

Immune networks: multitasking capabilities near saturation

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A minimal model

- $T: \sigma_i = \pm 1, i = 1, \dots, N_T$ (secreting or not)
- $B: b_\mu \sim \mathcal{N}(0, 1), \mu = 1, \dots, N_B$ (size relative to typical)
- $\xi_i^\mu = \pm 1, 0$ (elicitor, suppressor, no signal)

$$P(\xi_i^\mu) = \frac{d}{2}(\delta_{\xi_i^\mu, 1} + \delta_{\xi_i^\mu, -1}) + (1-d)\delta_{\xi_i^\mu, 0}$$

- $N_T > N_B$ but comparable: *near saturation*

$$N_B = \alpha N_T \quad (\alpha \leq 1)$$

- B-T interactions highly selective: *Finite connectivity*

$$d = \frac{c}{N_T}, \quad c = \mathcal{O}(N_T^0)$$

- Hamiltonian

$$H(\boldsymbol{\sigma}) = -\frac{1}{2c} \sum_{ij=1}^N \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j = -\frac{1}{2c} \sum_{\mu=1}^{\alpha N} M_\mu^2(\boldsymbol{\sigma})$$

- non-normalised overlaps:

$$M_\mu(\boldsymbol{\sigma}) = \sum_i \xi_i^\mu \sigma_i$$

- Add antigen fields $\{\psi_\mu\}$: $H \rightarrow H - \sum_\mu \psi_\mu M_\mu(\boldsymbol{\sigma})$

Network Analysis

- Giant component: $c/N_T > 1/\sqrt{N_T N_B}$ i.e. $\alpha c^2 > 1$

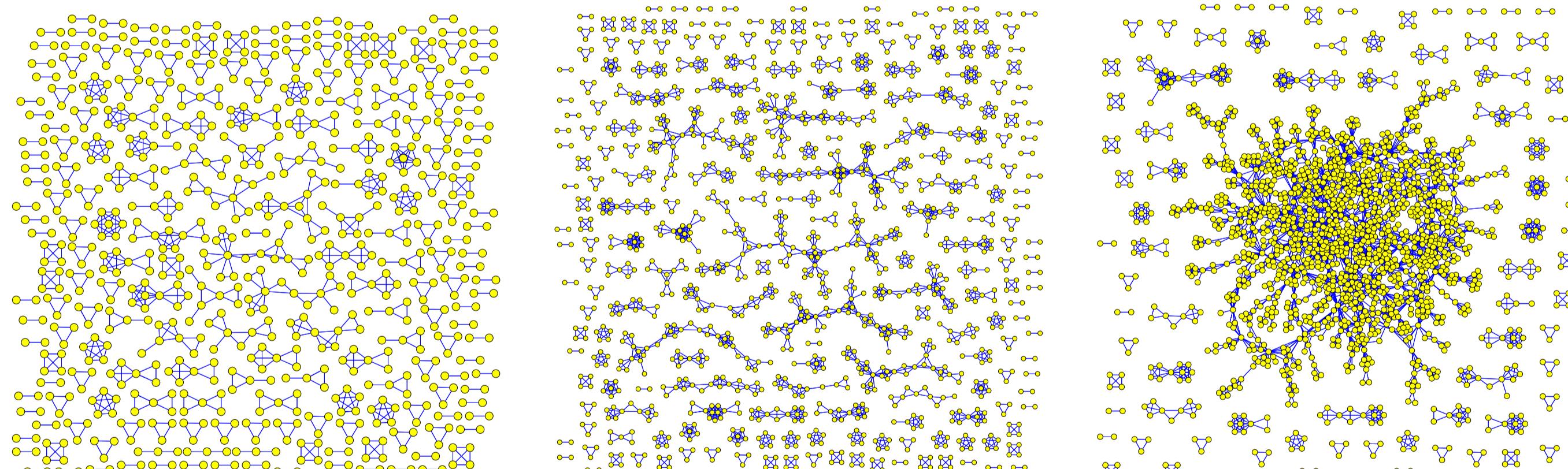


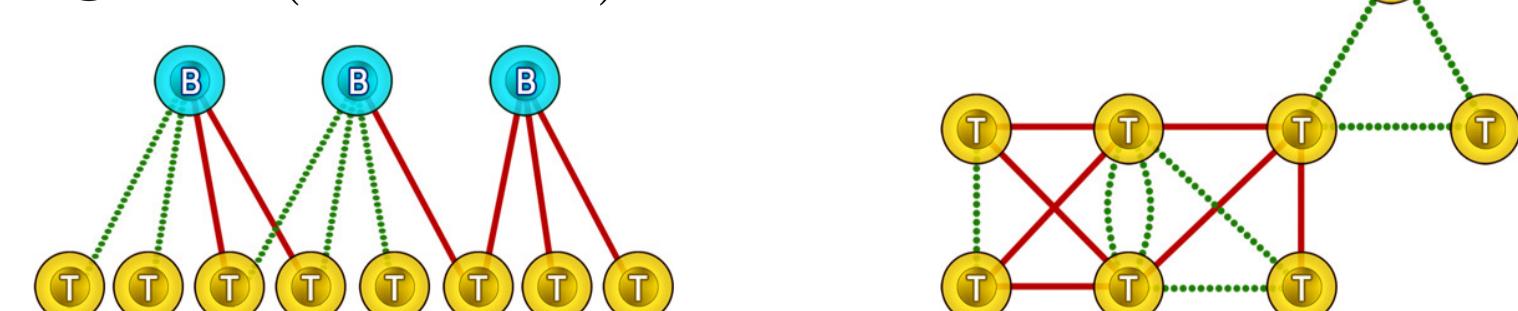
Figure: \mathcal{G} obtained for $N_T = 5 \times 10^3$ and $\alpha = 0.1$. From left to right:
 $\alpha c^2 < 1, = 1, > 1$

- under-percolated regime ($\alpha c^2 < 1$):



- σ_i, σ_j share at most one B -node μ : non-conflicting signals to μ

- over-percolated regime ($\alpha c^2 > 1$):



- loops in \mathcal{B} , several patterns in one \mathcal{G} component

Statistical mechanical analysis

$$P(M|\psi) = \sum_{k \geq 0} p_c(k) P(M|k, \psi) \quad \text{with} \quad p_c(k) = e^{-c} c^k / k!$$

$$P(M|k, \psi) = e^{-\alpha c k} \sum_{r \geq 0} \frac{(\alpha c)^r}{r!} \int_{-\infty}^{\infty} dh_1 \dots dh_r \left[\prod_{s \leq r} W(h_s) \right] \sum_{\ell_1 \dots \ell_r \leq k} \times \left\{ \frac{\langle \delta_{M, \sum_{\ell \leq k} \tau_\ell} e^{\beta(\sum_{\ell \leq k} \tau_\ell)^2/2c + \beta \psi \sum_{\ell \leq k} \tau_\ell + \beta \sum_{s \leq r} h_s \tau_s} \rangle_{\tau_1 \dots \tau_k = \pm 1}}{\langle e^{\beta(\sum_{\ell \leq k} \tau_\ell)^2/2c + \beta \psi \sum_{\ell \leq k} \tau_\ell + \beta \sum_{s \leq r} h_s \tau_s} \rangle_{\tau_1 \dots \tau_k = \pm 1}} \right\}$$

- $p_c(k)$ probability that a pattern has k non-zero entries

- $P(M|k, \psi)$ cond prob of overlap M given cytokine pattern has k non-zero entries

$$W(h) = e^{-c} \sum_{k \geq 0} \frac{c^k}{k!} e^{-\alpha c k} \sum_{r \geq 0} \frac{(\alpha c)^r}{r!} \int_{-\infty}^{\infty} dh_1 \dots dh_r \left[\prod_{s \leq r} W(h_s) \right] \sum_{\ell_1 \dots \ell_r \leq k} k^{-r} \langle\langle \delta \left[h - \tau \psi - \frac{1}{2\beta} \log \frac{\langle e^{\beta(\sum_{\ell \leq k} \tau_\ell)^2/2c + \beta(\sum_{\ell \leq k} \tau_\ell)(\psi + \tau/c) + \beta \sum_{s \leq r} h_s \tau_s} \rangle_{\tau_1 \dots \tau_k = \pm 1}}{\langle e^{\beta(\sum_{\ell \leq k} \tau_\ell)^2/2c + \beta(\sum_{\ell \leq k} \tau_\ell)(\psi - \tau/c) + \beta \sum_{s \leq r} h_s \tau_s} \rangle_{\tau_1 \dots \tau_k = \pm 1}} \right] \rangle\rangle_\psi \rangle_{\tau = \pm 1}.$$

- $W(h) = \delta(h)$ solution at any T ; $W(h) = W(-h)$, $m_1 = \int dh W(h)h = 0$

- First bifurcation away from $\delta(h)$ expected in $m_2 = \int dh W(h)h^2$

$$1 = \alpha c^2 \sum_{k \geq 0} e^{-c} \frac{c^k}{k!} \left\{ \frac{\int Dz \tanh(z\sqrt{\beta/c} + \beta/c) \cosh^{k+1}(z\sqrt{\beta/c} + \beta/c)}{\int Dz \cosh^{k+1}(z\sqrt{\beta/c} + \beta/c)} \right\}^2$$

- $\lim_{\beta \rightarrow 0} \text{RHS} = 0; \lim_{\beta \rightarrow \infty} \text{RHS} = \alpha c^2$

- A transition at finite temperature $T_c(\alpha, c) > 0$ exists as soon as $\alpha c^2 > 1$.

- For $\alpha c^2 = 1$, $T_c(\alpha, 1/\sqrt{\alpha}) = 0$

- No transition below percolation threshold $\alpha c^2 < 1$

Above and below critical line

- $\alpha c^2 < 1 \Rightarrow W(h) = \delta(h)$: only source of noise is thermal

$$P(M|k, \psi) = \left\langle \delta_{M, \sum_{\ell \leq k} \tau_\ell} \right\rangle_\tau \frac{e^{\beta M^2/2c + \beta \psi M}}{Z(\beta, k)} = \frac{e^{\beta \chi(M, \psi)} \binom{k}{\frac{k-M}{2}}}{\sum_{M=-k}^k \binom{k}{\frac{k-M}{2}} e^{\beta \chi(M, \psi)}}$$

- $\beta = 0$

$$P(M|k) = \frac{1}{2^k} \binom{k}{\frac{k-M}{2}} \quad \text{maximum at } M = 0 \forall k$$

- $\beta \rightarrow \infty$

$$\psi \neq 0: \quad P(M|k, \psi) = \delta_{|M|, k} \theta(\psi M) \\ \psi = 0: \quad P(M|k, \psi) = \frac{1}{2} \delta_{|M|, k} \quad \text{perfect retrieval}$$

- $T < T_c(\alpha, c)$

$$P(M|k, \psi) = \left\langle \left\langle \delta_{M, \sum_{\ell \leq k} \tau_\ell} e^{\beta(\sum_{\ell \leq k} \tau_\ell)^2/2c + \beta \psi \sum_{\ell \leq k} \tau_\ell + \beta \sum_{s \leq r} h_s \tau_s} \right\rangle_{\tau_1 \dots \tau_k = \pm 1} \right\rangle_{r, h, \ell, \psi}$$

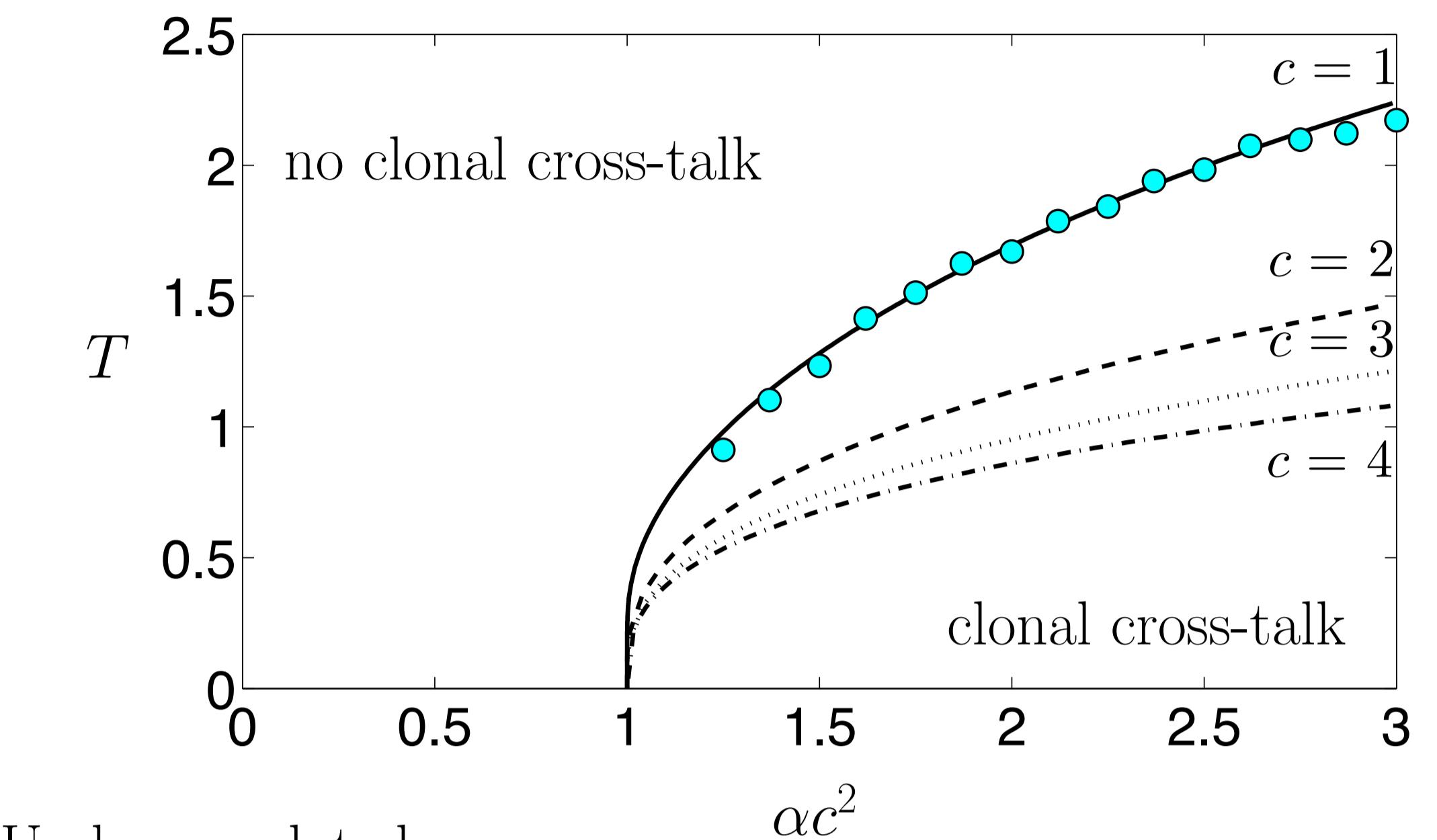
- r : number of interferences (Poissonian, with mean $\alpha c k$)

- $\ell_s, s = 1, \dots, r$: spins on which interference acts (uniform in $\{1, \dots, k\}$)

- h_s : clonal interference field acting on spin ℓ_s

Plots

- Critical line



- Underpercolated

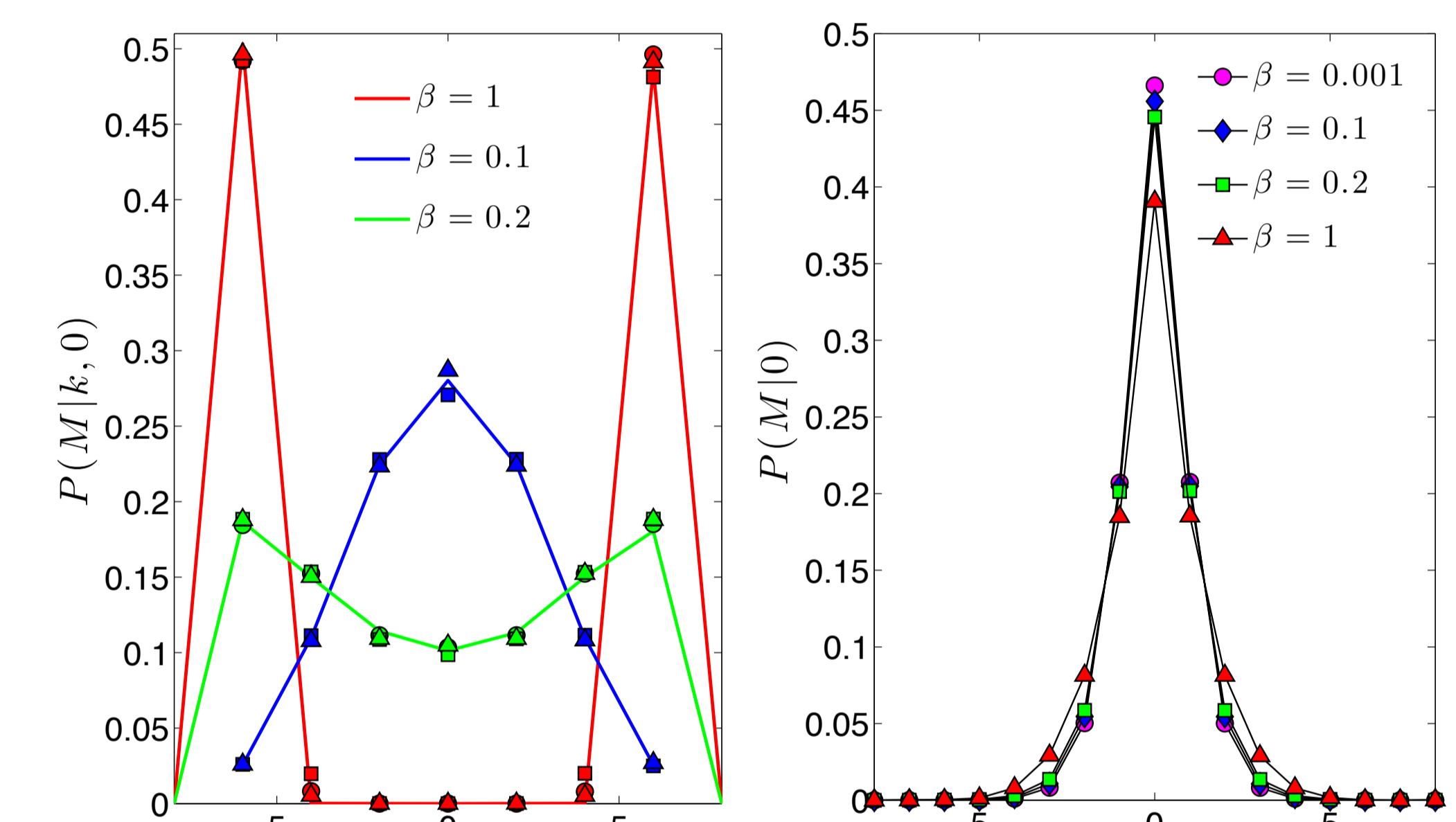


Figure: $k = 6, c = 1, \psi = 0$ Population dynamics (Solid) and MC simulations (Markers) with $N = 3.10^4$. Left: $\alpha = 0.005$ (bullets), $\alpha = 0.008$ (squares) and $\alpha = 0.011$ (triangles). Right: $\alpha = 0.5$.

- Fast switching from expansion to contraction: homeostasis?

- Overpercolated

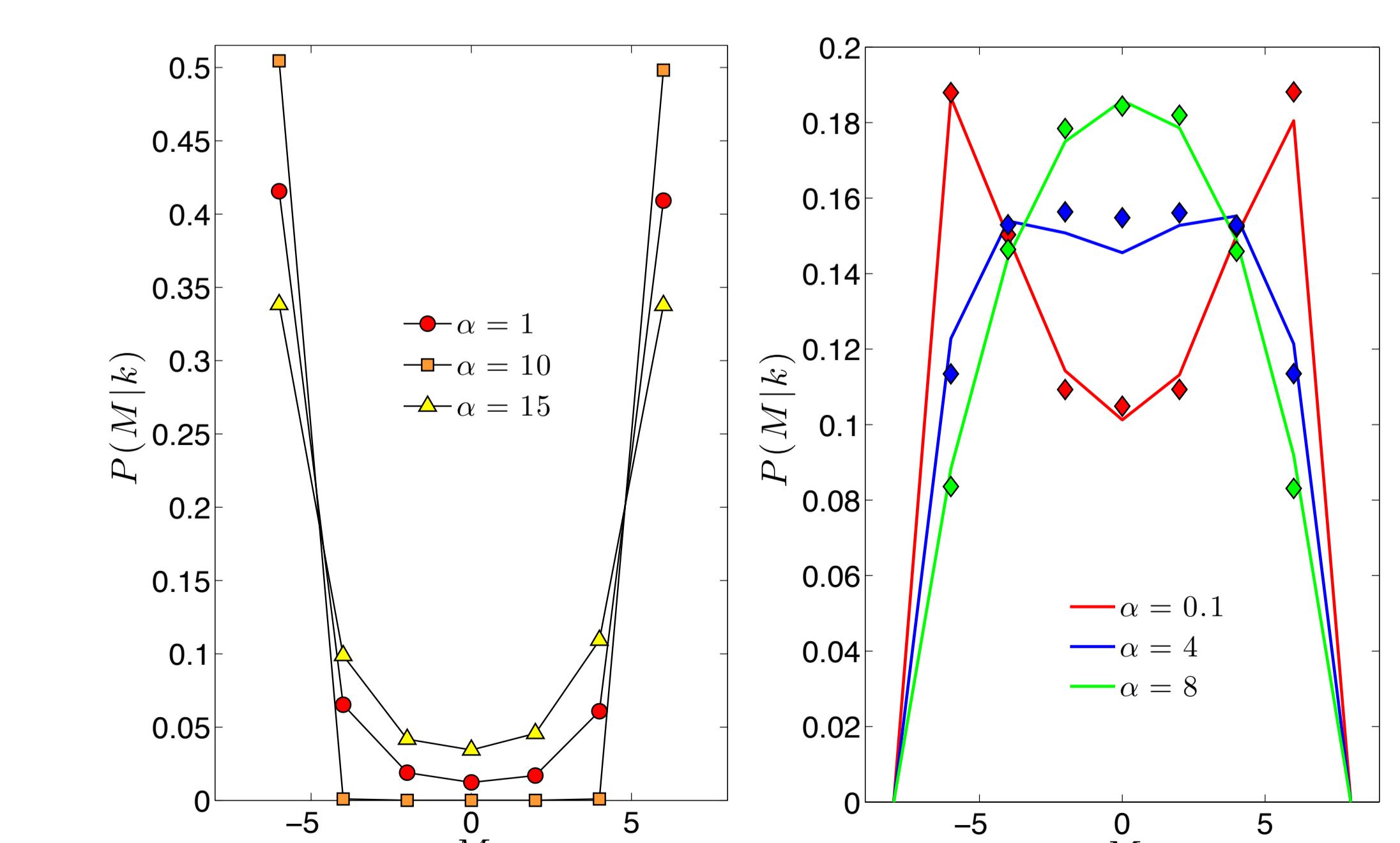


Figure: $k = 6$. Left: $c = 1$ and $\beta = 0.8$. Right: $c = 3$. Population dynamics (Solid) and MC simulations (Markers), $N = 3.10^4$. Note dependence on α .

- cross-talk diminishes parallel retrieval, but gracefully