An explication of Yakovlev's results in the C_{p^2} case

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1 Outline

In this note we apply the results of [Yak96] to classify all $\mathbb{Z}_p[C_{p^2}]$-indecomposable lattices (see Table 1). This was originally done in [HR62].

Some aspects of this are slightly painful and it is not clear that a Yakovlev diagram only classification would be an efficient approach for larger groups. On the other hand, if one is willing to use Yakovlev's results in tandem with the classification of [HR62], some there are significant gains (cf. Remark 3.3).

As an application, we show how to find the minimal index cyclic submodules of each of the indecomposable $\mathbb{Z}_p[C_{p^2}]$-lattices.

In Section 2) we recall Yakovlev's work, in Section 3) we classify indecomposable Yakovlev diagrams for C_{p^2} before finding their corresponding indecomposable lattices in Section 4), completing the classification. Section 5) describes the aforementioned application and in Section 6) we ask a question which if true would simplify the method.

2 Yakovlev's results

Yakovlev's results apply to a finite group G with a cyclic Sylow p-subgroup $P \cong C_{p^r}$. We shall assume G, P are of this form for the remainder of this section.

NOTATION 2.1 Let $P_i \leq P$ denote the subgroup of order p^i and let $N_i = N_G(P_i)$ denote the normaliser of P_i.

DEFINITION 2.2 We define the category $\mathfrak{M}(G)$ as follows. An object of $\mathfrak{M}(G)$ consists of a diagram

$$A_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} A_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \quad \cdots \quad \underset{\beta_{r-1}}{\overset{\alpha_{r-1}}{\rightleftarrows}} A_r.$$

with

- each A_i is a finite $(\mathbb{Z}/p^i\mathbb{Z})[N_i/P_i]$-module,
- the α_i, β_i are morphisms of $\mathbb{Z}_p[N_{i+1}]$-modules (note $N_{i+1} \leq N_i$),
- $\beta_i \circ \alpha_i = N_{P_{i+1}/P_i}$ (note $P \leq N_i$), whilst $\alpha_i \circ \beta_i = [p]$ is multiplication by p.

Morphisms are then given by maps $\{\gamma_i\}$ of $\mathbb{Z}[N_i/P_i]$-modules that commute with the choices of α_i and β_{i-1} separately. We call $\mathfrak{M}(G)$ the category of *Yakovlev diagrams*.

The category $\mathfrak{M}(G)$ is chosen so that we have:

LEMMA 2.3 *Let M be a $\mathbb{Z}_p[G]$-lattice, then*

$$H^1(P_1, M) \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} H^1(P_2, M) \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} \quad \dots \quad \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} H^1(P_r, M).$$

defines an element of $\mathfrak{M}(G)$. More generally, there is a functor Yak: $\mathbb{Z}_p[G]$*-lattices* $\to \mathfrak{M}(G)$.

Proof. [Yak96, Prop. 1.3] or [NSW08, Cor. 1.5.7]. □

We refer to Yak(M) as the *Yakovlev diagram* of M.

We can now state Yakovlev's result.

THEOREM 2.4 (Yakovlev [Yak96, Thm. 2.2, 2.4]) *Let G be a finite group, with a cyclic Sylow p-subgroup $P \cong C_{p^r}$. Write P_i for the subgroup of order p^i. Then*

i) the isomorphism class of a $\mathbb{Z}_p[G]$-lattice M is determined up to the addition of trivial source modules by the isomorphism class of its Yakovlev diagram:

$$H^1(P_1, M) \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} H^1(P_2, M) \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} \quad \dots \quad \underset{\text{res}_*}{\overset{\text{cores}^*}{\rightleftarrows}} H^1(P_r, M).$$

ii) all diagrams of $\mathfrak{M}(G)$ arise from the cohomology of some module, i.e. Yak *is essentially surjective.*

Trivial source modules are $\mathbb{Z}_p[G]$-lattices which are summands of permutation modules. In particular, they are cohomologically trivial in degree one. Amongst $\mathbb{Z}_p[G]$-lattices, trivial source modules are fairly mild and, as such, their invisibility to cohomology is often not a big loss. This makes Yakovlev's result useful in "black-box" lattice identification. This and techniques specifically for dealing with trivial source modules are discussed in my thesis [Tor18].

It is the second part which makes Yakovlev's result particularly useful for classifications of indecomposables. In particular, there is a bijection between non-trivial source indecomposable $\mathbb{Z}_p[G]$-lattices and indecomposable Yakovlev diagrams.

3 Yakovlev diagrams for C_{p^2}

In the case of C_{p^2}, a Yakovlev diagram consists of a diagram:

$$A_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A_2$$

with

- A_1 an $\mathbb{F}_p[C_{p^2}/C_p]$-module,
- A_2 an abelian group of exponent at most p^2,
- $\alpha : A_1 \to A_2$ a map of abelian groups factoring through the coinvariants $(A_1)_{C_{p^2}/C_p}$,
- $\beta : A_2 \to A_1$ a map of abelian groups with image within $A_1^{C_{p^2}/C_p}$,
- $\beta \circ \alpha = N_{C_{p^2}/C_p}$,

- $\alpha \circ \beta = [p]$.

We shall repeatedly use:

LEMMA 3.1 *Up to isomorphism, there is a unique indecomposable $\mathbb{F}_p[C_p]$-module U_k of dimension k for all $1 \leq k \leq p$ and any indecomposables is isomorphic to some U_k. The modules U_k are iterated extensions of U_1, the trivial module. Let i (resp. pr) denote the inclusion $U_1 \to U_k$ (resp. projection $U_k \to U_1$). The norm map $N_{C_p} : U_k \to U_k$ is zero unless $k = p$, in which case it is $i \circ \mathrm{pr}$.*

PROPOSITION 3.2 *The indecomposable Yakovlev diagrams for C_{p^2} are given by*

A) $U_1 \underset{i}{\overset{[p]\circ \mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p^2\mathbb{Z}$,

B) $0 \rightleftarrows \mathbb{Z}/p\mathbb{Z}$,

C) $U_p \underset{i}{\overset{\mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$,

D) $U_k \underset{i}{\overset{0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$ *for* $1 \leq k \leq p-1$,

E) $U_k \rightleftarrows 0$ *for* $1 \leq k \leq p-1$,

F) $U_k \underset{0\oplus i}{\overset{\mathrm{pr}\oplus 0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ *for* $2 \leq k \leq p-1$,

G) $U_k \underset{0}{\overset{\mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$ *for* $1 \leq k \leq p-1$.

REMARK 3.3 There are $4p-2$ indecomposable Yakovlev diagrams. Since there are three indecomposable trivial source $\mathbb{Z}_p[C_{p^2}]$-modules, $\mathbb{Z}_p, \mathbb{Z}_p[C_{p^2}/C_p], \mathbb{Z}_p[C_{p^2}]$, Theorem 2.4 predicts that there are $4p+1$ indecomposable C_{p^2}-lattices. This agrees with [HR62, Sec. 4]. The hard part of Proposition 3.2 is checking that the list is exhaustive. Combining Theorem 2.4 and [HR62] gives an alternative proof of this fact.

Proof of Proposition 3.2. It is easy to verify that the above diagrams define indecomposable Yakovlev diagrams. Suppose we have an arbitrary non-zero diagram:

$$A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \tag{1}$$

We now show that there is a summand isomorphic to one of the above types.

First suppose B has exponent p^2, and arbitrarily decompose $B = B_{p^2} \oplus B_p$ with B_{p^2} a direct sum of $\mathbb{Z}/p^2\mathbb{Z}$-terms and B_p of exponent p. Since $\beta \circ \alpha$ acts as multiplication by p on B_{p^2}, $\beta(B_{p^2})$ must be a non-trivial summand of A (isomorphic to $U_1^{\oplus \mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}} B_{p^2}}$). After choosing a complement, we may rewrite (1) in the form:

$$B_{p^2}/pB_{p^2} \oplus A' \underset{\begin{pmatrix} \mathrm{pr} & y \\ 0 & z \end{pmatrix}}{\overset{\begin{pmatrix} i & g \\ 0 & h \end{pmatrix}}{\rightleftarrows}} B_{p^2} \oplus B_p.$$

If we consider $g: A \to B_{p^2}$ as a map $A \xrightarrow{\tilde{g}} B_{p^2}/pB_{p^2} \xrightarrow{i} B_{p^2}$, then the change of basis

$$\begin{pmatrix} \mathrm{id} & \tilde{g} \\ 0 & \mathrm{id} \end{pmatrix} : B_{p^2}/pB_{p^2} \oplus A' \to B_{p^2}/pB_{p^2} \oplus A'$$

has the effect of diagonalising the action of α. However, it must also diagonalise β as $(\alpha \circ \beta)(B_p) = 0$, so the map $B_p \to B_{p^2}/pB_{p^2}$ must be zero also. In other words, there must be a summand of type A).

Now suppose B has exponent p and that A has a summand isomorphic to U_p. Then $\alpha(U_p) = \mathbb{Z}/p\mathbb{Z}$ and we can rewrite (1) in the form

$$U_p \oplus A' \underset{\begin{pmatrix} i & y \\ 0 & z \end{pmatrix}}{\overset{\begin{pmatrix} \mathrm{pr} & g \\ 0 & h \end{pmatrix}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z} \oplus B'.$$

In this case, the change of basis

$$\begin{pmatrix} \mathrm{id} & y \\ 0 & \mathrm{id} \end{pmatrix} : \mathbb{Z}/p\mathbb{Z} \oplus B' \to \mathbb{Z}/p\mathbb{Z} \oplus B'$$

diagonalises β, but it also diagonalises α given the requirement that the map $A' \to \mathbb{Z}/p\mathbb{Z} \to U_p$ must then be zero. So there is a summand of type C).

Thus, we may assume that B has exponent p and A has no U_p-summands. This forces $\alpha \circ \beta$ and $\beta \circ \alpha$ to both be trivial.

Claim *Let T be a submodule of $A^{C_{p^2}}$, then there exists a summand M of A for which $M^{C_{p^2}} = T$.*

Proof of Claim. Writing $A = \bigoplus_i U_{k_i}$ and $A^{C_{p^2}} = \bigoplus_i U_1 \leq \bigoplus_i U_{k_i} = A$, we can explicitly cut out the summand. □

Let $\overline{\mathrm{im}\,\beta}$ denote the above summand containing $\mathrm{im}\,\beta$. By arbitrarily choosing complements, we may rewrite (1) as

$$A' \oplus \overline{\mathrm{im}\,\beta} \underset{\begin{pmatrix} 0 & 0 \\ 0 & \beta' \end{pmatrix}}{\overset{\begin{pmatrix} f & g \\ 0 & 0 \end{pmatrix}}{\rightleftarrows}} \mathrm{im}\,\alpha \oplus B', \tag{2}$$

If $A' \neq 0$, arbitrarily choose a summand U_k of A' and consider $M = g^{-1}(\alpha(U_k))$. If $M = 0$, then $U_k \rightleftarrows \alpha(U_k)$ is a summand of type E) or G). Suppose M has a summand isomorphic to $U_{k'}$ for $k' > k$. Up to scaling we can assume that α acts as projection $U_k \oplus U_{k'} \xrightarrow{\mathrm{pr} \oplus \mathrm{pr}} \mathbb{Z}/p\mathbb{Z}$. Then define an isomorphism $A \to A$ which on $U_k \oplus U_{k'}$ is given by $\begin{pmatrix} \mathrm{id} & \mathrm{pr} \\ 0 & \mathrm{id} \end{pmatrix}$ and is the identity elsewhere. Under this change of basis, $\alpha(U_{k'}) = 0$ whilst β is preserved. Thus $U_{k'} \overset{0}{\rightleftarrows} \beta'^{-1}(U_{k'})$ is a summand, which itself has a summand of type D).

Now suppose that $U_{k'}$ is a summand of M with k' maximal and that $k' \leq k$. Under the change of basis $A \to A$ which on $U_k \oplus U_{k'}$ is given by $\begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{pr} & \mathrm{id} \end{pmatrix}$ and is the identity on the complement, α becomes trivial on U_k and preserves β. Thus $U_k \rightleftarrows 0$ is a summand of type E).

how big can M be?

Thus, we may assume that in (2) $A' = 0$. Now, if $\beta : B' \to \overline{\operatorname{im}\beta}$ is not injective, then we obtain a summand of type B). Thus β defines an isomorphism of B' onto $(\overline{\operatorname{im}\beta})^{C_{p^2}}$. In this case, for any choice of summand U_k of $\overline{\operatorname{im}\beta}$, $U_k \rightleftarrows \alpha(U_k) \oplus \mathbb{Z}/p\mathbb{Z}$ is a summand of type D) or F). □

4 Classifying representations

In this section, we show how to describe the non-trivial source representations which correspond to the diagrams of Proposition 3.2 via Theorem 2.4. In other words, we classify the indecomposable $\mathbb{Z}_p[G]$-lattices.

We shall repeatedly use the following description of the cohomology of a cyclic group:

LEMMA 4.1 *Let $C = \langle\sigma\rangle$ be a cyclic group and M a $\mathbb{Z}[G]$-module. Then*

i)

$$H^1(C, M) = {}_N M/(\sigma - 1)M,$$
$$H^2(C, M) = M^C/NM,$$

where $N : M \to M$ denotes the norm map and ${}_N M$, the kernel of the norm map.

ii) If $C' \leq C$ is a subgroup, corestriction and restriction are given by inclusion and norm respectively.

iii) (We shall not need this) If $C \lhd G$ is a normal subgroup and M is a G-module, then the isomorphisms of i) are as G/C-modules if the RHS is tensored by $H^2(C, \mathbb{Z})$.

NOTATION 4.2 From now on, σ will denote a generator of C_{p^2}. We denote by

- A the trivial representation of C_{p^2},
- $E = \mathbb{Z}_p[C_{p^2}/C_p]$,
- B the inflation of the augmentation ideal of C_{p^2}/C_p,
- C the module $\mathbb{Z}_p[C_{p^2}]/\Phi_{p^2}(g)$, where $\Phi_{p^2}(-)$ denotes the p^2-cyclotomic polynomial.
- I the augmentation ideal (which is isomorphic to $\mathbb{Z}_p[C_{p^2}]/\mathbb{Z}_p$)

This notation is chosen to coincide with that of [HR62] (but unfortunately A,E are trivial source, so not of types A), E)).

Type A)

Using Lemma 4.1 or the short exact sequence $0 \to I \to \mathbb{Z}_p[C_{p^2}] \to \mathbb{Z} \to 0$ we find that $H^1(C_{p^2}, I) = \mathbb{Z}/p^2\mathbb{Z}$, $H^1(C_p, I) \cong \mathbb{Z}/p\mathbb{Z}$. This forces the Yakovlev diagram of I to be of type A). Since I is indecomposable it must be the indecomposable lattice corresponding to $U_1 \underset{i}{\overset{[p]\circ \mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p^2\mathbb{Z}$.

Type B)

The lattice B is indecomposable. We claim Yak(B) is of type B). Indeed $H^1(C_p, B) = H^1(C_p, \mathbb{1}^{\oplus p-1}) = 0$, whilst $H^1(C_{p^2}, B) = B/(\sigma - 1)B = \mathbb{Z}/p\mathbb{Z}$.

Type C)

We claim that $\mathrm{Yak}(C)$ is of type C). Indeed, $H^1(C_{p^2}, C) = C/(\sigma - 1)C = \mathbb{Z}/p\mathbb{Z}$, $H^1(C_p, C) = H^1(C_p, I_{C_p})^{\oplus p} \oplus H^1(C_p, \mathbb{1})^{\oplus p-1} \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus p}$ with C_{p^2} acting as U_p. Then $\mathrm{Yak}(C)$ is the unique Yakovlev diagram with these groups, namely $U_p \underset{i}{\overset{\mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$.

Type D)

Consider extensions of the form

$$0 \to E \to X \to C \to 0. \tag{3}$$

Claim *If the extension is non-split, then X is indecomposable.*

Proof. Since all irreducible summands of $X \otimes \mathbb{Q}_p$ have multiplicity one, any decomposition must preserve isotypical components. This forces any non-trivial decomposition to induce a non-trivial decomposition of E, a contradiction. □

We claim that the isomorphism classes of indecomposable extensions given by (3) precisely correspond to Yakovlev diagrams of type C).

Since E is trivial source, it is cohomologically trivial in dimension one. As a result, $\mathrm{Yak}(X)$ is the kernel of the morphism from $\mathrm{Yak}(C)$ to the analogous diagram for $H^2(-, E)$. Using the explicit description of Lemma 4.1, we find that the diagram for $H^2(-, E)$ is also of type C). Suppose that the map $H^1(C_p, C) \to H^2(C_p, E)$ is given by $\mathrm{pr}_k \colon U_p \to U_p$ ($k \neq p$, pr_k denoting the composite $U_p \twoheadrightarrow U_k \hookrightarrow U_p$), then $\mathrm{Yak}(X)$ is forced to be given by

$$\begin{array}{cccc}
\mathrm{Yak}(E) & \mathrm{Yak}(X) & \mathrm{Yak}(C) & H^2(-,E) \\
\hline
0 \longrightarrow & \mathbb{Z}/p\mathbb{Z} = & \mathbb{Z}/p\mathbb{Z} \xrightarrow{\;0\;} & \mathbb{Z}/p\mathbb{Z} \\
\uparrow\downarrow & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle i} & {\scriptstyle \mathrm{pr}}\uparrow\downarrow{\scriptstyle i} & {\scriptstyle \mathrm{pr}}\uparrow\downarrow{\scriptstyle i} \\
0 \longrightarrow & U_{p-k} \xrightarrow[i_{p-k}]{} & U_p \xrightarrow[\mathrm{pr}_k]{} & U_p
\end{array}$$

Thus in order to show that all diagrams of type E) arise from such extensions it suffices to check that there exists extensions with boundary maps given by pr_k. (Note, in the case of $k = 0$, the extension is split.)

Claim *For all $0 \le k \le p-1$, there is an element of $\mathrm{Ext}^1(C, E)$ for which the corresponding extension X has induced map $U_p = H^1(C_p, X) \to H^2(C_p, E) = U_p$ is pr_k.*

Proof. Omitted. I would hope that that this can be straightforwardly checked by explicating Frobenius reciprocity and using that the boundary homomorphism is given by the canonical map $\mathrm{Hom}_{D(\mathrm{Rep}(C_{p^2}))}(C, E[1]) \to \mathrm{Hom}_{D(\mathrm{Rep}(C_p))}(C, E[1]) \to \mathrm{Hom}(H^1(C_p, C), H^2(C_p, E))$ (i.e. X is the mapping cone of the map $M \to N[1]$). □

Type E)

We claim that the non-trivial extensions of the form

$$0 \to A \oplus E \to X \to C \to 0 \tag{4}$$

correspond to Yakovlev diagrams of type E), except in the case of $k = p-1$. In which case, the extension decomposes and $U_{p-1} \rightleftarrows 0$ is the Yakovlev diagram of the non-trivial extension of the form

$$0 \to A \to X \to C \to 0.$$

To obtain the last statement, consider the boundary maps in the diagram:

$$\begin{array}{ccccccc}
\mathrm{Yak}(A) & & \mathrm{Yak}(X) & & \mathrm{Yak}(C) & & H^2(-,E) \\
\hline
0 & \longrightarrow & H^1(C_{p^2},X) & = & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & \mathbb{Z}/p^2\mathbb{Z} \\
 & & \uparrow & & \uparrow \scriptstyle{\mathrm{pr}} & & \uparrow \scriptstyle{i} \\
0 & \longrightarrow & H^1(C_p,X) & \longrightarrow & U_p & \xrightarrow[\partial]{} & \mathbb{Z}/p\mathbb{Z}
\end{array}$$

If $\partial = 0$, the sequence splits, so in the non-split case, ∂ is projection $U_p \to \mathbb{Z}/p\mathbb{Z}$ and we find $H^1(C_{p^2},X) = 0$, whilst $H^1(C_p,X) = U_{p-1}$.

The $0 \le k \le p-2$ case follows similarly from the following claim:

CLAIM *There is an extension* (4) *for which the induced map*

$$U_p \cong H^1(C_p,C) \to H^2(C_p, A \oplus C) = U_p \oplus \mathbb{Z}/p\mathbb{Z}$$

is $(\mathrm{pr}_k, \mathrm{pr})$*. Moreover, this is indecomposable.*

This is similar to the case of type D).

Type F)

We claim there are $p-1$ indecomposable extensions of the form

$$0 \to B \to X \to C \to 0,$$

one of which is the augmentation ideal and the remainder corresponding to Yakovlev diagrams of type F).

Type G)

These correspond to indecomposable extensions of the form

$$0 \to A \oplus B \to X \to C \to 0.$$

The above information can be summarised by the following table:

Type	Yakovlev diagram	Representation
A)	$U_1 \underset{i}{\overset{[p]\circ \mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p^2\mathbb{Z}$	I
B)	$0 \rightleftarrows \mathbb{Z}/p\mathbb{Z}$	B
C)	$U_p \underset{i}{\overset{\mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$	C
D)	$U_k \underset{i}{\overset{0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$ for $1 \le k \le p-1$	non-split extensions $0 \to E \to X \to C \to 0$
E)	$U_k \rightleftarrows 0$ for $1 \le k \le p-1$	indecomposable extensions $0 \to A \oplus E \to X \to C \to 0$
F)	$U_k \underset{0\oplus i}{\overset{\mathrm{pr}\oplus 0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ for $2 \le k \le p-1$	non-split extensions $0 \to B \to X \to C \to 0$, except for the augmentation ideal I.
G)	$U_k \underset{0}{\overset{\mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z}$ for $1 \le k \le p-1$	indecomposable extensions $0 \to A \oplus B \to X \to C \to 0$

Table 1: Indecomposable representations and their Yakovlev diagrams.

5 Minimal index cyclic submodules

In this section we calculate the smallest possible index cyclic submodules for all lattices lying within the augmentation ideal $I_{\mathbb{Q}_p}$ of $\mathbb{Q}_p[G]$. Recall a (sub)module is cyclic if it is a quotient of $\mathbb{Z}_p[G]$. Given two full rank lattices M, N lying within a fixed $\mathbb{Q}_p[G]$-representation T, there is some power of p for which $p^k M \le N$ and vice versa. As a result, M has a cyclic submodule of finite index whenever T is cyclic as a $\mathbb{Q}_p[G]$-module. Moreover, in the case of $G = C_{p^2}$ for any choice of cyclic T there is at most one choice of cyclic sublattice, up to isomorphism (this uses that C_{p^2} is abelian).

We can read off the possible sublattices of $I_{\mathbb{Q}_p}$ from Table 1. Namely, they are

- $B \oplus C$, which has Yakovlev diagram $U_p \underset{i\oplus 0}{\overset{\mathrm{pr}\oplus 0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$,
- the indecomposable lattices X_k with Yakovlev diagram $U_k \underset{0\oplus i}{\overset{\mathrm{pr}\oplus 0}{\rightleftarrows}} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ for $2 \le k \le p-1$,
- I, which has Yakovlev diagram $U_1 \underset{i}{\overset{[p]\circ \mathrm{pr}}{\rightleftarrows}} \mathbb{Z}/p^2\mathbb{Z}$.

Amongst these, the augmentation ideal is the cyclic lattice, so we wish to describe a minimal index inclusion $I \hookrightarrow M$ for the other $p-1$ lattices.

LEMMA 5.1 *Any inclusion $I \hookrightarrow X_k$ has index divisible by p^{k-1}. Any inclusion $I \hookrightarrow B \oplus C$ has index divisible by p^{p-1}.*

Proof. Given an inclusion $\iota\colon I \hookrightarrow X_k$ the induced morphism $H^1(C_p, I) \cong U_1 \xrightarrow{\iota_*} H^1(C_p, X_k) \cong U_k$ has cokernel of order at least p^{k-1}. On the otherhand, since the norm map is zero on I and X_k we have $H^1(C_p, I) = I/(\sigma-1)I$, $H^1(C_p, X_k) = X_k/(\sigma-1)X_k$, with ι_* given by applying ι to I. As a result, the index of the image of $I/(\sigma-1)I$ in $X_k/(\sigma-1)X_k$ divides $[X_k : I]$. The same is true for $B \oplus C$. □

We now exhibit cyclic submodules with the above indices.

Claim *Given $1 \le k \le p-2$, there is an inclusion $X_k \hookrightarrow X_{k+1}$ with index p.*

Proof. The map $(\sigma-1)\colon B \to B$ is a homomorphism of $\mathbb{Z}_p[C_{p^2}]$-lattices with image of index p. Pushing out by $(\sigma-1)$ yields a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_k & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle(\sigma-1)}\big\downarrow & \ulcorner & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & B & \longrightarrow & \tilde{X}_k & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

We claim that $\mathrm{Yak}(\tilde{X}_k) \cong \mathrm{Yak}(X_k)$, so that $\tilde{X}_k \cong X_k$. This is a diagram chase forced by the fact that $(\sigma-1)\colon H^1(C_{p^2}, B) \to H^1(C_{p^2}, B)$ is the zero map. Explicitly, consider:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(C_{p^2}, B) & \longrightarrow & H^1(C_{p^2}, X_k) & \longrightarrow & H^1(C_{p^2}, C) & \longrightarrow & 0 \\ & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow\wr & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\mathrm{pr}_2} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle 0\oplus\mathrm{id}} & & \big\| & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i_1} & \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\mathrm{pr}_2} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow\wr & & \\ 0 & \longrightarrow & H^1(C_{p^2}, B) & \longrightarrow & H^1(C_{p^2}, \tilde{X}_k) & \longrightarrow & H^1(C_{p^2}, C) & \longrightarrow & 0 \end{array}$$

Where the middle lower group and its maps are forced by the rest of the diagram. The non-injectivity of $0 \oplus \mathrm{id}\colon \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ suffices to force $H^1(C_p, \tilde{X}_k) \cong U_{k+1}$ and $\mathrm{Yak}(\tilde{X}_k) \cong \mathrm{Yak}(X_{k+1})$. □

Pushout by $(\sigma-1)$ also shows:

Claim *There are inclusions $I \hookrightarrow X_2$, $X_{p-1} \hookrightarrow B \oplus C$ with index p.*

Combining this with Lemma 5.1 gives:

Lemma 5.2 *The lattice X_k has a cyclic submodule of index p^{k-1} and this is minimal amongst cyclic submodules. The lattice $B \oplus C$ has a cyclic submodule of index p^{p-1} and this is minimal.*

Remark 5.3 Nothing interesting happens by repeating the same pushout operation on $B \oplus C$. As a direct sum, the pushout occurs in the first factor to give the index p inclusion $B \oplus C \subset B \oplus C$ given by $B \overset{(\sigma-1)}{\subset} B$.

$$I \subset X_2 \subset X_3 \subset \ldots \subset X_{p-1} \subset B \oplus C \subset B \oplus C \subset B \oplus C \subset \ldots$$

$$\xrightarrow[\text{Pushout by } B \overset{(\sigma-1)}{\to} B]{\qquad\qquad\qquad\qquad\qquad\qquad}$$

In the other direction, we can look at sublattices of I. It is fairly easy to see, from Yakovlev style arguments, that there does not exist any lattice $\tilde{I}$ for which I is the pushout of $\tilde{I}$ by $B \xrightarrow{\sigma-1}$. On the other hand, it can be checked that by the fibre product of $I \to C$ with the map $(\sigma-1)\colon C \to C$ is X_2 yields an index p inclusion $X_2 \subset I$.

CLAIM *The pullback X_k for $2 \leq k \leq p-2$ by $C \xrightarrow{\sigma-1} C$ is isomorphic to X_{k+1}. The pullback of I is X_2, the pullback of X_{p-1} is $B \oplus C$.*

Proof. Since this is a bit more tricky than the pushout case we give some detail. We prove that the pullback of X_k is X_{k+1}.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & \tilde{X}_k & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & \lrcorner & \downarrow{\scriptstyle \sigma-1} & & \\ 0 & \longrightarrow & B & \longrightarrow & X_k & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Taking $H^1(C_p,-)$ yields:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & H^1(C_p,\tilde{X}_k) & \longrightarrow & U_p & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{p-1}} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & U_k & \longrightarrow & U_p & \longrightarrow & \ldots \end{array}$$

Working under the constraints that the induced map $H^1(C_p,\tilde{X}_k) \to U_k$ must be the restriction of pr_{p-1} and must have cokernel of order at most p, the only options for $H^1(C_p,\tilde{X}_k)$ are U_{k+1} or U_k. In the U_{k+1}-case, the classification (Table 1) forces $\tilde{X}_k$ to be X_{p+1}.

We now wish to derive a contradiction in the U_k-case. If we consider $H^1(C_{p^2},-)$ cohomology we obtain:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i\oplus 0} & \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z} & \xrightarrow{0\oplus \mathrm{pr}} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{i\oplus 0} & \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z} & \xrightarrow{0\oplus \mathrm{pr}} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \end{array}$$

(that $H^1(C_{p^2},\tilde{X}_k) \cong \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}$ follows from the classification or a diagram chase). We deduce that the middle vertical map is $\begin{pmatrix} i & * \\ 0 & 0 \end{pmatrix}$. If we consider the induced map of Yakovlev diagrams, this is enough to yield a contradiction to $H^1(C_p,\tilde{X}_k) = U_k$ the composite $U_k \xrightarrow{\mathrm{pr}_{p-1}} U_k \xrightarrow{\mathrm{pr}\oplus 0} \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}$ is zero whereas $U_k \xrightarrow{\mathrm{pr}\oplus 0} \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z} \xrightarrow{\begin{pmatrix} i & * \\ 0 & 0 \end{pmatrix}} \mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}$ is not. □

Combining our previous inclusions yields:

$$\underbrace{B\oplus C \subset X_{p-1} \subset X_{p-2} \subset \ldots \subset X_2 \subset}_{\text{index } p^{p-1}} I \underbrace{\subset X_2 \subset \ldots \subset X_{p-2} \subset X_{p-1} \subset B\oplus C}_{\text{index } p^{p-1}} \tag{5}$$

Alternatively, these inclusions fit into the following diagram:

$$
\begin{array}{cccccccccccccccccc}
\ddots & \vdots & & \vdots & & \vdots & & & & & & & & & & & & \\
 & \cap & & \cap & & \cap & & & & & & & & & & & & \\
\ddots & X_{p-1} & \subset & B\oplus C & \subset & B\oplus C & \subset & \dots & & & & & & & & & & \\
 & & & \cap & & \cap & & & & & & & & & & & & \\
\ddots & & & X_{p-1} & \subset & B\oplus C & \subset & \dots & & & & & & & & & & \\
 & & & \cap & & \cap & & & & & & & & & & & & \\
\ddots & \vdots & & \vdots & & \vdots & & & & \ddots & & & & & & & & \\
 & \cap & & \cap & & \cap & & & & & & & & & & & & \\
 & I & \subset & X_2 & \subset & X_3 & \subset & \dots & & \dots\subset X_{p-1} & \subset & B\oplus C & \subset & \dots & & & & \\
 & & & \cap & & \cap & & & & & & \cap & & & & & & \\
 & & & I & \subset & X_2 & \subset & X_3 & \subset\dots & \dots\subset & & X_{p-1} & \subset & B\oplus C & \subset & B\oplus C & \subset & \dots \\
 & & & & & \cap & & \cap & & & & \cap & & \cap & & & & \\
 & & & & & \ddots & & \ddots & & & & \ddots & & \ddots & & & &
\end{array}
$$

The vertical inclusions are given by fibre product with $C \xrightarrow{\sigma-1} C$ and the horizontal maps by pushout with $B \xrightarrow{\sigma-1} B$. Note, as discussed there is no sublattice of I of index p for which I is the pushout along B. The commutativity of the diagram is justified as for any given extension X, there is at most one map fitting into the following diagram, as there are no non-zero maps $C \to B$,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\
 & & {\scriptstyle\sigma-1}\downarrow & & \downarrow & & \downarrow{\scriptstyle\sigma-1} & & \\
0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

In particular, the inclusion $I \subset X_2 \subset I$ is the canonical inclusion of I as a subset $|G|$-sublattice (which exists for any group), namely by $(\sigma-1)\cdot I \subset I$.

Finally, it is interesting to consider all sublattices lying between $p\cdot I$ and I. It turns out this is given by

$$I \subset X_2 \subset \ldots \subset X_{p-1} \subset B \underbrace{\oplus C \subset B\oplus C \subset \ldots \subset B\oplus C \subset B \oplus}_{p^2-2p+2} C \subset X_{p-1} \subset \ldots X_2 \subset I.$$

This can be seen by first noting that I/pI is the (p^2-1)-dimensional C_{p^2} indecomposable, so has a unique nested sublattice of each dimension together with Yakovlev style considerations. Here, the middle inclusions are given by $B\oplus(\sigma-1)\cdot C \subset B\oplus C$. This point can be seen as a consequence of the fact that the induced map is $\mathbb{Z}/p^2\mathbb{Z} \cong H^1(C_{p^2}, I) \to H^1(C_{p^2}, I) \cong \mathbb{Z}/p^2\mathbb{Z}$ is multiplication by p and so a copy of $\mathbb{Z}/p\mathbb{Z}$ must survive through all the inclusions. On the other hand, the inclusions $B\oplus C \subset B \oplus C$ must all be inclusions on only one of the factors, but only C has p^2-2p+2 sublattices lying between pC and C.

REMARK 5.4 As an aside, we can deduce that there are index p^{p-1} inclusions $I \subset B\oplus C$, $B\oplus C \subset I$ (see (5)) from Jacobinski's Theorem [CR94, Thm. (27.8)], as we now sketch. Let $\Lambda = \mathbb{Z}\times\mathbb{Z}[\zeta_p]\times\mathbb{Z}[\zeta_{p^2}]$ denote the maximal order of $\mathbb{Q}[C_{p^2}] \cong \prod_i A_i = \mathbb{Q}\times\mathbb{Q}(\zeta_p)\times\mathbb{Q}(\zeta_{p^2})$. Then Jacobinski's Theorem implies that

$$[\Lambda : \mathbb{Z}[G]] = \left(\frac{|G|^{|G|}}{\prod_i (n_i)^{\dim_{\mathbb{Q}} A_i}\Delta_i}\right)^{\frac{1}{2}}, \tag{6}$$

where n_i is the dimension of A_i over its centre, which is one in the abelian case, and Δ_i denotes the discriminant of the maximal order of A_i with respect to the reduced trace. In the case of C_{p^2} we obtain

$$\begin{aligned}[\Lambda : \mathbb{Z}[G]] &= \left(\frac{(p^2)^{p^2}}{1 \times p^{p-2} \times p^{p(2p-3)}} \right)^{\frac{1}{2}} \\ &= \left(\frac{(p^2)^{p^2}}{p^{2p^2-2p-2}} \right)^{\frac{1}{2}} \\ &= \left((p^2)^{p+1} \right)^{\frac{1}{2}} \\ &= p^{p+1}.\end{aligned}$$

So there is an index p^{p+1} inclusion $\mathbb{Z}[G] \subset \Lambda$. Quotienting by $\mathbb{1}$ there is then an inclusion

$$I = \mathbb{Z}[G]/\mathbb{1}_{\mathbb{Z}} \hookrightarrow \Lambda/(\Lambda \cap \mathbb{1}_{\mathbb{Q}}) = B \oplus C.$$

We can calculate the index using the formula

$$[(\Lambda \cap \mathbb{1}_{\mathbb{Q}}) : \mathbb{1}_{\mathbb{Z}}] \cdot [I : B \oplus C] = [\Lambda : \mathbb{Z}[G]] = p^{p+1}.$$

The index $[(\Lambda \cap \mathbb{1}_{\mathbb{Q}}) : \mathbb{1}_{\mathbb{Z}}] = p^2$ as it is generated by $\frac{1}{|G|} \sum_{g \in G} g$. We then obtain $I \subset B \oplus C$ with index p^{p-1}.

The reverse inclusion is similar using that Jacobinski's Theorem also implies that $[\mathbb{Z}[G] : (\Lambda : \mathbb{Z}[G])]$ is equal to (6), where $(\Lambda : \mathbb{Z}[G])$ denotes the largest Λ-submodule of $\mathbb{Z}[G]$ (the conductor).

6 Question

A question I have not thought much about:

QUESTION 6.1 Does a morphism of Yakovlev diagrams come from a morphism of lattices?

EXAMPLE 6.2 If so, this would provide a shortcut to the proof of Lemma 5.2 as follows. There is an obvious map $\mathrm{Yak}(X_k) \to \mathrm{Yak}(X_{k+1})$. Moreover, this map is not induced by any of the "degenerate" morphisms $X_k \to X_{k+1}$, e.g. $X_k \to C \to X_{k+1}$ (the composite $U_k \to U_p \to U_{k+1}$ is wrong). So the morphism provided by $\mathrm{Yak}(X_k) \to \mathrm{Yak}(X_{k+1})$ has image of finite index. Similarly to Lemma 5.1, we can then show that the index must be minimal.

Yakovlev provides an explicit construction of indecomposables with a given Yakovlev diagram [Yak96, Sec. 4]. It would suffice to check that this construction is functorial. Yakovlev already has results in this direction [Yak96, Lemma 5.2].

Note that there will never be an equivalence of categories between non-trivial source modules and Yakovlev diagrams.

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