STUDY GROUP ON p-ADIC HODGE THEORY

1. What is p-adic Hodge theory?

The objects of study of p-adic Hodge theory are continuous p-adic representations of absolute Galois groups of p-adic fields. One of the first results in this direction was due to Tate [4] where he proved that if A is an abelian variety over $\mathbb{Q}_p$, then the Galois representation $H^1_{\text{et}}(A, \mathbb{Q}_p) \otimes \mathbb{C}_p$ has a decomposition matching that given by Hodge theory. This result was later generalised by Faltings to $H^n_{\text{et}}(X, \mathbb{Q}_p) \otimes \mathbb{C}_p$, where X is any smooth proper scheme over $\mathbb{Q}_p$.

In p-adic Hodge theory, one associates to such p-adic Galois representations certain vector spaces enriched with some extra structure. Once we have this structure, we can specify nice constraints on these structures and in doing so cut out certain Galois representations whose associated spaces have these properties. What is remarkable is that these conditions can be chosen so that the corresponding Galois representations encapsulate important concepts coming from geometry. One example is that of crystalline representations, which play a similar role in p-local p-adic (i.e., $\ell = p$) Galois representations of abelian varieties to the role played by unramified representations in the case of ℓ-adic representations when $\ell \neq p$ (i.e., an abelian variety A over a p-adic field K has good reduction if and only if $H^1_{\text{et}}(A, \mathbb{Q}_p)$ is crystalline). Working with these enriched vector spaces is often easier than working with the Galois representations themselves and understanding what kind of enriched vector spaces arise from certain classes of p-adic Galois representations will ultimately help us understand Galois representations.

2. Plan of the talks

The aim of this study group is to give a (mostly proofless) introduction to the fundamental concepts and results of p-adic Hodge theory. The main sources are going to be [4] and parts I and II of [1].

The study group will run every Wednesday (starting on the 15th of January and ending on the 25th of March) from 12pm to 1.30pm. We will meet in room 505 of UCL's Mathematics department, except on the 18th of March, where we will meet in room G03, 26 Bedford Way.

(1) **Introduction and p-divisible groups I** (15/01, Alex)
   (a) Introduction
   (b) Definition of p-divisible groups
   (c) Examples
(2) **p-divisible groups II** (22/01, Pedro)
   (a) Theorems of Tate on p-divisible groups
   (b) Dieudonné modules and Honda systems

(3) **Faltings's theorem and Tate–Sen theorem** (29/01, Lambert)
Source: [1], from section 1.2 to 2.2.
(a) Definition of p-adic Galois representation
The objective is to define p-adic fields, p-adic Galois representations and $\mathbb{C}_K$-representations, where K is a p-adic field.
(b) Faltings's Hodge type decomposition of $H^n_{\text{et}}(X, \mathbb{Q}_p)$
This is theorem 2.2.3 in [1].
(c) Tate–Sen theorem
This is theorem 2.2.7 in [1].

(4) **Hodge–Tate representations** (05/02, Ambrus)
Source: [1], section 2.3.
(a) Serre–Tate theorem
This is theorem 2.3.1 in [1] (maybe prove it?).
(b) Definitions of Hodge–Tate representation and Hodge–Tate weights
(c) Examples
Cover examples 2.3.3 and 2.3.5.

(5) **Formalism of Hodge–Tate representations** (12/02, Ambrus)
Source: [1], section 2.4.
(a) Definitions of B_{HT} and D_{HT}
This B_{HT} is the first example of period ring we will see and the only one that we are actually going to define. Later, we will mention other period rings, but will content ourselves with stating some of their main properties.
(b) The comparison morphism and Theorem 2.4.11.
A more general version of this result will be stated in the next talk.

(6) **Formalism of admissible representations** (19/02, Pol)
Source: [1], section 5.
(a) Definition of (F, G)-regular domain
Apart from the definition, it should be shown that B_{HT} is a $(\mathbb{Q}_p, G_K)$-regular domain. We will meet other two $(\mathbb{Q}_p, G_K)$-regular domains: B_{dR} and B_{cris}.
(b) Definition of B-admissibility
This is how the classes of p-adic representations are specified: they are the B-admissible classes for some period ring B. So, for instance, the Hodge–Tate representations are precisely the B_{HT}-admissible representations, the de Rham representations are the B_{dR}-admissible representations, and the crystalline representations are the B_{cris}-admissible representations.
(c) The comparison morphism and Theorem 5.2.1

(7) **de Rham representations** (26/02, Gabriel)
Source: [1], sections 4 and 6; [3], theorem 3.1.4 in page 56.
(a) Main properties of B^+_{dR} and B_{dR}
The aim is *not* to define either of these rings. Some of the properties that should be mentioned are the ones pointed out in Proposition 4.4.6, that it canonically contains a copy of K_0, the fact that B^+_{dR} has a natural action of

G_K on it, that B_{dR}^+ contains a uniformiser t, canonical up to $\mathbb{Z}_p^\times$-multiple, on which G_K acts via the cyclotomic character. Then B_{dR} is defined to be the fraction field of B_{dR}^+, and it should be shown that $\mathrm{Gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ and that B_{dR} is $(\mathbb{Q}_p, G_K)$-regular. The insensitivity of the construction of B_{dR} to the replacement of K by a finite extension or by $K^{\hat{\mathrm{unr}}}$ should also be mentioned (this is the final paragraph of section 4).

(b) Discussion about de Rham representations
de Rham representations are then defined to be the B_{dR}-admissible p-adic representations. Note that the natural filtration on B_{dR} induces a filtration on $D_{\mathrm{dR}}(V)$. One of the things that could be shown is that de Rham representations are Hodge–Tate (assuming properties of B_{dR}). Note that the $\mathbb{Q}_p(n)$ are de Rham (maybe mention Corollary 6.3.4). Maybe mention Proposition 6.3.8 (which implies, for example, that the de Rham property will not distinguish between good and bad potentially good reduction of abelian varieties).

(c) The de Rham comparison theorem
State theorem 3.1.4 in page 56 of [3].

(8) **Isocrystals and the Dieudonné–Manin theorem** (04/03, Lorenzo)
Source: [1], sections 7.3 and 8.1.

(a) Definition of filtered isocrystals
Define isocrystals (definition 7.3.1) and give example 7.3.2.

(b) Statement of the Dieudonné–Manin theorem
This is theorem 8.1.4.

(c) Definition of slopes of an isocrystal
This is definition 8.1.5. The discussion in pages 104 and 105 should also be covered.

(d) The filtered isocrystal associated to an abelian variety
This is essentially the Dieudonné module of the p-divisible group of the abelian variety. Compute the possible filtered isocrystals associated to an elliptic curve $E/\mathbb{Q}_p$ with good reduction ($p \geq 5$). Deduce that the analogue of the Tate conjecture fails for local fields.

(9) **Weakly admissible filtered isocrystals** (11/03, Domenico)
Source: [1], sections 8.1 and 8.2 (of 8.2., only page 110 and first paragraph of page 111).

(a) Definition of Hodge and Newton polygons

(b) Examples
The case of elliptic curves $E/\mathbb{Q}_p$ with good reduction.

(c) The theorem of Berthelot–Ogus

(d) Definition of weakly admissible filtered isocrystal
At least lemma 8.1.13 and definition 8.2.1.

(10) **Crystalline representations** (18/03, Andy)
Source: [1], sections 8.3 and 9; [3], theorem 3.2.3 in page 57; [2], theorem A.

(a) Main properties of B_{cris}

As in the case of B_{dR}, we are not supposed to construct this ring. Some properties that should be covered are: B_{cris} is a subring of B_{dR} stable under the action of G_K; it comes equipped with a special injective endomorphism ϕ known as the *Frobenius endomorphism*; B_{cris} contains the canonical uniformiser t of B_{dR}^+ (see properties of B_{dR}^+ above) and $\phi(t) = pt$. Also, theorem 9.1.5, proposition 9.1.6 and theorem 9.1.10 should be covered.

(b) Crystalline representations
Mention properties of crystalline representations like the insensitivity to base change from K to an unramified extension (see discussion in section 9.3 of [1]).

(c) Colmez–Fontaine theorem
This is theorem A of [2]. As we are only concerned with the case of crystalline representations, it should be stated as follows: the functor D_{cris} : $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \to \mathrm{MF}_K^{\phi,\mathrm{wa}}$ is an equivalence of categories.

(d) 1-dimensional crystalline representations
Use the Colmez–Fontaine theorem to show that the 1-dimensional crystalline representations are precisely Tate twists of unramified representations. This is in section 8.3 of [1].

(e) The crystalline comparison theorem
State theorem 3.2.3 in page 57 of [3] and the following paragraph.

(11) **Fontaine–Mazur or Bloch–Kato conjecture** (25/03)

References

[1] O. Brinon and B. Conrad. CMI Summer School notes on p-adic Hodge theory.

[2] P. Colmez and J. Fontaine. Construction des représentations p-adiques semi-stables. *Inventiones mathematicae*, 140:1–43, 2000.

[3] U. Jannsen, S. Kleiman, and J. Serre. *Motives*. Motives. American Mathematical Society, 1994.

[4] J. T. Tate. p-divisible groups. In T. A. Springer, editor, *Proceedings of a Conference on Local Fields*, pages 158–183, Berlin, Heidelberg, 1967. Springer Berlin Heidelberg.