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ON AXIOMATISING PRODUCTS OF KRIPKE FRAMES

ÁGNES KURUCZ

Abstract. It is shown that the many-dimensional modal logic $\boldsymbol{K}^n$, determined by products of n-many Kripke frames, is not finitely axiomatisable in the n-modal language, for any $n > 2$. On the other hand, $\boldsymbol{K}^n$ is determined by a class of frames satisfying a single first-order sentence.

§1. Introduction. In this paper we show that the multi-modal logic $\boldsymbol{K}^n$, determined by the class of Cartesian products of n-many Kripke frames, is not finitely axiomatisable, whenever $n > 2$. It is also shown that $\boldsymbol{K}^n$ is determined by a first-order definable class of frames.

The formation of products is a standard mathematical way of introducing new dimensions. In modal logic products are used for constructing systems with several modal operators (say, temporal, epistemic, spatial). Modal products appear both in theoretical studies (e.g., [13], [14], [15]) and applications ([2], [4]). They are also closely related to finite variable fragments of classical first-order logic and to the corresponding classes of algebras (cylindric and polyadic) ([1], [9]).

In general, products of modal logics do not inherit the 'nice' axiomatisability properties of their components. One can find already two dimensional 'nasty' examples: e.g., though the well-known modal logic of the frame $(\omega, <)$ is finitely axiomatisable [12], the bi-modal logic of $(\omega, <) \times (\omega, <)$ is not even recursively enumerable [15]. On the other hand, some two-dimensional products of standard modal systems, such as $\boldsymbol{S5}\times\boldsymbol{S5}$ and $\boldsymbol{K}\times\boldsymbol{K}$, remain finitely axiomatisable (see [13], [7]). Not too much is known about axiomatisability properties of higher dimensional products. As an exception, $\boldsymbol{S5}^n$ is known to be non-finitely axiomatisable, whenever $n > 2$ ([11]).

Notation. Our notation is mostly standard. We consider binary relations as sets of ordered pairs, and write them in the infix form xRy.

Basic definitions. For any non-zero natural number n, let $\mathscr{G}_0 = (G_0, R^{\mathscr{G}_0})$, $\mathscr{G}_1 = (G_1, R^{\mathscr{G}_1}), \dots, \mathscr{G}_{n-1} = (G_{n-1}, R^{\mathscr{G}_{n-1}})$ be usual Kripke frames — that is, relational structures having one binary relation. Their *product* $\mathscr{G} \overset{\text{def}}{=} \mathscr{G}_0 \times \mathscr{G}_1 \times \cdots \times \mathscr{G}_{n-1}$ is defined to be the relational structure $(G, R_0^{\mathscr{G}}, R_1^{\mathscr{G}}, \dots, R_{n-1}^{\mathscr{G}})$ where G is the Cartesian

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product $G_0 \times G_1 \times \cdots \times G_{n-1}$ and, for each $\ell < n$, $R_\ell^{\mathscr{G}}$ is the following binary relation on G: for all $\bar{u} = (u_0, \dots, u_{n-1})$, $\bar{v} = (v_0, \dots, v_{n-1}) \in G$,

$$\bar{u} R_\ell^{\mathscr{G}} \bar{v} \quad \text{iff} \quad u_\ell R^{\mathscr{G}_\ell} v_\ell, \quad \text{and} \quad u_k = v_k \quad \text{whenever } k \neq \ell .$$

Such a product frame $\mathscr{G}$ will be called an *n-cube*.

Since n-cubes have n accessibility relations, the *n-modal language* which corresponds to them has to have n modal operators $\Diamond_0, \dots, \Diamond_{n-1}$ (and their duals $\Box_0, \dots, \Box_{n-1}$). Formulas of this language, using propositional variables from some fixed countable set P, will be called *n-formulas*. Also, the first-order language which is able to speak about n-cubes, the *n-frame language*, has n binary predicates $R_0, \dots, R_{n-1}$ (and no equality). One can expand the n-frame language with countably many unary predicates p ($p \in P$), and define the *standard first-order translation* $\varphi_A(x)$ of each n-formula A as follows.

$$\varphi_p(x) \stackrel{\text{def}}{=} \mathsf{p}(x), \quad \text{for } p \in P;$$
$$\varphi_{B \wedge C}(x) \stackrel{\text{def}}{=} \varphi_B(x) \wedge \varphi_C(x); \qquad \varphi_{\neg B}(x) \stackrel{\text{def}}{=} \neg\varphi_B(x);$$
$$\varphi_{\Diamond_\ell B}(x) \stackrel{\text{def}}{=} \exists y \ (x R_\ell y \ \wedge \ \varphi_B(y/x)), \quad \text{for } \ell < n \quad \text{(here } y \text{ is a fresh variable)}.$$

An *n-frame* is a relational structure $\mathscr{F} = (F, R_\ell^{\mathscr{F}})_{\ell<n}$ where for each $\ell < n$, $R_\ell^{\mathscr{F}}$ is a binary (*accessibility*) relation on the *set F of (possible) worlds*. Therefore, n-cubes are special n-frames. Throughout, n-frames are denoted by script letters with the corresponding roman letter denoting the set of worlds. A *model* $\mathscr{M} = (\mathscr{F}, \upsilon)$ *based on an n-frame* $\mathscr{F}$ is defined in the usual way, by giving a subset $\upsilon(p)$ of F (a *valuation*), for each propositional variable p. We also say that $\mathscr{F}$ is the *underlying n-frame of* $\mathscr{M}$. Truth and validity of n-formulas in models and n-frames are defined as usual. Note that a model $\mathscr{M}$ can be considered as a first-order model of the n-frame language expanded with countably many unary predicates. It is routine to check that, for any n-formula A, A is valid in $\mathscr{M}$ (considered as a modal model) iff its standard first-order translation φ_A is valid in $\mathscr{M}$ (considered as a first-order model).

The usual operations on frames can be defined on n-frames as well. In particular, given two n-frames $\mathscr{F} = (F, R_\ell^{\mathscr{F}})_{\ell<n}$ and $\mathscr{G} = (G, R_\ell^{\mathscr{G}})_{\ell<n}$, a function $h : F \to G$ is called a *p-morphism from $\mathscr{F}$ to $\mathscr{G}$* if it satisfies the following conditions, for all $u, v \in F$, $y \in G$, $\ell < n$.

- $u R_\ell^{\mathscr{F}} v$ implies $h(u) R_\ell^{\mathscr{G}} h(v)$ (*forward condition*)
- $h(u) R_\ell^{\mathscr{G}} y$ implies $(\exists w \in F)\ h(w) = y$ and $u R_\ell^{\mathscr{F}} w$ (*backward condition*).

If h is onto then we say that $\mathscr{G}$ *is a p-morphic image of* $\mathscr{F}$. $\mathscr{F}$ is a *subframe of* $\mathscr{G}$ if $F \subseteq G$ and, for all $\ell < n$, $R_\ell^{\mathscr{F}} = R_\ell^{\mathscr{G}} \cap (F \times F)$. Given some $w \in G$, the *subframe $\mathscr{G}^w$ of $\mathscr{G}$ generated by point w* is the subframe of $\mathscr{G}$ with the following set G^w of worlds:

$$G^w = \{w\} \cup \{u \in G : u \text{ is accessible from } w \text{ by the transitive closure of } \textstyle\bigcup_{\ell<n} R_\ell^{\mathscr{G}}\} .$$

Similarly to the mono-modal case, validity in frames is preserved under taking p-morphic images, point-generated subframes and disjoint unions as well.

An *n-modal logic* is a set of n-formulas closed under the rules of Substitution, Modus Ponens, and Necessitation $A/\Box_\ell A$ ($\ell < n$), and containing all propositional tautologies and all formulas $\Box_\ell(p \to q) \to (\Box_\ell p \to \Box_\ell q)$ ($\ell < n,\ p, q \in P$). We say that an n-modal logic $\boldsymbol{L}$ is *axiomatised by* some set Σ of n-formulas, if $\boldsymbol{L}$ is the smallest n-modal logic which contains Σ. An n-modal logic $\boldsymbol{L}$ is *determined by* a class $\mathbf{C}$ of n-frames, if $\boldsymbol{L}$ is the set of all n-formulas which are valid in each member of $\mathbf{C}$. $\boldsymbol{K}^n$ is the n-modal logic determined by the class of all n-cubes.

Main results. It is proved in [7] that the logic $\boldsymbol{K}^2$ can be axiomatised by the following two Sahlqvist-type 2-formulas (both are followed by their first-order correspondents).

Commutativity: $\Box_0\Box_1 p \leftrightarrow \Box_1\Box_0 p$

$$\forall xyz[(xR_0y \wedge yR_1z \to \exists u(xR_1u \wedge uR_0z)) \\ \wedge (xR_1y \wedge yR_0z \to \exists u(xR_0u \wedge uR_1z))]$$

Church–Rosser property: $\Diamond_0\Box_1 p \to \Box_1\Diamond_0 p$

$$\forall xyz[xR_0y \wedge xR_1z \to \exists u(yR_1u \wedge zR_0u)]$$

Our main result says that in higher dimensions an axiomatisation must be infinite:

THEOREM 1.1. *For any natural number $n > 2$, $\boldsymbol{K}^n$ is not finitely axiomatisable in the n-modal language.*

Theorem 1.1 answers negatively the first part of question Q16.163 of [6] namely, whether $\boldsymbol{K}^3$ is finitely axiomatisable. A negative answer to Question 23 posed in [7] also follows: $\boldsymbol{K}^3$ is not axiomatisable with the commutativity and Church–Rosser axioms (each of them is stated now for all pairs of coordinates).

PROOF OF THEOREM 1.1. We define a series $(\mathscr{F}_k : k \in \omega)$ of n-frames with the following properties:

(I) For every k, $\mathscr{F}_k$ does not validate $\boldsymbol{K}^n$ (Lemma 3.4 in §3).

(II) For any series of models $\mathscr{M}_k$ based on $\mathscr{F}_k$ ($k \in \omega$), there is some model $\mathscr{M}'$ such that (i) $\mathscr{M}'$ is an elementary substructure[1] of some nontrivial ultraproduct of the $\mathscr{M}_k$'s, and (ii) the underlying frame of $\mathscr{M}'$ validates $\boldsymbol{K}^n$ (Lemma 4.6 in §4).

Given such $\mathscr{F}_k$'s, assume now that there is some n-formula Ax axiomatising $\boldsymbol{K}^n$. Then, by (I), for each k there is some model $\mathscr{M}_k$ based on $\mathscr{F}_k$ such that Ax fails in $\mathscr{M}_k$. Then, considering now $\mathscr{M}_k$ as a first-order structure of the language having binary predicates $R_0, \dots, R_{n-1}$ and countably many unary predicates, the standard first-order translation φ_{Ax} of Ax fails in $\mathscr{M}_k$. Then, by (II)(i), φ_{Ax} fails in $\mathscr{M}'$ as well. Thus, considering now $\mathscr{M}'$ as a modal model, Ax fails in $\mathscr{M}'$. But this contradicts (II)(ii) namely, that the underlying frame of $\mathscr{M}'$ validates $\boldsymbol{K}^n$, so it must validate Ax. ⊣

Since $\boldsymbol{K}^2$ is axiomatised by Sahlqvist formulas, $\boldsymbol{K}^2$ is determined by a first-order definable class of frames. Our second result says that this latter property also holds in higher dimensions.

THEOREM 1.2. *For any natural number $n > 0$, $\boldsymbol{K}^n$ is determined by a class of n-frames satisfying a single first-order sentence of the n-frame language.*

[1] Here modal models are considered as relational structures of the first-order language (without equality) having binary predicates $R_0, \dots, R_{n-1}$ and countably many unary predicates.

Questions. (all are for $2 < n \in \omega$)

Q1. $\boldsymbol{K}^n$ is known to be recursively enumerable, see Cor.5.8 of [7]. Find a modal axiomatisation for $\boldsymbol{K}^n$. Is there an axiomatisation using only finitely many propositional variables?

Q2. Is $\boldsymbol{K}^n$ Sahlqvist?

Q3. As it is mentioned, $\boldsymbol{S5}^n$ is also known to be non-finitely axiomatisable (a result of Johnson [11], proved in an algebraic setting). Is $\boldsymbol{S5}^n$ finitely axiomatisable over $\boldsymbol{K}^n$?

Plan of paper. The next section gives the definition of the n-frames $\mathscr{F}_k$. We try to demonstrate the 'geometrical reason' behind properties (I) and (II) above: While 'not too large' pieces of $\mathscr{F}_k$ are always 'representable' in the sense that they are p-morphic images of n-cubes, $\mathscr{F}_k$ itself is not representable. However, taking larger and larger frames, the 'distance' between the two 'clashing patterns' which cause the non-representability becomes 'infinite in the limit'.

In §3 a series of $\boldsymbol{K}^n$-valid n-formulas is introduced which shows that the $\mathscr{F}_k$'s are not only non-representable, but in fact they do not validate $\boldsymbol{K}^n$ (property (I) above).

In §4 a certain two-player game is defined on so-called 'networks' over an n-frame $\mathscr{F}$. This game is similar to the ones played on networks over various atomic algebras of n-ary relations in the papers of Hirsch and Hodkinson, see e.g., [10]. In our case, a network is a 'semi p-morphism' (satisfying only the forward condition) from some n-cube (not necessarily on)to $\mathscr{F}$. Playing the game over some countable n-frame $\mathscr{F}$, the second, 'existential' player has a winning strategy in the ω-length game over $\mathscr{F}$ iff every point-generated subframe of $\mathscr{F}$ is a p-morphic image of some n-cube. Also, for any sequence $(\mathscr{F}_k : k \in \omega)$ of n-frames, if the existential player has a winning strategy in longer and longer finite games over $\mathscr{F}_k$ as k increases then she has a winning strategy in the ω-length game over any nontrivial ultraproduct of the $\mathscr{F}_k$'s. Therefore, given such n-frames $\mathscr{F}_k$, an argument similar to one in [10] shows that there is some model $\mathscr{M}$ having property (II) above, which completes the proof of Theorem 1.1.

In §5 we prove that, playing the above game over n-frames satisfying a certain first-order sentence Φ_n (of the n-frame language), the existential player always has a winning strategy in the ω-length game. This implies that $\boldsymbol{K}^n$ is determined by the class of all n-frames satisfying Φ_n, thus proves Theorem 1.2.

Finally, in §6 we discuss some possible generalisation of the results, and some related open problems.

§2. Frames.

In this section we construct the n-frames $\mathscr{F}_k$ $(k \in \omega)$. These frames are obtained by sticking together copies of three small 'gadgets' $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$. Below we first define these gadgets and show that they are 'representable' in the sense that they are p-morphic images of n-cubes. Next we define the $\mathscr{F}_k$'s and show that certain not too large subframes of them are representable. Here we also illustrate that the $\mathscr{F}_k$'s themselves are not representable, and later in §3 we prove that they do not even validate $\boldsymbol{K}^n$.

About the drawings. All the n-frames to be defined in this section are such that relations $R_\ell = \emptyset$ whenever $2 < \ell < n$. Therefore they can be illustrated with the help of pictures showing 3-dimensional objects. In those figures which show

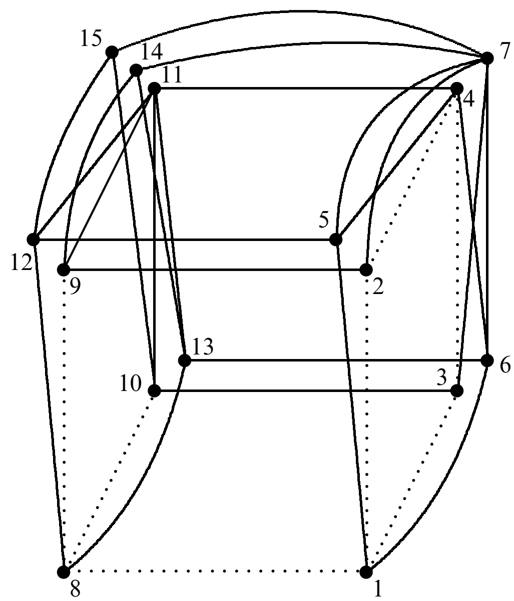
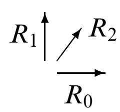


FIGURE 1. Gadget $\mathscr{G}_1$.

'abstract' (i.e., non-cubic) 3-frames, our convention in drawing R_0, R_1 and R_2 is the following.



We usually draw lines instead of arrows. However, when we draw 3-cubes (Figures 2, 4, 6, 8, 9), we always indicate the three 1-frames whose product the 3-cube in question is. The dotted lines in the figures indicate some patterns which will be explained at the end of this section and in §3.

DEFINITION 2.1. **(gadget $\mathscr{G}_1$)** Let $\mathscr{G}_1 = (G_1, R_\ell)_{\ell<n}$ be the following n-frame:

$$
\begin{aligned}
G_1 &\stackrel{\text{def}}{=} \{1, 2, 3, \ldots, 15\}\\
R_0 &\stackrel{\text{def}}{=} \{(8,1), (9,2), (10,3), (11,4), (12,5), (13,6), (14,7), (15,7)\}\\
R_1 &\stackrel{\text{def}}{=} \{(1,2), (1,5), (3,4), (3,7), (6,4), (6,7),\\
&\qquad (8,9), (8,12), (10,11), (10,15), (13,11), (13,14)\}\\
R_2 &\stackrel{\text{def}}{=} \{(1,3), (1,6), (2,4), (2,7), (5,4), (5,7),\\
&\qquad (8,10), (8,13), (9,11), (9,14), (12,11), (12,15)\}\\
R_\ell &\stackrel{\text{def}}{=} \emptyset, \ \text{ for } 2 < \ell < n \ \text{ (see Figure 1).}
\end{aligned}
$$

PROPOSITION 2.1. *There is a p-morphism h_1 from some n-cube $\mathscr{H}_1$ onto $\mathscr{G}_1$.*

PROOF. Consider the following three small 1-frames, $\mathscr{V}$, $\mathscr{A}$ and $\mathscr{E}$.

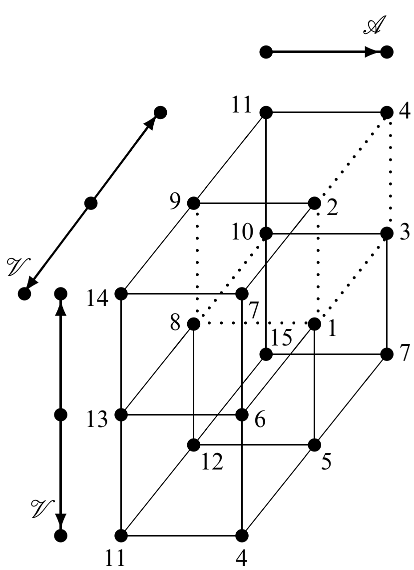
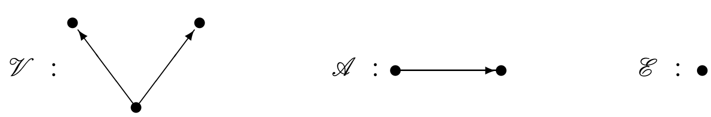


FIGURE 2. The p-morphism $h_1 : \mathscr{H}_1 \to \mathscr{G}_1$.



Let $\mathscr{H}_1$ be $\mathscr{A}\times\mathscr{V}\times\mathscr{V}$ for $n=3$, and $\mathscr{A}\times\mathscr{V}\times\mathscr{V}\times\mathscr{E}^{n-3}$ for $n>3$. Figure 2 illustrates $\mathscr{H}_1$ and defines a function h_1 from $\mathscr{H}_1$ to $\mathscr{G}_1$ by labelling each world of $\mathscr{H}_1$ with its h_1-image. It is routine to check that h_1 is a p-morphism onto $\mathscr{G}_1$. ⊣

DEFINITION 2.2. **(gadget $\mathscr{G}_2$)** Let $\mathscr{G}_2 = (G_2, R_\ell)_{\ell<n}$ be the following n-frame:

$$
\begin{aligned}
G_2 &\stackrel{\text{def}}{=} \{1,2,3,4,5,6,7,\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e},\mathsf{f},\mathsf{g},\mathsf{h},\mathsf{i},\mathsf{j},\mathsf{k}\}\\
R_0 &\stackrel{\text{def}}{=} \{(1,\mathsf{a}),(1,\mathsf{e}),(2,\mathsf{b}),(2,\mathsf{g}),(3,\mathsf{c}),(3,\mathsf{i}),\\
&\quad (4,\mathsf{j}),(4,\mathsf{k}),(5,\mathsf{b}),(5,\mathsf{f}),(6,\mathsf{c}),(6,\mathsf{h}),(7,\mathsf{d}),(7,\mathsf{k})\}\\
R_1 &\stackrel{\text{def}}{=} \{(1,2),(1,5),(3,4),(3,7),(6,4),(6,7),(\mathsf{a},\mathsf{b}),(\mathsf{a},\mathsf{f}),\\
&\quad (\mathsf{c},\mathsf{d}),(\mathsf{c},\mathsf{k}),(\mathsf{e},\mathsf{b}),(\mathsf{e},\mathsf{g}),(\mathsf{h},\mathsf{j}),(\mathsf{h},\mathsf{k}),(\mathsf{i},\mathsf{j}),(\mathsf{i},\mathsf{k})\}\\
R_2 &\stackrel{\text{def}}{=} \{(1,3),(1,6),(2,4),(2,7),(5,4),(5,7),\\
&\quad (\mathsf{a},\mathsf{c}),(\mathsf{a},\mathsf{h}),(\mathsf{b},\mathsf{d}),(\mathsf{b},\mathsf{j}),(\mathsf{e},\mathsf{c}),(\mathsf{e},\mathsf{i}),(\mathsf{f},\mathsf{k}),(\mathsf{g},\mathsf{k})\}\\
R_\ell &\stackrel{\text{def}}{=} \emptyset, \quad \text{for } 2<\ell<n \ \text{ (see Figure 3).}
\end{aligned}
$$

PROPOSITION 2.2. *There is a p-morphism h_2 from some n-cube $\mathscr{H}_2$ onto $\mathscr{G}_2$.*

PROOF. Take the 1-frames $\mathscr{V}$ and $\mathscr{E}$ defined in the proof of Prop. 2.1, and let $\mathscr{H}_2$ be $\mathscr{V}\times\mathscr{V}\times\mathscr{V}$ for $n=3$, and $\mathscr{V}\times\mathscr{V}\times\mathscr{V}\times\mathscr{E}^{n-3}$ for $n>3$. Figure 4 illustrates $\mathscr{H}_2$ and defines the p-morphism h_2. ⊣

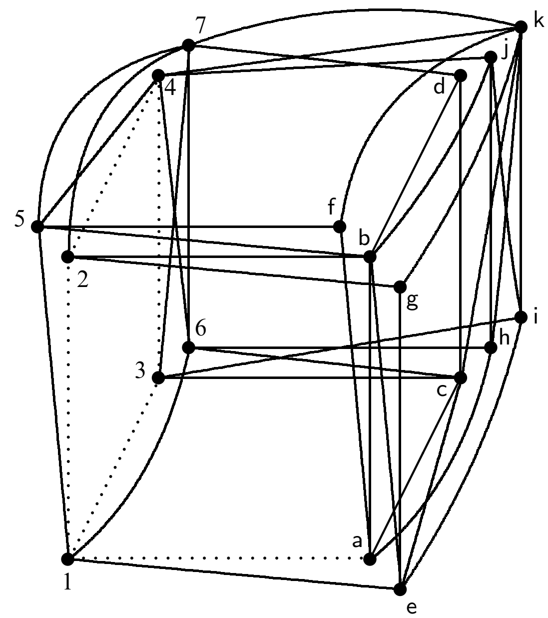


FIGURE 3. Gadget $\mathscr{G}_2$.

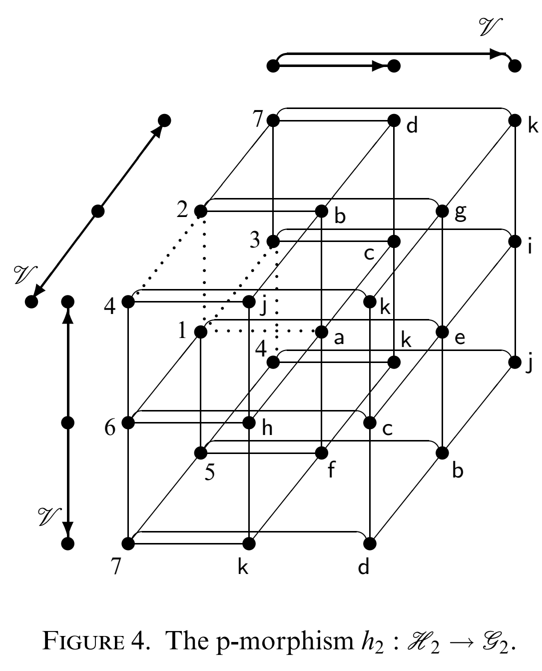


FIGURE 4. The p-morphism $h_2 : \mathscr{H}_2 \to \mathscr{G}_2$.

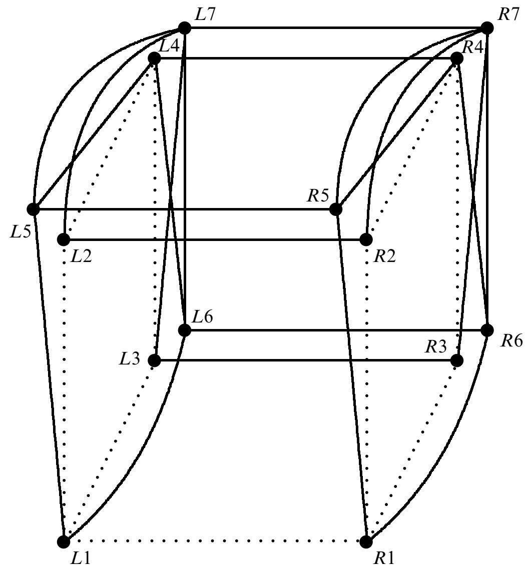


FIGURE 5. Gadget $\mathscr{G}_3$.

DEFINITION 2.3. **(gadget $\mathscr{G}_3$)** Let $\mathscr{G}_3 = (G_3, R_\ell)_{\ell<n}$ be the following n-frame:

$$
\begin{aligned}
G_3 &\stackrel{\text{def}}{=} \{L1, L2, L3, L4, L5, L6, L7, R1, R2, R3, R4, R5, R6, R7\}\\
R_0 &\stackrel{\text{def}}{=} \{(L1, R1), (L2, R2), (L3, R3), (L4, R4), (L5, R5), (L6, R6), (L7, R7)\}\\
R_1 &\stackrel{\text{def}}{=} \{(L1, L2), (L1, L5), (L3, L4), (L3, L7), (L6, L4), (L6, L7),\\
&\quad (R1, R2), (R1, R5), (R3, R4), (R3, R7), (R6, R4), (R6, R7)\}\\
R_2 &\stackrel{\text{def}}{=} \{(L1, L3), (L1, L6), (L2, L4), (L2, L7), (L5, L4), (L5, L7),\\
&\quad (R1, R3), (R1, R6), (R2, R4), (R2, R7), (R5, R4), (R5, R7)\}\\
R_\ell &\stackrel{\text{def}}{=} \emptyset, \quad \text{for } 2 < \ell < n \quad \text{(see Figure 5).}
\end{aligned}
$$

PROPOSITION 2.3. *There are two different p-morphisms h_3^1 and h_3^2 onto $\mathscr{G}_3$, both are coming from the same n-cube.*

PROOF. It is easy to see that $\mathscr{G}_1$ and $\mathscr{G}_3$ are p-morphic images of the same n-cube $\mathscr{H}_1$, defined in the proof of Prop. 2.1. However, in case of $\mathscr{G}_3$ one can give two different p-morphisms h_3^1 and h_3^2 from $\mathscr{H}_1$. See Figure 6 for the definitions of h_3^1 and h_3^2, again by labelling the worlds of $\mathscr{H}_1$ with their p-morphic images in $\mathscr{G}_3$. $\dashv$

Now we are in a position to define the n-frames $\mathscr{F}_k$, for $k \in \omega$. Observe that the 'right face' of gadget $\mathscr{G}_1$ (i.e., the subframe consisting of worlds $1, 2, 3, 4, 5, 6, 7$) is isomorphic to the 'left face' of gadget $\mathscr{G}_2$, and also to both the left and right faces of gadget $\mathscr{G}_3$. $\mathscr{F}_k$ will be the n-frame obtained by 'sticking together' $\mathscr{G}_1$, then k-many $\mathscr{G}_3$'s, and then $\mathscr{G}_2$, always identifying the corresponding '$1, 2, 3, 4, 5, 6, 7$'-faces. This 'sticking' process can be defined in general as follows. Assume that two arbitrary n-frames $\mathscr{A}$ and $\mathscr{B}$ are given, together with subframes $\mathscr{A}' \subseteq \mathscr{A}$, $\mathscr{B}' \subseteq \mathscr{B}$ such that

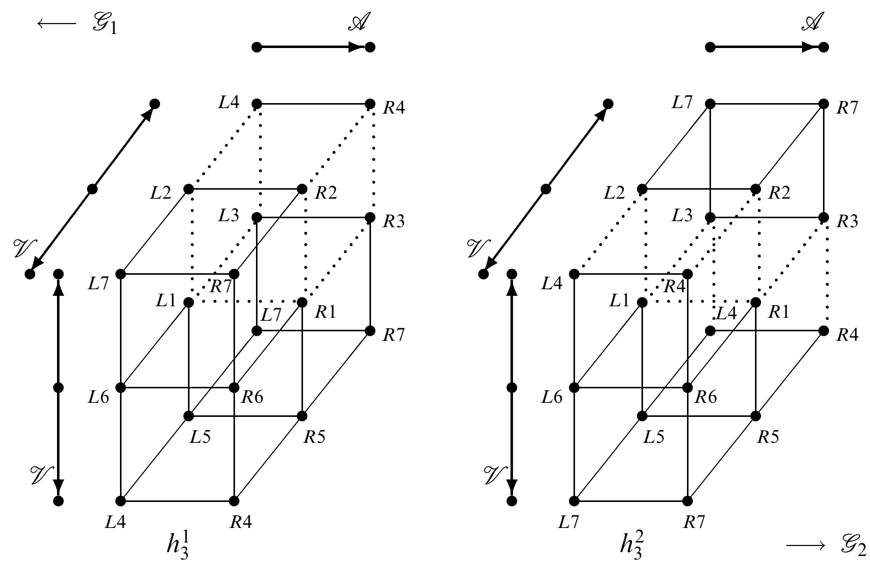


FIGURE 6. The two different p-morphisms onto $\mathscr{G}_3$.

there is some isomorphism f between $\mathscr{A}'$ and $\mathscr{B}'$. First, take an isomorphic copy $\mathscr{A}^*$ of $\mathscr{A}$ along some isomorphism g such that

- g extends f
- $A^* \cap B = B'$.

Next, define the *amalgam* $\mathsf{Am}(\mathscr{A}, \mathscr{B}, f)$ *of* $\mathscr{A}$ *and* $\mathscr{B}$ *along* f to be the union (as relational structures) of $\mathscr{A}^*$ and $\mathscr{B}$ that is, let

$$\mathsf{Am}(\mathscr{A}, \mathscr{B}, f) \stackrel{\text{def}}{=} (A^* \cup B, R_\ell^{\mathscr{A}^*} \cup R_\ell^{\mathscr{B}})_{\ell<n}$$

(which is now defined up to isomorphism).

For each $0 < k \in \omega$, we define the n-frame $\mathscr{G}_3(k)$ as follows. Let $\mathscr{G}_3(1) \stackrel{\text{def}}{=} \mathscr{G}_3$, and for each $0 < k \in \omega$, let $\mathscr{G}_3(k+1) \stackrel{\text{def}}{=} \mathsf{Am}(\mathscr{G}_3, \mathscr{G}_3(k), f_{33})$, where f_{33} is the function taking world Ri to (some isomorphic copy of) Li, for $i = 1, \ldots, 7$.

DEFINITION 2.4. **(frames $\mathscr{F}_k^{left}$, $\mathscr{F}_k^{right}$ and $\mathscr{F}_k$)** Let

$$\mathscr{F}_0^{left} \stackrel{\text{def}}{=} \mathscr{G}_1,\ \mathscr{F}_0^{right} \stackrel{\text{def}}{=} \mathscr{G}_2, \text{ and } \mathscr{F}_0 \stackrel{\text{def}}{=} \mathsf{Am}(\mathscr{G}_1, \mathscr{G}_2, f_{12}),$$

where f_{12} is the identity on the set $\{1, 2, 3, 4, 5, 6, 7\}$. For $k > 0$, let

$$\mathscr{F}_k^{left} \stackrel{\text{def}}{=} \mathsf{Am}(\mathscr{G}_1, \mathscr{G}_3(k), f_{13}),$$

where f_{13} is the function taking world i to (some isomorphic copy of) Li, for $i = 1, \ldots, 7$; let

$$\mathscr{F}_k^{right} \stackrel{\text{def}}{=} \mathsf{Am}(\mathscr{G}_3(k), \mathscr{G}_2, f_{32}),$$

where f_{32} is the function taking world Ri to i, for $i = 1, \ldots, 7$; and let

$$\mathscr{F}_k \stackrel{\text{def}}{=} \mathsf{Am}(\mathscr{F}_k^{left}, \mathscr{G}_2, f_{32}) = \mathsf{Am}(\mathscr{G}_1, \mathscr{F}_k^{right}, f_{13}) \text{ (see Figure 7).}$$

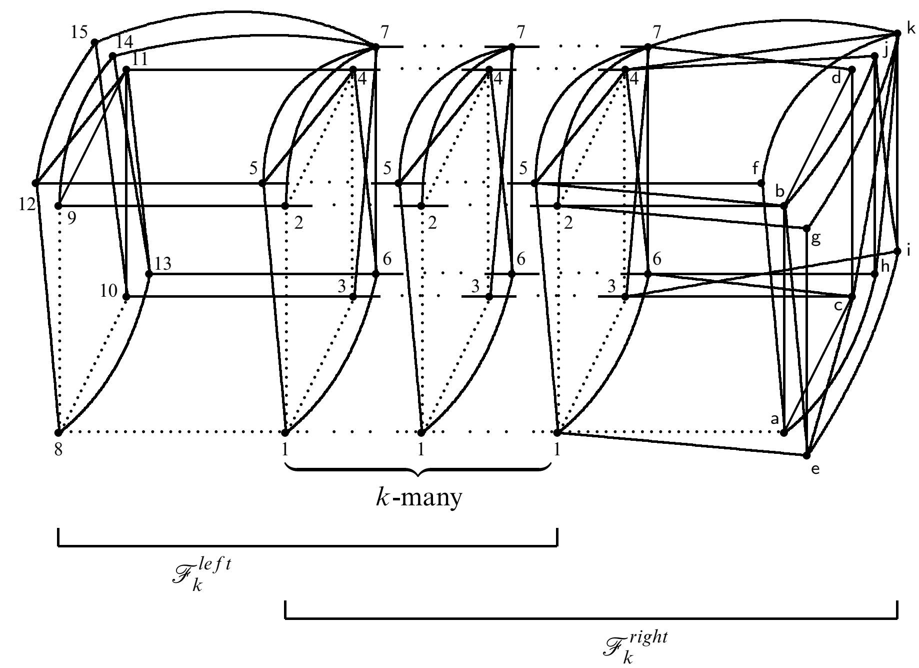


FIGURE 7. Frames $\mathscr{F}_k$, $\mathscr{F}_k^{left}$ and $\mathscr{F}_k^{right}$.

PROPOSITION 2.4. *Both* $\mathscr{F}_k^{left}$ *and* $\mathscr{F}_k^{right}$ *are p-morphic images of* n*-cubes.*

PROOF. See Figures 8 and 9. ⊣

In the next section we will show that $\mathscr{F}_k$ does not validate $\boldsymbol{K}^n$, thus it cannot be a p-morphic image of any n-cube. One way of seeing the reason for this is as follows. $\mathscr{G}_2$ can be considered as a gadget forcing a p-morphism where there is no pre-image of world 4 which is R-accessible from some pre-images of both 2 and 3. On the other hand, $\mathscr{G}_1$ forces a p-morphism where all pre-images of 4 are R-accessible from some pre-images of both 2 and 3. Since $\mathscr{G}_3$ only transfers these 'forces' to some distance, wherever they meet they clash: there cannot be a p-morphism with both properties.

§**3. Formulas.** For each natural number $k > 0$, we define φ_k to be the following first-order sentence of the n-frame language (in fact, only predicates R_0, R_1 and R_2 are used), see also Figure 10:

$$\begin{aligned}\varphi_k :\ & \forall x_1 \dots x_k xyz\ \big[xR_0x_1 \wedge x_1R_0x_2 \wedge \cdots \wedge x_{k-1}R_0x_k \wedge xR_1y \wedge xR_2z \ \rightarrow \\ & \exists\, uy_1 \dots y_k z_1 \dots z_k u_1 \dots u_k\ \big(yR_2u \wedge zR_1u \wedge yR_0y_1 \wedge \\ & y_1R_0y_2 \wedge \cdots \wedge y_{k-1}R_0y_k \wedge zR_0z_1 \wedge z_1R_0z_2 \wedge \cdots \wedge z_{k-1}R_0z_k \wedge uR_0u_1 \wedge \\ & u_1R_0u_2 \wedge \cdots \wedge u_{k-1}R_0u_k \wedge x_kR_1y_k \wedge x_kR_2z_k \wedge y_kR_2u_k \wedge z_kR_1u_k\big)\big]\,.\end{aligned}$$

It is easy to check the following claim.

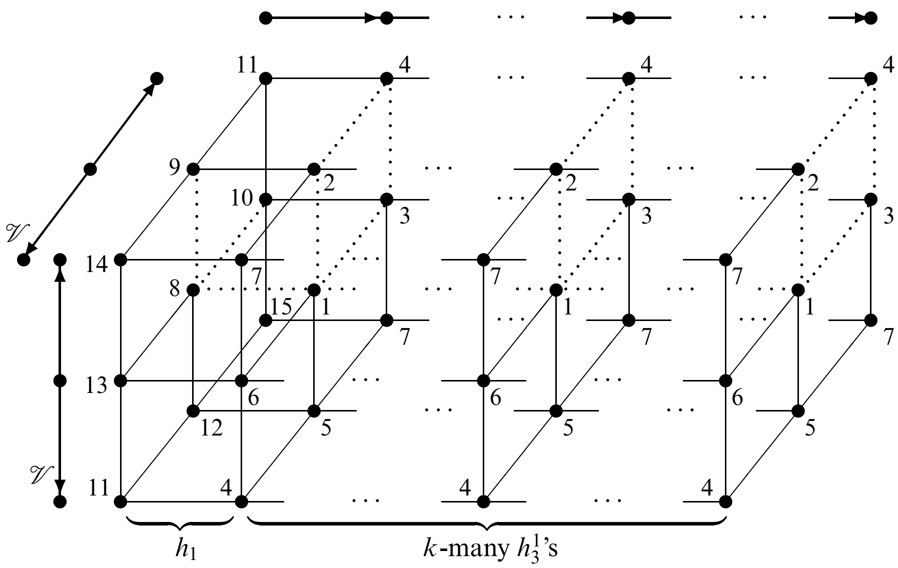


FIGURE 8. A p-morphism onto $\mathscr{F}_k^{left}$.

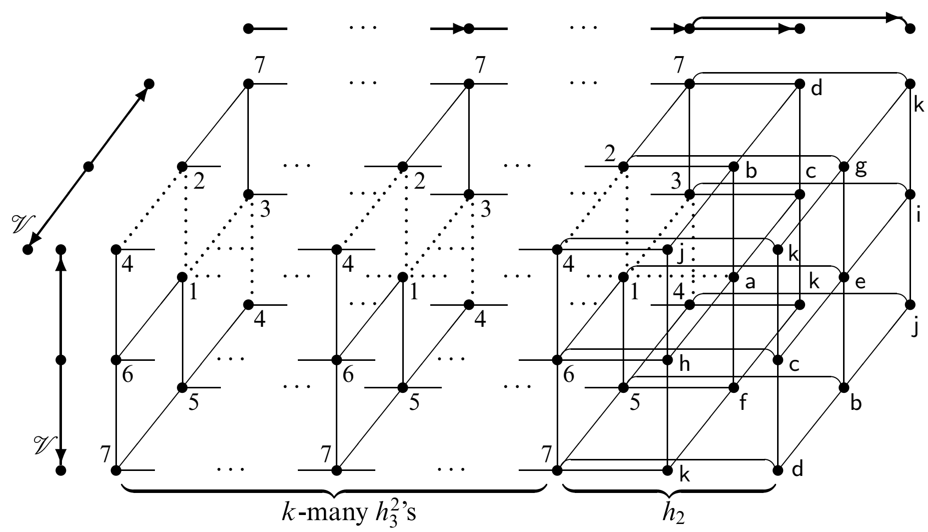


FIGURE 9. A p-morphism onto $\mathscr{F}_k^{right}$.

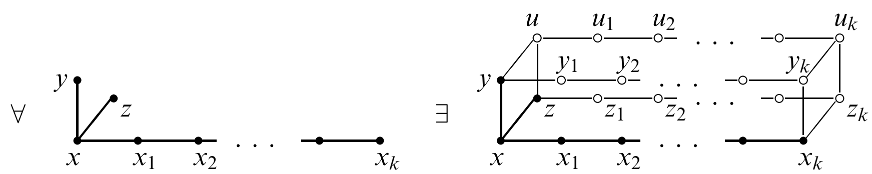


FIGURE 10. The formula φ_k.

CLAIM 3.1. *For any* $0 < k \in \omega$, φ_k *is valid in every* n*-cube.*

These first-order properties are modally definable. Namely, for each $0 < k \in \omega$, consider the following n-formula V_k:

$$\begin{aligned} V_k : \Big[&\Diamond_0^k(\Box_1 p_{01} \wedge \Box_2 p_{02}) \ \wedge\ \Diamond_1(\Box_0^k p_{10} \wedge \Box_2 p_{12}) \ \wedge\ \Diamond_2(\Box_0^k p_{20} \wedge \Box_1 p_{21}) \wedge \\ &\wedge \Box_0^k \Box_1 (p_{01} \wedge p_{10} \to \Box_2 q_2) \ \wedge\ \Box_0^k \Box_2 (p_{02} \wedge p_{20} \to \Box_1 q_1) \ \wedge \\ &\wedge \Box_1 \Box_2 (p_{12} \wedge p_{21} \to \Box_0^k q_0) \Big] \ \longrightarrow\ \Diamond_0^k \Diamond_1 \Diamond_2 (q_0 \wedge q_1 \wedge q_2) \ . \end{aligned}$$

(Here $\Diamond_0^k$ and $\Box_0^k$ abbreviate k-length sequences of $\Diamond_0$'s and $\Box_0$'s, respectively.)

CLAIM 3.2. *For any* $0 < k \in \omega$, *and for any* n*-frame* $\mathscr{F}$, φ_k *is valid in* $\mathscr{F}$ *iff* V_k *is valid in* $\mathscr{F}$.

PROOF. We prove the harder right-to-left direction only. Fix some $0 < k \in \omega$. Assume $\mathscr{F} = (F, R_\ell^{\mathscr{F}})_{\ell < n}$ is an n-frame validating V_k, and let $x, y, z, x_1, \dots, x_k \in F$ be given as in φ_k. In order to 'cubify' them, we define a model $\mathscr{M} = (\mathscr{F}, \upsilon)$ on $\mathscr{F}$ as follows.

- $\upsilon(p_{01}) \stackrel{\text{def}}{=} \{v \in F : x_k R_1^{\mathscr{F}} v\}$
- $\upsilon(p_{02}) \stackrel{\text{def}}{=} \{v \in F : x_k R_2^{\mathscr{F}} v\}$
- $\upsilon(p_{10}) \stackrel{\text{def}}{=} \{v \in F : v \text{ is } R_0^{\mathscr{F}}\text{-accessible in } k \text{ steps from } y\}$
- $\upsilon(p_{12}) \stackrel{\text{def}}{=} \{v \in F : y R_2^{\mathscr{F}} v\}$
- $\upsilon(p_{20}) \stackrel{\text{def}}{=} \{v \in F : v \text{ is } R_0^{\mathscr{F}}\text{-accessible in } k \text{ steps from } z\}$
- $\upsilon(p_{21}) \stackrel{\text{def}}{=} \{v \in F : z R_1^{\mathscr{F}} v\}$
- $\upsilon(q_0) \stackrel{\text{def}}{=} \{v \in F : v \text{ is } R_0^{\mathscr{F}}\text{-accessible in } k \text{ steps from an } s \in \upsilon(p_{02}) \cap \upsilon(p_{20})\}$
- $\upsilon(q_1) \stackrel{\text{def}}{=} \{v \in F : \exists s \in \upsilon(p_{02}) \cap \upsilon(p_{20})\ s R_1^{\mathscr{F}} v\}$
- $\upsilon(q_2) \stackrel{\text{def}}{=} \{v \in F : \exists s \in \upsilon(p_{01}) \cap \upsilon(p_{10})\ s R_2^{\mathscr{F}} v\}$

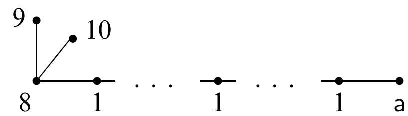
It is routine to check that the antecedent of V_k holds in $\mathscr{M}$ at world x. Thus, by assumption, $\Diamond_0^k \Diamond_1 \Diamond_2 (q_0 \wedge q_1 \wedge q_2)$ also holds in $\mathscr{M}$ at x which implies that there are worlds $x_1', \dots, x_k', y_k', u_k$ with $x R_0^{\mathscr{F}} x_1', \dots, x_{k-1}' R_0^{\mathscr{F}} x_k'$, $x_k' R_1^{\mathscr{F}} y_k'$, $y_k' R_2^{\mathscr{F}} u_k$, and $q_0 \wedge q_1 \wedge q_2$ holds in $\mathscr{M}$ at u_k. Unfolding the definitions of $\upsilon(q_0)$, $\upsilon(q_1)$ and $\upsilon(q_2)$, we obtain worlds $u, y_1, \dots, y_k, z_1, \dots, z_k, u_1, \dots, u_{k-1}$ as required. $\dashv$

COROLLARY 3.3. *For any* $0 < k \in \omega$, V_k *is a* $\boldsymbol{K}^n$*-validity.*

Next, we prove that the frames $\mathscr{F}_k$ $(k \in \omega)$, defined in §2, do not validate $\boldsymbol{K}^n$.

LEMMA 3.4. *For any* $k \in \omega$, V_{k+2} *fails in* $\mathscr{F}_k$. *Thus,* $\mathscr{F}_k$ *does not validate* $\boldsymbol{K}^n$.

PROOF. Take the following 'fork' of $\mathscr{F}_k$:



(dotted lines in Figure 7). Then

- the only world which is both R_2-accessible from 9 and R_1-accessible from 10 is 11;
- the only world which is both R_1-accessible from a and R_0-accessible in $k+2$ steps from 9 is b;
- the only world which is both R_2-accessible from a and R_0-accessible in $k+2$ steps from 10 is c;
- the only world which is both R_2-accessible from b and R_1-accessible from c is d;

but d is not R_0-accessible in $k+2$ steps from 11. ⊣

Remark. The formula V_1 was shown to me by V. Shehtman. It was Sz. Mikulás who pointed out that this formula does not follow from the commutativity and Church-Rosser properties. A 3-frame (of 33 worlds) showing this independence of V_1 can be found in 3.2.68 of [9]. As far as I know, V_1 is the simplest $\boldsymbol{K}^n$-validity which does not follow from the above obvious properties. Note that, by Prop.2.4, Cor.3.3, and Lemma 3.4, V_k does not follow from commutativity and Church-Rosser properties plus $\{V_j : j < k\}$, for any $k > 1$.

§4. Games.

Notation. Throughout, we use notation $\bar{u} = (u_0, \ldots, u_{n-1})$ for n-tuples. Given $\ell < n$, and two n-tuples $\bar{u}$ and $\bar{v}$, $\bar{u} \equiv_\ell \bar{v}$ denotes that $u_k = v_k$ whenever $k \neq \ell$. In case h is a function and X is a subset of its domain then $h|_X$ denotes the restriction of h to X.

Fix, for this section, an arbitrary n-frame $\mathscr{F} = (F, R_\ell^{\mathscr{F}})_{\ell<n}$. We will define a game between two players $\forall$ (male) and $\exists$ (female) over $\mathscr{F}$. In this game, $\exists$ intends to construct, step-by-step, a p-morphism from some n-cube onto $\mathscr{F}$, and $\forall$ tries to challenge her by showing possible defects of her construction. Our game and its properties are similar to those of Hirsch and Hodkinson in [10] where games are played on networks over various atomic algebras of n-ary relations.

We define an $\mathscr{F}$*-network* to be a tuple

$$N = (U_0^N, \ldots, U_{n-1}^N, R_0^N, \ldots, R_{n-1}^N, h^N),$$

where for each $\ell < n$, U_ℓ^N is a non-empty set, $R_\ell^N \subseteq U_\ell^N \times U_\ell^N$, and $h^N : U_0^N \times \cdots \times U_{n-1}^N \to F$ is a function such that for all $\bar{u}, \bar{v} \in U_0^N \times \cdots \times U_{n-1}^N$, for all $\ell < n$,

$$\text{if} \quad \bar{u} \equiv_\ell \bar{v} \quad \text{and} \quad u_\ell R_\ell^N v_\ell \quad \text{then} \quad h^N(\bar{u}) R_\ell^{\mathscr{F}} h^N(\bar{v})$$

(that is, h^N is a homomorphism[2] from the n-cube $(U_0^N, R_0^N) \times \cdots \times (U_{n-1}^N, R_{n-1}^N)$ to the n-frame $\mathscr{F}$). An $\mathscr{F}$-network N is called *finite* if each of the sets U_ℓ^N $(\ell < n)$ is finite.

[2]A homomorphism satisfies the forward condition (concerning p-morphisms) only.

We define a *game* $G_\omega(\mathscr{F})$ between $\forall$ and $\exists$. They build a countable sequence of finite $\mathscr{F}$-networks

$$N_0 \subseteq N_1 \subseteq \dots \subseteq N_i \subseteq \dots .$$

(Here $N_{i-1} \subseteq N_i$ means that, for each $\ell < n$, $U_\ell^{N_{i-1}} \subseteq U_\ell^{N_i}$, $R_\ell^{N_{i-1}} \subseteq R_\ell^{N_i}$, and $h^{N_{i-1}} \subseteq h^{N_i}$.)

In the 0^{th} round, $\forall$ picks any world $a \in F$. $\exists$ responds with some $\mathscr{F}$-network N_0 with $U_0^{N_0} \times \dots \times U_{n-1}^{N_0} = \{\bar{u}\}$ for some n-tuple $\bar{u}$, $R_\ell^{N_0} = \emptyset$ $(\ell < n)$, and $h^{N_0}(\bar{u}) = a$.

In the i^{th} round $(0 < i \in \omega)$, some sequence $N_0 \subseteq \dots \subseteq N_{i-1}$ of $\mathscr{F}$-networks is already built. $\forall$ picks

- an n-tuple $\bar{v} \in U_0^{N_{i-1}} \times \dots \times U_{n-1}^{N_{i-1}}$
- an index $\ell < n$
- a world $b \in F$ such that $h^{N_{i-1}}(\bar{v}) R_\ell^{\mathscr{F}} b$.

$\exists$ can respond in two ways. If there is some n-tuple $\bar{w} \in U_0^{N_{i-1}} \times \dots \times U_{n-1}^{N_{i-1}}$ with $\bar{v} \equiv_\ell \bar{w}$, $v_\ell R_\ell^{N_{i-1}} w_\ell$ and $h^{N_{i-1}}(\bar{w}) = b$ then she responds with $N_i = N_{i-1}$. Otherwise, she responds (if she can) with some $\mathscr{F}$-network N_i extending N_{i-1} such that

- $U_\ell^{N_i} = U_\ell^{N_{i-1}} \cup \{u^+\}$ (where u^+ is some fresh point); $R_\ell^{N_i} = R_\ell^{N_{i-1}} \cup \{(v_\ell, u^+)\}$;
- $U_k^{N_i} = U_k^{N_{i-1}}$, $R_k^{N_i} = R_k^{N_{i-1}}$ whenever $k \neq \ell$; and
- $h^{N_i}(v_0, \dots, v_{\ell-1}, u^+, v_{\ell+1}, \dots, v_{n-1}) = b$.

If $\exists$ can respond in each round i for $i \in \omega$ then *she wins the play*. We say that $\exists$ *has a winning strategy in* $G_\omega(\mathscr{F})$ if she can win all plays, whatever moves $\forall$ takes in the rounds.

The game $G_k(\mathscr{F})$ (for $k \in \omega$) is similar to $G_\omega(\mathscr{F})$, but there are only k rounds. If $\exists$ can successfully respond in all rounds up to the k^{th} round, she has won the play. Similarly, we say that $\exists$ *has a winning strategy in* $G_k(\mathscr{F})$, if she can win all plays of length k. Clearly, if $\exists$ has a winning strategy in $G_k(\mathscr{F})$ for some $k \leq \omega$, then she has a winning strategy in $G_i(\mathscr{F})$ as well, for any $i < k$.

Claim 4.1. *If $\mathscr{F}$ is a p-morphic image of some n-cube then $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$.*

Proof. Suppose that there is some p-morphism h onto $\mathscr{F}$, coming from some n-cube $\mathscr{G} = (U_0, R_0) \times \dots \times (U_{n-1}, R_{n-1})$. $\exists$ can use this h to determine her winning strategy in $\mathscr{G}_\omega(\mathscr{F})$ as follows. In the 0^{th} round, suppose $\forall$ picks some $a \in F$. Then let $\exists$ choose some h-preimage $\bar{u} = (u_0, \dots, u_{n-1})$ of a and let her respond with the $\mathscr{F}$-network $N_0 = (\{u_0\}, \dots, \{u_{n-1}\}, \emptyset, \dots, \emptyset, h|_{\{\bar{u}\}})$. We may assume that in the i^{th} round $(i > 0)$ some $\mathscr{F}$-network N_{i-1} has been already constructed with the following properties:

$$\begin{aligned}&(\forall \ell < n)\ U_\ell^{N_{i-1}} \subseteq U_\ell,\ R_\ell^{N_{i-1}} \subseteq R_\ell, \text{and}\\ &h^{N_{i-1}} = h|_{U_0^{N_{i-1}} \times \dots \times U_{n-1}^{N_{i-1}}}\ .\end{aligned} \tag{1}$$

Let $\forall$ pick some n-tuple $\bar{v} \in U_0^{N_{i-1}} \times \dots \times U_{n-1}^{N_{i-1}}$, $\ell < n$, and $b \in F$ with $h^{N_{i-1}}(\bar{v}) R_\ell^{\mathscr{F}} b$. Then, by (1), $h(\bar{v}) R_\ell^{\mathscr{F}} b$ holds. Let $\exists$ choose some $\bar{w} \in U_0 \times \dots \times U_{n-1}$ with $h(\bar{w}) = b$, $\bar{v} \equiv_\ell \bar{w}$ and $v_\ell R_\ell w_\ell$. If $\bar{w}$ can be chosen from $U_0^{N_{i-1}} \times \dots \times U_{n-1}^{N_{i-1}}$ then let $\exists$

respond with $N_i = N_{i-1}$. Otherwise, $w_\ell \notin U_\ell^{N_{i-1}}$ must hold, thus let her take $U_\ell^{N_i} = U_\ell^{N_{i-1}} \cup \{w_\ell\}$; $R_\ell^{N_i} = R_\ell^{N_{i-1}} \cup \{(v_\ell, w_\ell)\}$; $U_k^{N_i} = U_k^{N_{i-1}}$, $R_k^{N_i} = R_k^{N_{i-1}}$, whenever $k \neq \ell$; and $h^{N_i} = h|_{U_0^{N_i} \times \cdots \times U_{n-1}^{N_i}}$. Clearly, N_i is an $\mathscr{F}$-network extending N_{i-1}, in both cases. Moreover, since (1) is still satisfied, $\exists$ can continue to play in this way forever. ⊣

CLAIM 4.2. *Let $\mathscr{F}$ be a countable n-frame.*

(i) *If $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$ then every point-generated subframe of $\mathscr{F}$ is a p-morphic image of some n-cube.*
(ii) *If $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$ then $\mathscr{F}$ validates $\boldsymbol{K}^n$.*

PROOF. (i): Pick some subframe $\mathscr{F}^a$ of $\mathscr{F}$, generated by world a. Consider a play of the game $G_\omega(\mathscr{F})$ when $\forall$ eventually picks all possible n-tuples, and corresponding worlds and R_ℓ-connections of $\mathscr{F}^a$ (since $\mathscr{F}^a$ is countable, he can do this). If $\exists$ uses her strategy then she succeeds to construct a countable ascending chain of $\mathscr{F}$-networks whose union gives a p-morphism from some n-cube onto $\mathscr{F}^a$.

(ii): This follows from (i), since every frame is a p-morphic image of some disjoint union of its point-generated subframes. ⊣

CLAIM 4.3. *If, for every $k \in \omega$, $\mathscr{G}_k$ is an n-frame such that $\exists$ has a winning strategy in $G_k(\mathscr{G}_k)$ then $\exists$ has a winning strategy in $G_\omega(\mathscr{G})$, where $\mathscr{G}$ is any nontrivial ultraproduct of the $\mathscr{G}_k$'s.*

PROOF. Let $\mathscr{G}$ be an ultraproduct of the $\mathscr{G}_k$'s over some nonprincipal ultrafilter D over ω. One can define the 'ultraproduct' of the winning strategies in $G_k(\mathscr{G}_k)$ to obtain a winning strategy in $G_\omega(\mathscr{G})$ as follows. It is easy to see that, for any finite $\mathscr{G}$-network $N = (U_0, \ldots, U_{n-1}, R_0, \ldots, R_{n-1}, h)$, one can choose a series of functions

$$h^{(k)} : U_0 \times \cdots \times U_{n-1} \to G_k \quad (k \in \omega),$$

such that $h(\bar{u})$ equals to the D-class of the sequence $(h^{(k)}(\bar{u}) : k \in \omega)$, for each $\bar{u} \in U_0 \times \cdots \times U_{n-1}$; and

$$\{k \in \omega : N^{(k)} \stackrel{\text{def}}{=} (U_0, \ldots, U_{n-1}, R_0, \ldots, R_{n-1}, h^{(k)}) \text{ is a finite } \mathscr{G}_k\text{-network}\} \in D.$$

Also, it is easy to see that, in round 0 of any play of the game $G_\omega(\mathscr{G})$, $\exists$ can always respond with some finite $\mathscr{G}$-network N_0 such that

$$\{k \in \omega \ : \ N_0^{(k)} \text{ is a finite } \mathscr{G}_k\text{-network}\} \in D \ .$$

Thus we may assume that in round i $(0 < i \in \omega)$ some sequence $N_0 \subseteq \cdots \subseteq N_{i-1}$ of finite $\mathscr{G}$-networks is already defined such that

$$X_i \stackrel{\text{def}}{=} \{k \in \omega \ : \ N_0^{(k)} \subseteq \cdots \subseteq N_{i-1}^{(k)} \text{ is a sequence of finite } \mathscr{G}_k\text{-networks}\} \in D \ .$$

Let $N_{i-1} = (U_0, \ldots, U_{n-1}, R_0, \ldots, R_{n-1}, h)$ and assume that $\forall$ picks some $\bar{v} \in U_0 \times \cdots \times U_{n-1}$, $\ell < n$, $b \in G$ with $h(\bar{v}) R_\ell^{\mathscr{G}} b$. We show that $\exists$ can always respond properly with some finite $\mathscr{G}$-network $N_i \supseteq N_{i-1}$ such that

$$\{k \in \omega : N_0^{(k)} \subseteq \ldots N_{i-1}^{(k)} \subseteq N_i^{(k)} \text{ is a sequence of finite } \mathscr{G}_k\text{-networks}\} \in D \qquad (2)$$

holds. Assume that b is the D-class of some sequence $(b_k \in G_k : k \in \omega)$. Then

$$T_i \stackrel{\text{def}}{=} \{k \in \omega \ : \ h^{(k)}(\bar{v}) R_\ell^{\mathscr{G}_k} b_k\} \in D,$$

and, by assumption on the $\mathscr{G}_k$'s,

$$Y_i \stackrel{\text{def}}{=} \{k \in \omega \ : \ \exists \text{ has a winning strategy in } G_i(\mathscr{G}_k)\} \in D \ .$$

For any $k \in T_i \cap X_i \cap Y_i$, let $M_i^{(k)} = (\dots, f^{(k)})$ be $\exists$'s response to $\forall$'s move $\bar{v}, \ell, b_k$ in the play of the game $G_i(\mathscr{G}_k)$, having history $N_0^{(k)} \subseteq \cdots \subseteq N_{i-1}^{(k)}$. Then

(i) either $Z_i \stackrel{\text{def}}{=} \{k \in T_i \cap X_i \cap Y_i \ : \ M_i^{(k)} = N_{i-1}^{(k)}\} \in D$

(ii) or $Z_i^+ \stackrel{\text{def}}{=} \{k \in T_i \cap X_i \cap Y_i \ : \ M_i^{(k)} \supset N_{i-1}^{(k)}\} \in D$

hold. In case (i), let $\exists$ respond with $N_i = N_{i-1}$, and let $N_i^{(k)} \stackrel{\text{def}}{=} M_i^{(k)}$, for all $k \in Z_i$, and arbitrary otherwise. In case (ii), we may assume that $\exists$ used the same fresh point u^+ to extend U_ℓ for the $\mathscr{G}_k$-network $M_i^{(k)}$ $(k \in Z_i^+)$. Then let $N_i^{(k)} \stackrel{\text{def}}{=} M_i^{(k)}$, for all $k \in Z_i^+$, and arbitrary otherwise. Define a function f as follows. For all $\bar{u} \in U_0 \times \cdots \times (U_\ell \cup \{u^+\}) \times \ldots \times U_{n-1}$, for all $k \in \omega$, let

$$(f(\bar{u}))_k \stackrel{\text{def}}{=} \begin{cases} f^{(k)}(\bar{u}) \ , & \text{if } k \in Z_i^+ \\ \text{any } c \in G_k \ , & \text{otherwise .} \end{cases}$$

Finally, for all $\bar{u} \in U_0 \times \cdots \times (U_\ell \cup \{u^+\}) \times \ldots \times U_{n-1}$, let $h^{N_i}(\bar{u})$ be the D-class of the sequence $(f^{(k)}(\bar{u}) : k \in \omega)$, and let

$$N_i \stackrel{\text{def}}{=} (U_0, \cdots, U_\ell \cup \{u^+\}, \ldots, U_{n-1}, R_0, \cdots, R_\ell \cup \{(v_\ell, u^+)\}, \ldots, R_{n-1}, h^{N_i}) \ .$$

It is easily checked that, in both cases (i) and (ii), N_i is a proper response satisfying (2), thus $\exists$ can continue this way forever. ⊣

Claim 4.4. *If $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$ then for any model $\mathscr{M}$ on $\mathscr{F}$ there is some countable elementary substructure*[3] *$\mathscr{M}' = (\mathscr{F}', \dots)$ of $\mathscr{M}$ such that $\exists$ also has a winning strategy in $G_\omega(\mathscr{F}')$.*

Proof. We will build a countable elementary chain of countable, elementary substructures of $\mathscr{M}$, and define $\mathscr{M}'$ to be the union of the chain. Let $\mathscr{M}_0'$ be any countable, elementary substructure of $\mathscr{M}$ (which exists by the downward Löwenheim-Skolem-Tarski theorem). Suppose that we have already defined the countable, elementary substructure $\mathscr{M}_k' = (\mathscr{F}_k', \dots)$ of $\mathscr{M}$, for some $k \in \omega$. $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$: every move she takes according to this strategy depends on the actual move of $\forall$ and on the 'history' of that particular play — that is, on the previous moves of $\forall$ and the previous responses of her. We also know that in each round of any play the number of worlds of $\mathscr{F}$ mentioned by $\exists$ in that play so far is always finite. Now consider those plays of the game $G_\omega(\mathscr{F})$ where in each round $\forall$ can pick worlds from $\mathscr{F}_k'$ only. Then the set $S_k \subseteq F$ of all those worlds which are mentioned in some response of $\exists$ in some of these plays is countable. Let $\mathscr{M}_{k+1}'$ be a countable, elementary substructure of $\mathscr{M}$ containing $S_k \cup F_k'$ (again, it exists by the downward Löwenheim-Skolem-Tarski theorem). Finally, let $\mathscr{M}'$ be the union (as structures) of the M_k''s, $k \in \omega$. Then $\mathscr{M}'$ is an elementary substructure of $\mathscr{M}$, by the elementary chain theorem (see e.g., [3]). Clearly, $\mathscr{M}'$ is countable and $\exists$ has a winning strategy in $G_\omega(\mathscr{F}')$. ⊣

[3] Here modal models are considered as relational structures of the first-order language (without equality) having binary predicates $R_0, \dots, R_{n-1}$ and countably many unary predicates for the propositional variables.

Now consider the sequence $(\mathscr{F}_k : k \in \omega)$ of n-frames defined in §2. Below we show (Lemma 4.6) that they satisfy property (II), discussed in §1.

CLAIM 4.5. *For each* $k \in \omega$, $\exists$ *has a winning strategy in* $G_k(\mathscr{F}_k)$.

PROOF. Fix some $k \in \omega$. By Prop. 2.4, both $\mathscr{F}_k^{left}$ and $\mathscr{F}_k^{right}$ are p-morphic images of n-cubes. This implies, by Claim 4.1, that $\exists$ has a winning strategy in $G_\omega(\mathscr{F}_k^{left})$ and also in $G_\omega(\mathscr{F}_k^{right})$. Consider a play of the game $G_k(\mathscr{F}_k)$. In the 0^{th} round, $\forall$ picks some $a \in F_k$ such that either (i) $a \in F_k^{left} - F_k^{right}$ or (ii) $a \in F_k^{right}$. Thus from now on this play over $\mathscr{F}_k$ (which is of length less than k now) can be considered as either a play over $\mathscr{F}_k^{left}$ (in case (i)) or over $\mathscr{F}_k^{right}$ (in case (ii)). In both cases, $\exists$ is able to survive k rounds. ⊣

LEMMA 4.6. *For any series of models* $\mathscr{M}_k$ *on* $\mathscr{F}_k$ $(k \in \omega)$, *there is some model* $\mathscr{M}'$ *such that* (i) $\mathscr{M}'$ *is an elementary substructure of some nontrivial ultraproduct of the* $\mathscr{M}_k$*'s, and* (ii) *the underlying frame of* $\mathscr{M}'$ *validates* $\boldsymbol{K}^n$.

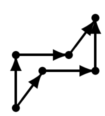
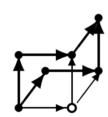
PROOF. Let $\mathscr{M}$ be some nontrivial ultraproduct of the $\mathscr{M}_k$'s. Then clearly, the underlying frame $\mathscr{F}$ of $\mathscr{M}$ is the ultraproduct of the $\mathscr{F}_k$'s. By Claims 4.5 and 4.3, $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$. Now one can use Claims 4.4 and 4.2(ii) to obtain a model $\mathscr{M}'$ as required. ⊣

Now, Lemmas 3.4 and 4.6 together complete the proof of Theorem 1.1.

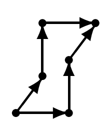
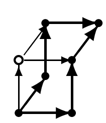
§5. First-order axiomatisability. In this section we prove Theorem 1.2. First, we discuss the case of $n = 3$ and then show how to generalise it for larger n's. Consider the following sentences of the 3-frame language: for all $i < j < 3$, let

$\psi_1^{ij} : \forall xyz(xR_jy \wedge yR_iz \rightarrow \exists u(xR_iu \wedge uR_jz))$ (commutativity$_1$);
$\psi_2^{ij} : \forall xyz(xR_iy \wedge yR_jz \rightarrow \exists u(xR_ju \wedge uR_iz))$ (commutativity$_2$);
$\psi_3^{ij} : \forall xyz(xR_iy \wedge xR_jz \rightarrow \exists u(yR_ju \wedge zR_iu))$ (Church–Rosser property);

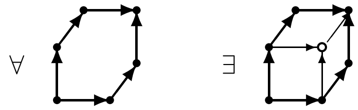
$$\chi_1 : \forall xyzstr(xR_1s \wedge sR_0y \wedge yR_2r \wedge xR_2t \wedge tR_0z \wedge zR_1r \rightarrow \exists u(xR_0u \wedge uR_1y \wedge uR_2z));$$

∀  ∃ 

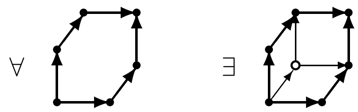
$$\chi_2 : \forall xyzstr(yR_0s \wedge sR_1x \wedge xR_2t \wedge yR_2r \wedge rR_1z \wedge zR_0t \rightarrow \exists u(uR_0x \wedge yR_1u \wedge uR_2z));$$

∀  ∃ 

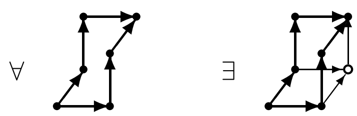
$$\chi_3 : \forall xyzstr(sR_0y \wedge yR_2r \wedge rR_1z \wedge sR_1x \wedge xR_2t \wedge tR_0z \rightarrow \exists u(xR_0u \wedge yR_1u \wedge uR_2z));$$



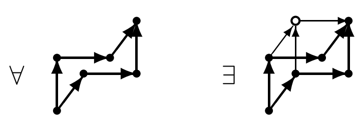
$$\chi_4 : \forall xyzstr(zR_0t \wedge tR_2x \wedge xR_1s \wedge zR_1r \wedge rR_2y \wedge yR_0s \rightarrow \exists u(uR_0x \wedge uR_1y \wedge zR_2u));$$



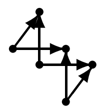
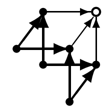
$$\chi_5 : \forall xyzstr(tR_0z \wedge zR_1r \wedge rR_2y \wedge tR_2x \wedge xR_1s \wedge sR_0y \rightarrow \exists u(xR_0u \wedge uR_1y \wedge zR_2u));$$



$$\chi_6 : \forall xyzstr(rR_1z \wedge zR_0t \wedge tR_2x \wedge rR_2y \wedge yR_0s \wedge sR_1x \rightarrow \exists u(uR_0x \wedge yR_1u \wedge zR_2u));$$



$$\chi_7 : \forall xyzstr(sR_0y \wedge sR_1x \wedge tR_0z \wedge tR_2x \wedge rR_1z \wedge rR_2y \rightarrow \exists u(xR_0u \wedge yR_1u \wedge zR_2u));$$

$\forall$  $\exists$ 

and let

$\Phi_3: \ \psi_1^{01} \wedge \psi_2^{01} \wedge \psi_3^{01} \wedge \psi_1^{02} \wedge \psi_2^{02} \wedge \psi_3^{02} \wedge \psi_1^{12} \wedge \psi_2^{12} \wedge \psi_3^{12} \wedge \chi_1 \wedge \chi_2 \wedge \chi_3 \wedge \chi_4 \wedge \chi_5 \wedge \chi_6 \wedge \chi_7$.

It is easy to check the following claim.

Claim 5.1. *Φ_3 holds in every 3-cube.*

Lemma 5.2. *If $\mathscr{F}$ is a 3-frame satisfying Φ_3 then $\exists$ has a winning strategy in $G_\omega(\mathscr{F})$.*

As a corollary we obtain that $\boldsymbol{K}^3$ is determined by a first-order definable class of frames.

THEOREM 5.3. *$\boldsymbol{K}^3$ is determined by the class of 3-frames satisfying Φ_3. That is, a 3-formula is valid in every 3-cube iff it is valid in every 3-frame satisfying Φ_3.*

Proof of Theorem 5.3. Claim 5.1 proves the right-to-left direction. For the other direction, it is enough to consider only countable 3-frames satisfying Φ_3, by a standard Löwenheim–Skolem–Tarski argument (see e.g., [7], Prop.5.4). However, countable 3-frames satisfying Φ_3 validate $\boldsymbol{K}^3$, by Lemma 5.2 and Claim 4.2(ii). $\dashv$

Proof of Lemma 5.2. Fix some 3-frame $\mathscr{F} = (F, R_0^{\mathscr{F}}, R_1^{\mathscr{F}}, R_2^{\mathscr{F}})$ satisfying Φ_3. We define the strategy $\exists$ should follow in each round i ($i \in \omega$) of the game over $\mathscr{F}$. In round 0, her response is determined by the rules of the game. In round i ($0 < i \in \omega$), some sequence $N_0 \subseteq \cdots \subseteq N_{i-1}$ of $\mathscr{F}$-networks is already constructed with, say, $N_{i-1} = (U_0, U_1, U_2, R_0, R_1, R_2, h)$. Assume that $\forall$ picks some $\bar{u} \in U_0 \times U_1 \times U_2$, $\ell < 3$ and $b \in F$ with $h(u_0, u_1, u_2) R_\ell^{\mathscr{F}} b$.

First, assume that $\ell = 0$. By the rules of the game, if there is some $\bar{v} \in U_0 \times U_1 \times U_2$ such that $\bar{u} \equiv_0 \bar{v}$ and $u_0 R_0 v_0$ then $\exists$ must respond with $N_i \stackrel{\text{def}}{=} N_{i-1}$. Otherwise, she has to take a fresh point u^+ and to respond with some $\mathscr{F}$-network

$$N_i = (U_0 \cup \{u^+\}, U_1, U_2, R_0^+, R_1^+, R_2^+, h^+)\ ,$$

where $R_0^+ = R_0 \cup \{(u_0, u^+)\}$, $R_1^+ = R_1$, $R_2^+ = R_2$, $h^+|_{U_0 \times U_1 \times U_2} = h$ and $h^+(u^+, u_1, u_2) = b$. If both U_1 and U_2 are one-element sets then there is nothing more to do, since N_i is defined. Otherwise, say, when $|U_1| > 1$, the remaining task is to define h^+ on all the 3-tuples of form (u^+, v, w), where $v \in U_1$, $w \in U_2$ and $(v, w) \neq (u_1, u_2)$. (These 3-tuples will be called *new 3-tuples*.) In order to do this, let us observe the following.

Claim 5.4. *For each $\ell < 3$, the structure (U_ℓ, R_ℓ) is a finite, irreflexive, intransitive tree.*

We intend to define a binary relation $\prec$ on $(U_0 \cup \{u^+\}) \times U_1 \times U_2$. To this end, recall the 3-tuple $\bar{u} = (u_0, u_1, u_2)$ which $\forall$ picked. Enumerate $U_1 = \{a^0, a^1, \ldots, a^{M_1}\}$ ($M_1 \geq 1$) in the following way: let $a^0 \stackrel{\text{def}}{=} u_1$ and then take the unique R_1^{-1}-path, starting from u_1 and ending with the root of the tree (U_1, R_1). (Call these points of U_1 as *downward points*.) Then continue with all the other points of U_1 in their order of 'creation' in the game (*upward points* of U_1). Enumerate $U_2 = \{b^0, \ldots, b^{M_2}\}$ in a similar way, starting from u_2. It is not hard to see that these enumerations have the following property.

Claim 5.5. *For all* $0 < j \leq M_1$, *there is a unique* 1-predecessor of a^j: *there is a unique* $k < j$ *such that either* $a^k R_1 a^j$ *or* $a^j R_1 a^k$. *In particular, if* a^j *is a downward point then its* 1-*predecessor is* a^{j-1} *and* $a^j R_1 a^{j-1}$; *if* a^j *is an upward point and* a^k *is its* 1-*predecessor then* $a^k R_1 a^j$. *Similarly, for all* $0 < j \leq M_2$, *there is a unique* 2-predecessor of b^j.

Now for all $\bar{v}, \bar{v}' \in (U_0 \cup \{u^+\}) \times U_1 \times U_2$, let

$$\begin{aligned}\bar{v} \prec \bar{v}' \iff & \text{ either } \bar{v} \in U_0 \times U_1 \times U_2 \text{ and } \bar{v} \neq \bar{v}';\\ & \text{or } v_0 = v_0' = u^+ \text{ and } v_1 = a^j,\ v_1' = a^k \text{ with } j < k;\\ & \text{or } v_0 = v_0' = u^+ \text{ and } v_1 = v_1' \text{ and } v_2 = b^j,\ v_2' = b^k \text{ with } j < k\ .\end{aligned}$$

Claim 5.6. *Any two distinct elements of* $U_0 \times U_1 \times U_2$ *are* $\prec$-*comparable*; *any element of* $U_0 \times U_1 \times U_2$ *is* $\prec$-*less than* (u^+, u_1, u_2); *and* $\prec$ *is an irreflexive, transitive linear ordering on* $\{u^+\} \times U_1 \times U_2$ *with* (u^+, u_1, u_2) *being the* $\prec$-*least element*.

Now we are in a position to define the function h^+ on $\{u^+\} \times U_1 \times U_2$. By Claim 5.6, we can proceed by induction on $\prec$. For all new 3-tuples $\bar{x}$, we will define $h^+(\bar{x})$ in such a way that the following always holds:

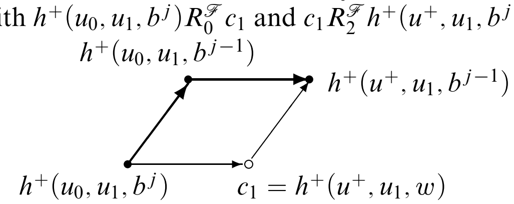
$$\begin{aligned}(\text{I.H.}(\bar{x}))\ (\forall \ell < 3)(\forall \bar{y} \prec \bar{x})\ & \bar{y} \equiv_\ell \bar{x} \text{ and } y_\ell R_\ell^+ x_\ell \text{ implies } h^+(\bar{y}) R_\ell^{\mathcal{F}} h^+(\bar{x})\\ & \bar{y} \equiv_\ell \bar{x} \text{ and } x_\ell R_\ell^+ y_\ell \text{ implies } h^+(\bar{x}) R_\ell^{\mathcal{F}} h^+(\bar{y}).\end{aligned}$$

This condition clearly holds for $\bar{x} = (u^+, u_1, u_2)$ because of the following. If $\bar{y} \prec \bar{x}$ then $\bar{y}$ must be an element of $U_0 \times U_1 \times U_2$. If $\bar{y} \equiv_0 \bar{x}$ and either $y_0 R_0^+ x_0$ or $x_0 R_0^+ y_0$ hold then the only possibility is $\bar{y} = (u_0, u_1, u_2)$ and $y_0 R_0^+ x_0$. But in this case $h^+(\bar{x}) = b$ and $h^+(\bar{y}) R_0^{\mathcal{F}} b$, by $h^+(\bar{y}) = h(\bar{y})$. In case $\ell = 1, 2$, there is no $\bar{y} \prec \bar{x}$ with $\bar{y} \equiv_\ell \bar{x}$, by Claim 5.6.

Now take some new 3-tuple $\bar{x}$ and assume that h^+ has been defined on all $\bar{z} \in \{u^+\} \times U_1 \times U_2$, $\bar{z} \prec \bar{x}$ such that $(\text{I.H.}(\bar{z}))$ hold. We distinguish three cases:

(1) $\bar{x} = (u^+, u_1, w)$ for some $w \neq u_2$;
(2) $\bar{x} = (u^+, v, u_2)$ for some $v \neq u_1$;
(3) $\bar{x} = (u^+, v, w)$ for some $v \neq u_1,\ w \neq u_2$.

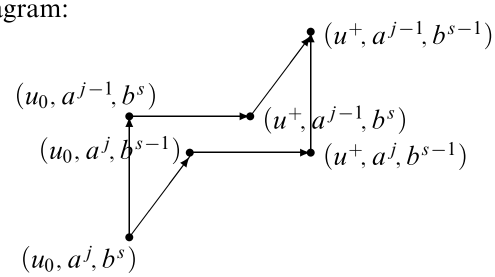
Case (1): Recall the enumeration $\{b^0, \dots, b^{M_2}\}$ of U_2. Assume $w = b^j$, for some $0 < j \leq M_2$. There are two cases. **Case (1a):** w is a downward point. Then, by Claim 5.5, the 2-predecessor of w is b^{j-1} and $b^j R_2 b^{j-1}$ holds. Since $h^+|_{U_0 \times U_1 \times U_2} = h$, $h^+(u_0, u_1, b^j) R_2^{\mathcal{F}} h^+(u_0, u_1, b^{j-1})$ follows. Also, since the new 3-tuple $(u^+, u_1, b^{j-1}) \prec \bar{x}$, we have $(\text{I.H.}(u^+, u_1, b^{j-1}))$. Thus, by $u_0 R_0^+ u^+$ and $(u_0, u_1, b^{j-1}) \prec (u^+, u_1, b^{j-1})$, $h^+(u_0, u_1, b^{j-1}) R_0^{\mathcal{F}} h^+(u^+, u_1, b^{j-1})$ follows. Thus there is a $c_1 \in F$ with $h^+(u_0, u_1, b^j) R_0^{\mathcal{F}} c_1$ and $c_1 R_2^{\mathcal{F}} h^+(u^+, u_1, b^{j-1})$, by ψ_1^{02}.



Let $h^+(\bar{x}) \stackrel{\text{def}}{=} c_1$. It remains to show that $(\text{I.H.}(\bar{x}))$ holds. The only $\bar{y} \prec \bar{x}$ such that $\bar{y} \equiv_0 \bar{x}$ and either $y_0 R_0^+ x_0$ or $x_0 R_0^+ y_0$ is $\bar{y} = (u_0, u_1, b^j)$. There is no $\bar{y} \prec \bar{x}$ with $\bar{y} \equiv_1 \bar{x}$. If $\bar{y} \prec \bar{x}$, $\bar{y} \equiv_2 \bar{x}$ and either $y_2 R_2^+ x_2$ or $x_2 R_2^+ y_2$ then $\bar{y}$ must be (u^+, u_1, b^{j-1}), by Claim 5.5. In all cases, $(\text{I.H.}(\bar{x}))$ holds. **Case (1b):** w is an upward point. Let b^k be the 2-predecessor of b^j. Then, by Claim 5.5, $k < j$ and $b^k R_2 b^j$. Everything is similar to case (1a), but this time one must use ψ_3^{02} to define $h^+(\bar{x})$.

Case (2): $\bar{x} = (u^+, v, u_2)$ for some $v \neq u_1$. **Case (2a):** v is a downward point. Use ψ_1^{01}. **Case (2b):** v is an upward point. Use ψ_3^{01}.

Case (3): $\bar{x} = (u^+, v, w)$ for some $v \neq u_1$, $w \neq u_2$. **Case (3a):** both v and w are downward points, say $v = a^j$, $w = b^s$, for some $0 < j \leq M_1$, $0 < s \leq M_2$. Then, by Claim 5.5, the 1-predecessor of v is a^{j-1}, the 2-predecessor of w is b^{s-1}, and we have the following diagram:



All new 3-tuples of this diagram are $\prec$-less than $\bar{x}$. Thus, by $h^+|_{U_0\times U_1\times U_2} = h$ and (I.H.), the h^+-images of the above 3-tuples are connected in the same way as they are. Therefore, by χ_1, there is some $c_2 \in F$ such that

$$h^+(u_0, a^j, b^s) R_0^{\mathscr{F}} c_2,\ c_2 R_1^{\mathscr{F}} h^+(u^+, a^{j-1}, b^s) \text{ and } c_2 R_2^{\mathscr{F}} h^+(u^+, a^j, b^{s-1}) .$$

Let $h^+(\bar{x}) = h^+(u^+, a^j, b^s) \stackrel{\text{def}}{=} c_2$. It remains to show that (I.H.$(\bar{x})$) holds. If $\bar{y} \prec \bar{x}$, $\bar{y} \equiv_0 \bar{x}$ and either $y_0 R_0^+ x_0$ or $x_0 R_0^+ y_0$ then $\bar{y}$ must be (u_0, a^j, b^s). If $\bar{y} \prec \bar{x}$, $\bar{y} \equiv_1 \bar{x}$ and either $y_1 R_1^+ x_1$ or $x_1 R_1^+ y_1$ then $\bar{y}$ must be (u^+, a^{j-1}, b^s). If $\bar{y} \prec \bar{x}$, $\bar{y} \equiv_2 \bar{x}$ and either $y_2 R_2^+ x_2$ or $x_2 R_2^+ y_2$ then $\bar{y}$ must be (u^+, a^j, b^{s-1}). In all cases, (I.H.$(\bar{x})$) holds. **Case (3b):** v is an upward point, w is a downward point. It is similar to case (3a), but now use χ_3. **Case (3c):** v is a downward point, w is an upward point. Use χ_5. **Case (3d):** both v and w are upward points. Use χ_7.

This way we defined $h^+(\bar{x})$, for all new 3-tuples $\bar{x}$.

Claim 5.7. *$N_i = (U_0 \cup \{u^+\}, U_1, U_2, R_0^+, R_1^+, R_2^+, h^+)$ is an $\mathscr{F}$-network extending N_{i-1}.*

Proof. $N_i \supseteq N_{i-1}$ by definition. Take some $\ell < 3$ and $\bar{x}, \bar{y} \in (U_0 \cup \{u^+\}) \times U_1 \times U_2$ with $\bar{x} \equiv_\ell \bar{y}$ and $x_\ell R_\ell^+ y_\ell$. Let, say, $\bar{y} \prec \bar{x}$. If $\bar{x} \notin \{u^+\} \times U_1 \times U_2$ then $\bar{y} \notin \{u^+\} \times U_1 \times U_2$ as well, thus $h^+(\bar{x}) R_\ell^{\mathscr{F}} h^+(\bar{y})$ holds by $h^+|_{U_0\times U_1\times U_2} = h$. If $\bar{x} \in \{u^+\} \times U_1 \times U_2$ then $h^+(\bar{x}) R_\ell^{\mathscr{F}} h^+(\bar{y})$ holds by (I.H.$(\bar{x})$). The case of $\bar{x} \prec \bar{y}$ is similar. $\dashv$

Claim 5.7 proves that we succeeded to define a response for $\exists$, in case $\forall$ picks the index $\ell = 0$. The cases when he picks 1 or 2 are similar: one has to use the first-order sentences

$$\psi_2^{01}, \psi_3^{01}, \psi_1^{12}, \psi_3^{12}, \chi_2, \chi_3, \chi_6, \chi_7, \text{ and } \psi_2^{02}, \psi_3^{02}, \psi_2^{12}, \psi_3^{12}, \chi_4, \chi_5, \chi_6, \chi_7,$$

respectively, in order to define a proper response for $\exists$. $\dashv$

Proof of Theorem 1.2. For any $0 < n \in \omega$, we are going to define a sentence Φ_n of the n-frame language. First, for any natural number $2 \leq k \leq n$, for any strictly increasing sequence $\bar{f} : k \to n$, and for any natural number $1 \leq \ell < 2^k$, we define a sentence $\Phi_\ell^{\bar f}$ as follows. Let $\bar\ell$ denote the first k digits of the number ℓ in binary in the reverse order, i.e., let $\bar\ell : k \to \{0,1\}$ be such that $\ell = \sum_{i=0}^{k-1}(2^{\ell_i} - 1)$. Let $\bar u^{(0)}, \bar u^{(1)}, \dots, \bar u^{(k-1)}$ be the k 'neighbours' of $\bar\ell$, i.e., the 0-1 sequences with $\bar u^{(i)} \equiv_i \bar\ell$, $\bar u^{(i)} \neq \bar\ell$ $(i < k)$. It is easy to see that for any $i < j < k$ there is some unique $\bar v^{(ij)} : k \to \{0,1\}$ such that $\bar v^{(ij)} \equiv_j \bar u^{(i)}$, $\bar v^{(ij)} \neq \bar u^{(i)}$, and $\bar v^{(ij)} \equiv_i \bar u^{(j)}$, $\bar v^{(ij)} \neq \bar u^{(j)}$. Now for each of $\bar u^{(i)}$ $(i<k)$, $\bar v^{(ij)}$ $(i<j<k)$, and $\bar\ell$ take some variable $u^{(i)}$, $v^{(ij)}$, ℓ, respectively, and let $\Phi_\ell^{\bar f}$ be

$$\forall u^{(0)} \dots u^{(k-1)} v^{(01)} \dots v^{(k-2,k-1)} \Big(\bigwedge_{i<j<k} v^{(ij)} R^{\pm}_{f_i} u^{(i)} \wedge v^{(ij)} R^{\pm}_{f_j} u^{(j)} \to \exists \ell \big(\bigwedge_{i<k} \ell R^{\pm}_{f_i} u^{(i)} \big) \Big),$$

where $x R^{\pm}_{f_i} u^{(i)}$ $(x \in \{v^{(01)}, \dots, v^{(k-2,k-1)}, \ell\},\ i<k)$ denotes $x R_{f_i} u^{(i)}$ if $x_i = 0$ and $u^{(i)}_i = 1$; and it denotes $u^{(i)} R_{f_i} x$ if $x_i = 1$ and $u^{(i)}_i = 0$.

Now, for $2 \le n \in \omega$, let Φ_n be the conjuction of all $\Phi_\ell^{\bar f}$'s, for any $k \in \omega$, $2 \le k \le n$, for any strictly increasing sequence $\bar f : k \to n$, and for any $\ell \in \omega$, $1 \le \ell < 2^k$. Note that, with this notation, ψ_ℓ^{ij} of the previous proof is just Φ_ℓ^{ij}, and χ_ℓ is Φ_ℓ^{012}, thus we obtain the same Φ_3. Also, Φ_2 is just the conjuction of commutativity and Church–Rosser properties, for R_0 and R_1. Φ_1 can be, say, $\forall x\ (x = x)$. A proof similar to the one of Theorem 5.3 shows that, for every $0 < n \in \omega$, $\boldsymbol{K}^n$ is determined by the class of n-frames satisfying Φ_n. Note that this way we obtain a new, step-by-step proof of the theorem in [7] stating that $\boldsymbol{K}^2$ is axiomatised by commutativity and Church–Rosser properties. ⊣

For any n-modal logic $\boldsymbol{L}$, a set Σ of n-formulas is said to be ***L**-consistent* if no negation of some finite conjuction of elements of Σ belongs to $\boldsymbol{L}$. The *canonical frame* $\mathscr{F}^{\mathbf{L}_P} = (F^{\mathbf{L}_P}, R_\ell^{\mathbf{L}_P})_{\ell<n}$ for an n-modal logic $\boldsymbol{L}$, *corresponding to some set P of propositional variables*, is the n-frame where $F^{\mathbf{L}_P}$ is the set of all maximal $\boldsymbol{L}$-consistent sets of n-formulas, using propositional variables from P; and for all $\ell < n$, Σ, Δ, $\Sigma R_\ell^{\mathbf{L}_P} \Delta$ iff for any n-formula A, $\Box_\ell A \in \Sigma$ implies $A \in \Delta$. An n-modal logic $\boldsymbol{L}$ is called *canonical* if its canonical frames $\mathscr{F}^{\mathbf{L}_P}$ validate $\boldsymbol{L}$, for all possible sets P. The well-known Fine–van Benthem theorem (cf. [5] and [16] for the mono-modal case) says that if an n-modal logic is determined by a first-order definable class of n-frames then it is canonical. Thus Theorem 1.2 yields the following corollary.

Corollary 5.8. *For any $0 < n \in \omega$, $\boldsymbol{K}^n$ is canonical.*

The canonicity of $\boldsymbol{K}^n$ can be proved in a simpler way as well, as it was pointed out by Y. Venema. Namely, it is straightforward to show that the class of isomorphic copies of n-cubes is closed under taking ultraproducts. Then one can use Thm.3.6.7 of Goldblatt [8], saying (in an algebraic setting) that if an n-modal logic is determined by a class of n-frames which is closed under ultraproducts then it is canonical.

§**6. Outlook.** Little modifications of the proof of Theorem 1.1 yield further non-finite axiomatisability results concerning products. In general, for any $0 < n \in \omega$, and Kripke complete mono-modal logics $\boldsymbol{L}_\ell$ $(\ell < n)$, define the *product logic*

$\boldsymbol{L}_0 \times \cdots \times \boldsymbol{L}_{n-1}$ as the set of all n-formulas which are valid in those n-cubes $(U_0, R_0) \times \cdots \times (U_{n-1}, R_{n-1})$ where, for each $\ell < n$, frame (U_ℓ, R_ℓ) validates $\boldsymbol{L}_\ell$.

The following mono-modal logics are considered here as components. $\boldsymbol{K}$ = all frames; $\boldsymbol{K4}$ = all transitive frames; $\boldsymbol{T}$ = all reflexive frames; $\boldsymbol{S4} = \boldsymbol{K4} + \boldsymbol{T}$; $\boldsymbol{KB}$ = all symmetric frames; $\boldsymbol{B} = \boldsymbol{KB} + \boldsymbol{T}$; $\boldsymbol{S5} = \boldsymbol{S4} + \boldsymbol{KB}$.

THEOREM 6.1. *For any* $3 \leq n \in \omega$, *if* $\boldsymbol{L}_0 \in \{\boldsymbol{K}, \boldsymbol{T}\}$, $\boldsymbol{L}_1, \boldsymbol{L}_2 \in \{\boldsymbol{K}, \boldsymbol{K4}, \boldsymbol{T}, \boldsymbol{S4}\}$, $\boldsymbol{L}_\ell \in \{\boldsymbol{K}, \boldsymbol{K4}, \boldsymbol{T}, \boldsymbol{S4}, \boldsymbol{B}, \boldsymbol{KB}, \boldsymbol{S5}\}$ $(3 \leq \ell < n)$ *then the product logic* $\boldsymbol{L}_0 \times \cdots \times \boldsymbol{L}_{n-1}$ *is not finitely axiomatisable in the n-modal language.*

PROOF. One has to modify the definition of a network and some rules of the game played over n-frames in order to build p-morphisms which come from appropriate n-cubes. In case some of the component logics are reflexive, one also have to modify the definition of the n-frames $\mathscr{F}_k$ $(k \in \omega)$ of §2 by postulating every node to be reflexive in the required coordinates. ⊣

There are many products of standard mono-modal logics which are out of the scope of Theorem 6.1 above. E.g., is the logic $\boldsymbol{KB}^3$ finitely axiomatisable?

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DEPARTMENT OF COMPUTING
IMPERIAL COLLEGE
LONDON, ENGLAND
E-mail: kuag@doc.ic.ac.uk