f_n\|^2 = \frac{1}{|E_n|} \int_{E_n} |a(x)|^2 \geq \varepsilon
\]
(here \( |E_n| \) is the Lebesgue measure of \( E_n \)). So \( M(a)f_n \) does not converge strongly to zero, which contradicts the compactness of \( M(a) \).

2.4. The spectrum of \( M(a) \). Let \( \mathcal{H} \) be a separable Hilbert space; we denote by \( \mathcal{B}(\mathcal{H}) \) the set of all bounded linear operators on \( \mathcal{H} \).

For \( M \in \mathcal{B}(\mathcal{H}) \), the spectrum of \( M \), denoted by \( \sigma(M) \), is the set of all \( \lambda \in \mathbb{C} \) such that the operator \( M - \lambda = M - \lambda I \) is not invertible. Recall that \( M - \lambda \) is invertible if and only if
\[
\text{Ker}(M - \lambda) = \{0\} \quad \text{and} \quad \text{Ran}(M - \lambda) = \mathcal{H}.
\]
The ‘only if’ part is trivial, and the ‘if’ part is the consequence of the deep Banach inverse map theorem.

If \( \lambda \in \sigma(M) \), then at least one of the following is true:
- \( \text{Ker}(M - \lambda) \neq \{0\} \), i.e. \( \lambda \) is an eigenvalue;
- \( \text{Ran}(M - \lambda)^\perp \neq \{0\} \), i.e. \( \text{Ran}(M - \lambda) \) is not dense in \( \mathcal{H} \);
- \( \text{Ran}(M - \lambda) \) is not closed.

In the finite dimensional case \( \text{dim} \mathcal{H} < \infty \), we have \( \lambda \in \sigma(M) \) if and only if \( \text{Ker}(M - \lambda) \neq \{0\} \); in the infinite dimensional case, the situation is more complex.