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Decidable and Undecidable Logics with a Binary Modality *

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Abstract. We give an overview of decidability results for modal logics having a binary modality. We put an emphasis on the demonstration of proof-techniques, and hope that this will also help in finding the borderlines between decidable and undecidable fragments of usual first-order logic.

Key words: poly-modal and multi-modal logics, decision problems, Arrow logics, algebraic logic, relation algebra, associativity, dynamic logics, action algebras, Boolean algebras with operators

We investigate here decidability problems concerning logics having an extra binary connective "$\circ$" beside the Boolean ones. These logics are strongly related to ordinary first-order logic, see Henkin *et al.* (1985, ch. 5.3) on this connection in an algebraic setting. Our most important aims are to give a transparent overview of the results and to stress the crucial points and ideas of the proofs, especially when the extra binary connective is associative. The emphasis is on those parts of the proof methods which have been well known for the specialists. (The reason for this is the didactic character of the present paper.) For the details of those ideas which are our own contributions we give references to more technical papers.

Since associativity of "$\circ$" corresponds to commutativity of first-order existential quantifiers in some sense (cf. op cit), our results and techniques can help to find decidable fragments of first-order logic as well (for such results see Németi (1985, 1987, 1992)).

1. Preliminaries

We say that $\mathcal{L}$ is a *logic with a binary modality* iff (1–3) below hold.

1. The language of $\mathcal{L}$ includes the usual Boolean connectives and a binary connective "$\circ$". "$\bullet$" denotes the dual of "$\circ$", i.e. $(\varphi \bullet \psi) \stackrel{\text{def}}{=} \neg(\neg\varphi \circ \neg\psi)$. We stress

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at this point that there may be further connectives in the language. Formulas of $\mathcal{L}$ ($F_{\mathcal{L}}$) are built up from an infinite set $\{p_0, p_1, \ldots\}$ of propositional variables with the help of the connectives in the usual way.

2. $\mathcal{L}$ has (pt) and (K) below among its axioms.
   (pt) all propositional tautologies;
   (K) $((p_0 \to p_1) \bullet p_2) \to ((p_0 \bullet p_2) \to (p_1 \bullet p_2))$
   $(p_0 \bullet (p_1 \to p_2)) \to ((p_0 \bullet p_1) \to (p_0 \bullet p_2))$.
3. The set of validities (theorems) of $\mathcal{L}$ is closed under substitution, modus ponens and the following "replacement of equivalents":

$$\frac{\varphi \leftrightarrow \psi}{(\varphi \bullet \chi) \leftrightarrow (\psi \bullet \chi)} \qquad \frac{\varphi \leftrightarrow \psi}{(\chi \bullet \varphi) \leftrightarrow (\chi \bullet \psi)}.$$

A logic with a binary modality is called *normal* iff the set of validities of $\mathcal{L}$ is also closed under the necessitation rules:

$$\frac{\varphi}{\varphi \bullet \psi} \qquad \frac{\psi}{\varphi \bullet \psi}.$$

We note that, according to the definition above, "$\circ$" is a "$\Diamond$-type" and "$\bullet$" is a "$\Box$-type" connective. We also note that, by a classical theorem of Jónsson and Tarski (1948), the minimal* normal logic with a binary modality is strongly sound and complete w.r.t. the following class $\mathcal{F}$ of Kripke-frames:

$$\mathcal{F} \overset{\text{def}}{=} \{\langle \mathcal{W}, \mathbf{C}\rangle : \mathcal{W} \text{ is a set, } \mathbf{C} \subseteq \mathcal{W} \times \mathcal{W} \times \mathcal{W}\}$$

(with the usual truth definition for "$\circ$":

$$w \Vdash \varphi \circ \psi \quad \text{iff} \quad (\exists u, v \in W)(\mathbf{C}(w, u, v) \text{ and } u \Vdash \varphi \text{ and } v \Vdash \psi).$$

There are special frames whose worlds are pairs and the accessibility relation $\mathbf{C}$ acts as composition between them. A *square frame* is a frame $\langle W, \mathbf{C}\rangle$, where
$W = U \times U$ for some set U and
$\mathbf{C} = \{\langle\langle a, c\rangle, \langle a, b\rangle, \langle b, c\rangle\rangle : a, b, c \in U\}$.
The set U is called the *base set* of the square frame. Similar frames are discussed in connection with Arrow Logics (which *are* logics with a binary modality), cf. e.g. van Benthem (1994), Venema (1992).

The $\circ$-*fragment* of a logic $\mathcal{L}$ with a binary modality is the set of those validities of $\mathcal{L}$ which include only "$\circ$" and Boolean connectives.

A logic $\mathcal{L}$ (perhaps in a language with more connectives) is called a *square extension* of the minimal logic with a binary modality iff the $\circ$-fragment of $\mathcal{L}$ is valid in all square frames. An example for a square extension is the logic with a

* By $\mathcal{L}$ being *minimal* among logics having some properties we always understand the following: (i) the language of $\mathcal{L}$ is the 'smallest possible' (e.g. in case of logics with a binary modality, the language consists of the Booleans and "$\circ$" only); (ii) for any logic $\mathcal{L}'$ having the particular properties, and for any $T \cup \{\varphi\} \subseteq F_{\mathcal{L}}$, $T \vdash_{\mathcal{L}} \varphi$ implies $T \vdash_{\mathcal{L}'} \varphi$.

binary modality (having only the Booleans and "$\circ$" as connectives) characterized by the class of all square frames.

A logic $\mathcal{L}$ is *decidable* iff there is an algorithm deciding whether a formula is valid in $\mathcal{L}$ or not.

* * *

In this paper all the logical results are proved in an algebraic "disguise", with the help of the correspondence between logics and their algebraic counterparts, see e.g. Andréka *et al.* (1994d), Blok and Pigozzi (1989), chapters 3.4, 5.6 of Henkin *et al.* (1985) about this connection in general. In case of logics with a binary modality this correspondence can be described as follows. To any such logic $\mathcal{L}$, one can associate an algebraic similarity type* $t_\mathcal{L}$ by considering any n-ary connective of $\mathcal{L}$ as an n-ary function symbol of $t_\mathcal{L}$. In this way a formula of $\mathcal{L}$ containing propositional variables, say, p_3, p_{25} can be considered as a term of type $t_\mathcal{L}$ having (algebraic) variables p_3, p_{25}. For any set T of formulas of $\mathcal{L}$, let $\mathbf{A}_\mathcal{L}^T \stackrel{\text{def}}{=} \langle A_\mathcal{L}^T, f\rangle_{f\in t_\mathcal{L}}$ be the *formula-algebra*** of T in $\mathcal{L}$, that is, $A_\mathcal{L}^T \stackrel{\text{def}}{=} \{(\varphi)_T : \varphi \in F_\mathcal{L}\}$, where $(\varphi)_T$ is the congruence class $\{\psi \in F_\mathcal{L} : T \vdash_\mathcal{L} (\varphi \leftrightarrow \psi)\}$. Then the algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of $\mathcal{L}$ is the following class:

$$\mathsf{Alg}(\mathcal{L}) \stackrel{\text{def}}{=} \{\mathbf{A}_\mathcal{L}^T : T \subseteq F_\mathcal{L}\}.$$

Now for each formula φ of $\mathcal{L}$,

$$\vdash_\mathcal{L} \varphi \quad\iff\quad \mathsf{Alg}(\mathcal{L}) \models (\varphi = 1)$$

holds, that is, φ is a validity (theorem) of $\mathcal{L}$ iff φ (as a term) is identically 1 in each algebra of $\mathsf{Alg}(\mathcal{L})$. As a consequence we obtain the following fact:

Fact 1. A logic $\mathcal{L}$ is decidable iff its algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ has a decidable equational theory, that is, the set

$$\{\langle\tau,\sigma\rangle : \tau, \sigma \text{ are terms and } \mathsf{Alg}(\mathcal{L}) \models (\tau = \sigma)\}$$

is decidable.

2. The weakest undecidability theorem

In this section we state and prove our results only in their weakest forms. The reason for this is that we want to stress the crucial points and ideas of the proofs. The statements of Section 3 are slightly stronger, it is also to demonstrate the necessary modifications in the proofs. The results in their full strength are stated

* i.e. a similarity type of usual first-order logic, having only function symbols

** also called Lindenbaum–Tarski algebra in Section 5. Most of the techniques on which the proof of Theorem 1 is based, are well known for the specialists. Our main contribution is a translation method for turning quasiequations* to equations, which works not only in discriminator varieties.

A logic $\mathcal{L}$ with a binary modality is called *associative* iff axiom (A) below is among its axioms.

(A) $((p_0 \circ p_1) \circ p_2) \leftrightarrow (p_0 \circ (p_1 \circ p_2))$.

Obviously, axiom (A) holds in every square frame.

THEOREM 1.

1. *The minimal associative logic with a binary modality is undecidable.*
2. *The minimal normal associative logic with a binary modality is undecidable.*
3. *Any square extension of the minimal associative logic with a binary modality is undecidable.*

First, we give the algebraic counterparts of the logics in the theorem and then prove that these classes all have undecidable equational theories.

The algebraic counterpart of the minimal associative logic with a binary modality is the class BSG of all Boolean-ordered semigroups, defined as follows. Let t_{BSG} be the first-order similarity type (i.e. language) consisting of the Boolean operation symbols and a binary operation symbol "$\circ$". An algebra $\mathbf{A} = \langle A, \wedge, \vee, -, 0, 1, \circ \rangle$ of type t_{BSG} is called a *Boolean-ordered semigroup* iff (1)–(3) below hold for $\mathbf{A}$:

(1) $\langle A, \wedge, \vee, -, 0, 1 \rangle$ is a Boolean algebra;
(2) $\langle A, \circ \rangle$ is a semigroup (i.e., "$\circ$" is associative);
(3) "$\circ$" distributes over "$\vee$", that is,

$\mathbf{A} \models (x \vee y) \circ z = (x \circ z) \vee (y \circ z)$ and
$\mathbf{A} \models z \circ (x \vee y) = (z \circ x) \vee (z \circ y)$.

Similarly, the algebraic counterpart of the minimal normal associative logic with a binary modality is the class of all *normal* members of BSG, i.e., those Boolean-ordered semigroups which satisfy the equations $0 \circ x = x \circ 0 = 0$.

A set-BSG is defined to be a Boolean-ordered semigroup such that it is a Boolean algebra consisting of some subsets of $U \times U$ for some set U, together with "$\circ$" as the usual composition of binary relations. Observe that, since 0 denotes the empty set in set-BSG's, any set-BSG is normal. Now let $\mathcal{L}$ be a square extension of the minimal associative logic with a binary modality. Then the algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of $\mathcal{L}$ is such that

(a) the similarity type of $\mathsf{Alg}(\mathcal{L})$ includes t_{BSG};

* A *quasiequation* is a formula of form $(e_1 \& \cdots \& e_n) \Rightarrow e_0$, where $e_0, e_1, \ldots, e_n$ are equations.

(b) the t_{BSG}-reduct of each algebra in $\mathsf{Alg}(\mathcal{L})$ is a Boolean-ordered semigroup;
(c) any set-BSG is embeddable into the t_{BSG}-reduct of some algebra in $\mathsf{Alg}(\mathcal{L})$.

We note that BSG and the class of normal BSG's are classes satisfying conditions (a)–(c) above. Thus it is enough to prove item 3 of Theorem 1.

Proof of Theorem 1, item 3. Let $\mathcal{L}$ be an arbitrary square extension of the minimal associative logic with a binary modality. We will prove that the equational theory of $\mathsf{Alg}(\mathcal{L})$ is undecidable. The proof is based on the fact that the *halting problem of Turing machines* can be "coded by the behaviour of equations" in $\mathsf{Alg}(\mathcal{L})$. There are three basic "building blocks" of this "coding". In this first proof some of the blocks contain simple or well-known statements. We include them here in order to be able to illustrate later how the more complicated proofs are built of these kinds of blocks.

Let $Queq(\circ)$ denote the set of all quasiequations of the language of semigroups (using $\circ$ for the semigroup operation) and let SG denote the class of all semigroups.

First block: By a well-known result of Post (cf. e.g. Davis (1977)), the halting problem of Turing machines is equivalent to the word problem of semigroups. That is, for every Turing machine T and possible input i there is some quasiequation $q_{T,i} \in Queq(\circ)$ such that

$$T \text{ halts with input } i \quad \Longleftrightarrow \quad \mathsf{SG} \models q_{T,i}.$$

This proves the following statement.

CLAIM 1.1 (Post's theorem) The quasiequational theory of semigroups (i.e. the set $\{q \in Queq(\circ) : \mathsf{SG} \models q\}$) is undecidable.

Second block: By definition, the $\circ$-reduct of any Boolean-ordered semigroup is a semigroup. Thus, by conditions (a), (b) on $\mathsf{Alg}(\mathcal{L})$, for any $q \in Queq(\circ)$, if $\mathsf{SG} \models q$ then $\mathsf{Alg}(\mathcal{L}) \models q$ also holds.* Since we are interested in deciding equations, we have to "translate" quasiequations to equations in a way which preserves "validity in $\mathsf{Alg}(\mathcal{L})$". Here we note that for certain "nice" classes of algebras (for *discriminator varieties*, cf. e.g. Burris and Sankappanavar (1981)) one can always translate any quasiequation q to an equation $e(q)$ in a "validity preserving" way.** Though $\mathsf{Alg}(\mathcal{L})$ is *not* necessarily a discriminator variety in this case, there is some recursive translation, which works at least for quasiequations of $Queq(\circ)$, that is, $\mathsf{SG} \models q \Leftrightarrow \mathsf{Alg}(\mathcal{L}) \models e(q)$ holds.

Namely, let the unary term $c(x)$ of type t_{BSG} be defined by

$$c(x) \stackrel{\text{def}}{=} x \vee (1 \circ x) \vee (x \circ 1) \vee (1 \circ x \circ 1).$$

* This idea was known to Tarski already around 1950.

** This method for proving undecidability of discriminator varieties was also well known a long time ago, cf. e.g. Pixley (1971).

For any quasiequation $q \in Queq(\circ)$ of form

$$[(\tau_1 = \sigma_1)\&\ldots\&(\tau_m = \sigma_m)] \Rightarrow (\tau_0 = \sigma_0)$$

an equation $e(q)$ of type t_{BSG} is defined as follows.

$$e(q) \stackrel{\mathrm{def}}{=} \quad \tau_0 \oplus \sigma_0 \leq c((\tau_1 \oplus \sigma_1) \vee \cdots \vee (\tau_m \oplus \sigma_m)),$$

where $\oplus$ denotes symmetric difference.

First, assume $\mathsf{Alg}(\mathcal{L}) \not\models e(q)$ for some $q \in Queq(\circ)$, that is, there is some $\mathbf{A} \in \mathsf{Alg}(\mathcal{L})$ and $\mathbf{a} \in {}^{\omega}A$ such that $\mathbf{A} \not\models e(q)[\mathbf{a}]$. Using the $\circ$-reduct of $\mathbf{A}$, we construct a semigroup $\mathbf{B}$ and an evaluation $\mathbf{a}' \in {}^{\omega}B$ such that $\mathbf{B} \not\models q[\mathbf{a}']$. Let $\tau \stackrel{\mathrm{def}}{=} (\tau_1 \oplus \sigma_1) \vee \cdots \vee (\tau_m \oplus \sigma_m)$, and let $\mathbf{B} \stackrel{\mathrm{def}}{=} \langle B, \circ^{\mathbf{B}} \rangle$ be the following algebra:

$$B \stackrel{\mathrm{def}}{=} \{x \vee c(\tau)^{\mathbf{A}}[\mathbf{a}] \ : \ x \in A\}$$

$$x \circ^{\mathbf{B}} y \stackrel{\mathrm{def}}{=} (x \circ^{\mathbf{A}} y) \vee c(\tau)^{\mathbf{A}}[\mathbf{a}],$$

for any $x, y \in B$.

Then it is easy to check that $x \mapsto x \vee c(\tau)^{\mathbf{A}}[a]$ is a $\circ$-homomorphism from the $\circ$-reduct of $\mathbf{A}$ onto $\mathbf{B}$. Thus $\mathbf{B}$ is a semigroup. Also, it can be shown that

$$\mathbf{B} \models (\tau_0 \neq \sigma_0)[\mathbf{a}'] \quad \text{and} \quad \mathbf{B} \models (\tau_k = \sigma_k)[\mathbf{a}'] \qquad (k = 1, \ldots, m),$$

where $\mathbf{a}' \stackrel{\mathrm{def}}{=} \langle \ldots, a_j \vee c(\tau)^{\mathbf{A}}[\mathbf{a}], \ldots \rangle_{j \in \omega}$. Thus $\mathbf{B} \not\models q[\mathbf{a}']$, proving the following claim.

CLAIM 1.2. For any $q \in Queq(\circ)$,

$$\mathsf{SG} \models q \quad \Longrightarrow \quad \mathsf{Alg}(\mathcal{L}) \models e(q).$$

Third block: It is left to prove that "$\mathsf{Alg}(\mathcal{L}) \models e(q) \Rightarrow \mathsf{SG} \models q$". Assume that $\mathbf{S} \not\models q$ for some semigroup $\mathbf{S}$. However, this semigroup $\mathbf{S} = \langle S, \cdot \rangle$ can be embedded into the $\circ$-reduct of some Boolean-ordered semigroup in the following way.

Let $e \notin S$ be a new element and let us define a new semigroup $\mathbf{S}^+ = \langle S^+, \cdot \rangle$ by $S^+ \stackrel{\mathrm{def}}{=} S \cup \{e\}$ and by postulating $a \cdot e = e \cdot a = a$ for all $a \in S^+$. Let $C_{\mathrm{S}} \stackrel{\mathrm{def}}{=} \{\{\langle b, b \cdot a \rangle : b \in S^+\} : a \in S^+\}$, and let $\mathbf{C}_{\mathrm{S}} \stackrel{\mathrm{def}}{=} \langle C_{\mathrm{S}}, \circ \rangle$, where "$\circ$" is the usual composition of relations. Then the following two statements are easy to prove.

- $\mathbf{C}_{\mathrm{S}}$ is a semigroup which embeds $\mathbf{S}$ (this is the well-known *Cayley representation* of $\mathbf{S}$).
- Let $\mathbf{B}$ denote the Boolean algebra of all subsets of $S^+ \times S^+$ expanded with relation composition as "$\circ$". Then $\mathbf{B} \in \mathsf{set\text{-}BSG}$ and $\mathbf{C}_{\mathrm{S}}$ can be embedded into the $\circ$-reduct of $\mathbf{B}$.

Thus we have the embeddings $\mathbf{S} \hookrightarrow \mathbf{C}_S \hookrightarrow \mathbf{B} \in \mathsf{set\text{-}BSG}$. Therefore $\mathbf{S} \not\models q$ implies $\mathbf{B} \not\models q$ and, since any $\mathsf{set\text{-}BSG}$ is normal, $\mathbf{B} \models c(0) = 0$. Then an easy computation shows that $\mathbf{B} \not\models e(q)$. Now, by condition (c) on $\mathsf{Alg}(\mathcal{L})$, $\mathbf{B}$ can be embedded into the t_{BSG}-reduct of some algebra $\mathbf{A}$ in $\mathsf{Alg}(\mathcal{L})$. Therefore $\mathbf{A} \not\models e(q)$ also holds, proving the following claim.

CLAIM 1.3. For any $q \in Queq(\circ)$,

$$\mathsf{Alg}(\mathcal{L}) \models e(q) \quad \Longrightarrow \quad \mathsf{SG} \models q.$$

Now, the undecidability proof can be built of the three blocks, that is, of Claims 1.1, 1.2 and 1.3. If one could decide the equational theory of $\mathsf{Alg}(\mathcal{L})$ then that decision procedure would also solve the halting problem of Turing machines. But the latter is well known to be unsolvable, completing the proof. ∎

3. More undecidable logics

In the previous section we showed that the minimal associative logic with a binary modality is "hereditarily" undecidable in some sense. The question naturally arises whether there are any more undecidable extensions? Certainly, there are some extensions which are decidable, any inconsistent logic or logics including the axiom $(p_0 \circ p_1) \leftrightarrow (p_0 \wedge p_1)$ are trivial examples.

Theorem 2 below is a strengthening of Theorem 1. It states the undecidability of more logics. We present it as a separate theorem because we want to make clear, which part of the above proof must be modified. Theorem 3 treats a parallel line, namely logics with an associative *and* commutative "$\circ$". For stronger results in both directions see Section 5 (see also Andréka *et al.* (1994c)). Finally, Theorem 4 states the undecidability of some non-associative logics, the proofs and stronger results can be found in Németi *et al.* (1995).

A logic $\mathcal{L}$ is called an *unbounded finite-square extension* of the minimal logic with a binary modality iff the $\circ$-fragment of $\mathcal{L}$ is valid in an arbitrarily large finite square frame (i.e., for every $n \in \omega$ there is some set U with $|U| \geq n$ such that the $\circ$-fragment of $\mathcal{L}$ is valid in a square frame with base set U). We note that the class of unbounded finite-square extensions is strictly bigger than that of square extensions, see the Remark after the proof of Theorem 2 below.

THEOREM 2. *Any unbounded finite-square extension of the minimal associative logic with a binary modality is undecidable.*

Proof of Theorem 2. let $\mathcal{L}$ be an arbitrary unbounded finite-square extension of the minimal associative logic with a binary modality. Then the algebraic counterpart $\mathsf{Alg}(\mathcal{L})$ of $\mathcal{L}$ is such that

(a) the similarity type of $\mathsf{Alg}(\mathcal{L})$ includes t_{BSG};

(b) the t_{BSG}-reduct of each algebra in $\mathsf{Alg}(\mathcal{L})$ is a Boolean-ordered semigroup;
(c) any finite (!) set-BSG is embeddable into the t_{BSG}-reduct of some algebra in $\mathsf{Alg}(\mathcal{L})$.

First block: Before formalizing the particular refinement of the halting problem we use here, let us fix some terminology. By a *Turing machine* (Tm for short) we understand a deterministic Turing machine, taking natural numbers as inputs. A *configuration* of T is a triple describing the tape-contents, finite automaton-state and head position of T at a certain time instance. A *computation* of T is a sequence of "subsequent" configurations, in the usual sense. By T being *deterministic* we mean that for any input x there is a unique computation of T. We will say that Turing machine T *terminates*, *diverges*, *loops*, etc. with input x if it does so in the usual sense. We say that the computation of T with input x is *cyclic* iff there is a configuration (of T) which recurs throughout the computation of T.

Note that if T terminates then we consider this terminating computation as cyclic, namely the halting configuration is the recurrent configuration. If T diverges then the diverging computation can be either cyclic or noncyclic. For more detail about Tm's see e.g. Davis (1977), Rogers (1967).

The following lemma differs from saying that the *halting problem of Turing machines* is undecidable, it says that the terminating and nonterminating cyclic Tm's are recursively inseparable.

LEMMA 1. *Let R be a set of pairs $\langle T, x\rangle$, where T is a Tm and $x \in \omega$ (i.e. x is a possible input for T). Assume that for any Tm T and $x \in \omega$, conditions (i–ii) below hold.*
(i) If T terminates for x then $\langle T, x\rangle \in R$.
(ii) If the computation of T with input x is cyclic and diverges then $\langle T, x\rangle \notin R$.
Then R is undecidable.

Proof of Lemma 1 The proof goes by a standard diagonalization argument, see e.g. the proof of Theorem XII(c) on p. 94 of Rogers (1967). ∎

Let FSG denote the class of all finite semigroups. As a special case of the results in Gurevich and Lewis (1984), Lemma 1 above yields the following statement concerning semigroups.

CLAIM 2.1. Let Q be a set such that

$$\{q \in Queq(\circ) \ : \ \mathsf{SG} \models q\} \subseteq Q \subseteq \{q \in Queq(\circ) \ : \ \mathsf{FSG} \models q\}.$$

Then Q is undecidable.

Second block: This block of the proof of Theorem 2 is the same as in case of square extensions (cf. Theorem 1).

Third block: Now, if we assume $\mathbf{S} \not\models q$ for some finite semigroup $\mathbf{S}$, then, by having the embeddings $\mathbf{S} \hookrightarrow \mathbf{C_S} \hookrightarrow \mathbf{B} \in$ "finite set-BSG", condition (c) on $\mathsf{Alg}(\mathcal{L})$ ensures that "$\mathsf{Alg}(\mathcal{L}) \models e(q) \;\Rightarrow\; \mathsf{FSG} \models q$".

Now we can put together the three blocks as follows. Let $Q \stackrel{\text{def}}{=} \{q \in Queq(\circ) : \mathsf{Alg}(\mathcal{L}) \models e(q)\}$. By the second and the third blocks,

$$\{q \in Queq(\circ) \;:\; \mathsf{SG} \models q\} \;\subseteq\; Q \;\subseteq\; \{q \in Queq(\circ) \;:\; \mathsf{FSG} \models q\}.$$

Thus, by the first block (Claim 2.1), Q cannot be decidable. Thus the equational theory of $\mathsf{Alg}(\mathcal{L})$ is also undecidable, completing the proof of Theorem 2. ∎

Remark 2.1. We show that there are strictly more unbounded finite-square extensions than square extensions. We give an equation which holds in every finite BSG but fails in some infinite one.

Consider the following quasiequation $q \in Queq(\circ)$:

$$[(x = e \circ x = x \circ e) \;\&\; (y = e \circ y = y \circ e) \;\&\; (z = e \circ z = z \circ e) \;\&$$
$$\& \; (v = e \circ v = v \circ e) \;\&\; (e = e \circ e) \;\&\; (x \circ y = e) \;\&$$
$$\& \; (x \circ z = x \circ v)] \;\Rightarrow\; (z = v)$$

It is easy to construct an infinite semigroup in which q fails. We claim that q holds in every finite semigroup. Indeed, assume that the first five equations of the premiss of q hold in some finite semigroup $\mathbf{S}$. Since $\mathbf{S}$ is finite, the subsemigroup of $\mathbf{S}$ generated by x, y, z, v and e is finite too, thus it is a (finite) monoid with unit e. But in every finite monoid, if an element x has a right-inverse then $(x \cdot z = x \cdot v \Rightarrow z = v)$ holds. Indeed, if x has a right-inverse then there is some $n \in \omega$ with $x^n = 1$, thus $z = 1 \cdot z = x^n \cdot z = x^n \cdot v = v$.

Now we claim that the "equational translation" $e(q)$ of q (see the proof of Theorem 1) gives the desired equation, that is, an equation separating "finite BSG" from BSG. Indeed, by the proof of Theorem 2 above, Claims 1.2 and 1.3 above also hold in the following forms:

1. $\mathsf{FSG} \models q \implies$ "finite BSG" $\models e(q)$.
2. $\mathsf{BSG} \models e(q) \implies \mathsf{SG} \models q$.

Moreover, the above example q separating FSG from SG (and its translation $e(q)$ separating "finite BSG" from BSG) can be improved the following way. There are infinitely many quasivarieties between FSG and SG. Similarly, there are infinitely many varieties between "finite BSG" and BSG. Thus, there are infinitely many (in fact, continuum many) logics to which Theorem 2 applies but Theorem 1 does not. ∎

There are lots of logics which are certainly missing from the class of undecidable logics discussed so far, namely those logics which take that kind of restrictions on

"$\circ$" which are not valid in arbitrarily large square frames. E.g., such logics are the *commutative* ones, that is, which contain the axiom

$$(p_0 \circ p_1) \leftrightarrow (p_1 \circ p_0).$$

THEOREM 3. *The minimal associative and commutative logic with a binary modality is undecidable.*

Proof of Theorem 3. The algebraic counterpart of this logic is the class of *commutative Boolean-ordered semigroups*, i.e., those members of BSG where "$\circ$" is commutative. It is certainly not true that every semigroup, or even every finite semigroup is embeddable into the $\circ$-fragment of such an algebra. Thus we have to look for semigroups "elsewhere" in the algebras. As it is proved in Andréka *et al.* (1994c), one can "associate" a semigroup to each Boolean-ordered semigroup in such a way that

- the universe of the semigroup is equationally definable in BSG;
- the semigroup-operation is term definable in t_{BSG};
- there is a commutative Boolean-ordered semigroup such that every finite semigroup is embeddable to the semigroup "associated" to this particular algebra.

With the help of these "associated" semigroups it is proved in Andréka *et al.* (1994c) that for any $q \in Queq(\circ)$ there is some quasiequation $q^\bullet$ in the language including "$\wedge$", "$\vee$" and "$\circ$" such that

$$\mathsf{SG} \models q \quad\Longrightarrow\quad \text{commutative } \mathsf{BSG} \models e(q^\bullet) \quad\Longrightarrow\quad \mathsf{FSG} \models q,$$

proving, by the usual pattern, that the equational theory of commutative BSG's is undecidable. ∎

So far we have seen that associativity yields undecidability of many logics. Now we discuss an other class of undecidable logics, namely Euclidean logics (see the definition below). First we extend the language with one constant "*Id*" and two binary connectives "$\rhd$", "$\lhd$" which will be the *conjugates* of "$\circ$". That is, a frame for a *logic with a binary conjugated modality* is of form $\langle W, \mathbf{C}, \mathbf{E}\rangle$, where $\mathbf{C} \subseteq W \times W \times W$, $\mathbf{E} \subseteq W$, and the new clauses in the truth definition are

$w \Vdash (\varphi \rhd \psi)$ iff $(\exists u, v)\ [\mathbf{C}(v, u, w)$ and $u \Vdash \varphi$ and $v \Vdash \psi]$
$w \Vdash (\varphi \lhd \psi)$ iff $(\exists u, v)\ [\mathbf{C}(u, w, v)$ and $u \Vdash \varphi$ and $v \Vdash \psi]$
$w \Vdash Id$ iff $\mathbf{E}(w)$.

Such a frame is called *totally symmetric* iff the following formulas hold in it:

$(p_0 \rhd Id) \leftrightarrow p_0$ and
$p_0 \rightarrow (p_0 \circ p_0)$.

Of course, the former theorems extend for logics with a binary conjugated modality, since already the $\circ$-fragments of the logics in question are undecidable.

A logic $\mathcal{L}$ with a binary conjugated modality is called *Euclidean* iff $\mathcal{L}$ contains the following axioms:

(Eucl) $((p_0 \triangleright p_1) \circ p_2) \to (p_0 \triangleright (p_1 \circ p_2))$
(Unit) $(p_0 \circ Id) \leftrightarrow p_0$ and $(Id \circ p_0) \leftrightarrow p_0$.

THEOREM 4.
1. *The minimal Euclidean logic with a binary conjugated modality is undecidable.*
2. *Any extension of the minimal commutative Euclidean logic with a binary conjugated modality whose $\langle \circ, \triangleright, \triangleleft, Id \rangle$-fragment* is valid in all totally symmetric frames is undecidable.*

The statements are proved in Németi *et al.* (1995) (see also Simon and Kurucz (1993)) by a method extending the one we discussed so far. Among others, they use results from the papers Andréka *et al.* (1994a, 1994b).

4. Decidable logics

So far we have seen that many extensions of the minimal associative logic with a binary modality are undecidable. What can we say about the sublogics? In this section we discuss some possible directions in which one can find decidable sublogics.

First consider the language including the Booleans, "$\circ$", "*Id*" and a unary connective "$^\smile$". We call a logic $\mathcal{L}$ of this language an *Arrow Logic* iff $\mathcal{L}$ is a normal logic with binary modality "$\circ$" and (the dual of) "$^\smile$" also satisfies the corresponding (K) and (Nec) (cf. van Benthem (1994), Venema (1992)). Let $\mathcal{AL}_{\min}$ denote the minimal Arrow Logic. It is again a consequence of the results of Jónsson and Tarski (1948) that $\mathcal{AL}_{\min}$ is strongly sound and complete w.r.t. the class of Kripke-frames (called *arrow frames*) $\langle W, \mathbf{C}, \mathbf{R}, \mathbf{E} \rangle$, where $\mathbf{C} \subseteq W \times W \times W$ interprets "$\circ$", $\mathbf{R} \subseteq W \times W$ interprets "$^\smile$", and $\mathbf{E} \subseteq W$ interprets "*Id*". That is, all possible choices of a ternary $\mathbf{C}$, a binary $\mathbf{R}$ and a unary $\mathbf{E}$ are allowed as accessibility relations in arrow frames for $\mathcal{AL}_{\min}$.

Stronger Arrow Logics can be obtained by adding new axioms, i.e., by restricting the class of arrow frames. For example, consider the axiom

$$\neg(p_0{}^\smile) \leftrightarrow (\neg p_0)^\smile.$$

This axiom ensures that the accessibility relation $\mathbf{R}$ is actually a function $\mathbf{R} : W \to W$ in all frames. So if we add this axiom to $\mathcal{AL}_{\min}$ then we obtain a stronger Arrow Logic in which all arrow frames satisfy that $\mathbf{R} : W \to W$ is a function.

Below we list seven potential axioms (AL1)–(AL7) from the paper van Benthem (1994), together with the corresponding frame conditions.

(AL1) $\neg p_0{}^\smile \to (\neg p_0)^\smile$ iff $\forall x \exists y\, \mathbf{R}(x,y)$
(AL2) $(\neg p_0)^\smile \to \neg p_0{}^\smile$ iff $\forall x, y, z\, [\mathbf{R}(x,y) \text{ and } \mathbf{R}(x,z) \to y = z]$

* i.e. those theorems of $\mathcal{L}$ which include only "$\circ$", "$\triangleright$", "$\triangleleft$", "*Id*" and the Booleans.

(AL3) $p_0{}^{\smile\smile} \leftrightarrow p_0$ iff $\forall x \exists y [\mathbf{R}(x,y)$ and $\mathbf{R}(y,x)]$ and
and $\forall x,y,z\,[\mathbf{R}(x,y)$ and $\mathbf{R}(y,z) \rightarrow x = z]$

(AL4) $(p_0 \circ p_1)^{\smile} \rightarrow p_1{}^{\smile} \circ p_0{}^{\smile}$ iff $\forall x,y,z,x'\,[\mathbf{C}(x,y,z)$ and $\mathbf{R}(x,x') \rightarrow$
$\rightarrow \exists y',z'\,(\mathbf{R}(y,y')$ and $\mathbf{R}(z,z') \rightarrow \mathbf{C}(x',z',y'))]$

(AL5) $p_0 \circ \neg(p_0{}^{\smile} \circ p_1) \rightarrow \neg p_1$ iff $\forall x,y,z\,[\mathbf{C}(x,y,z) \rightarrow \exists y'\,(\mathbf{R}(y,y')$ and
and $\mathbf{C}(z,y',x))]$

(AL6) $Id \rightarrow Id^{\smile}$ iff $\forall x\,[\mathbf{E}(x) \rightarrow \exists y\,(\mathbf{R}(x,y)$ and $\mathbf{E}(y))]$

(AL7) $Id \circ p_0 \rightarrow p_0$ iff $\forall x,y,z\,[\mathbf{E}(y)$ and $\mathbf{C}(x,y,z) \rightarrow x = z]$.

We note that the minimal associative Arrow Logic $\mathcal{AL}_{\min} + (\mathrm{A})$ as well as any Arrow Logic which is obtained by adding some of the above axioms (AL1)–(AL7) to $\mathcal{AL}_{\min} + (\mathrm{A})$ is undecidable by the third statement of Theorem 1. However, we can weaken associativity in the following way:

(A$^-$) $((p_1 \wedge Id) \circ \top) \circ \top \leftrightarrow (p_1 \wedge Id) \circ (\top \circ \top)$,

where $\top$ abbreviates the formula "$p_0 \rightarrow p_0$". (This weakening of associativity was investigated first by Maddux (1978) in connection with Relation Algebras.)

An Arrow Logic $\mathcal{L}$ is called *weakly associative* iff (A$^-$) is an axiom of $\mathcal{L}$. Since (A$^-$) is a consequence of (A), associative Arrow Logics are also weakly associative.

Now let $\mathcal{AL}_{1\text{-}5} \stackrel{\text{def}}{=} \mathcal{AL}_{\min} + \{(\mathrm{AL1}),\ldots,(\mathrm{AL5})\}$ and let $\mathcal{AL}_{1\text{-}7} \stackrel{\text{def}}{=} \mathcal{AL}_{1\text{-}5} + \{(\mathrm{AL6}),(\mathrm{AL7})\}$.

THEOREM 5.

1. *$\mathcal{AL}_{1\text{-}5}$ is decidable.*
2. *$\mathcal{AL}_{1\text{-}5} + (A^-)$ is decidable.*
3. *$\mathcal{AL}_{1\text{-}7} + (A^-)$ is decidable.*

It is proved in Németi (1987) (cf. also Marx *et al.* (1995), Mikulás *et al.* (1995) that the equational theories of the algebraic counterparts of the logics in Theorem 5 are decidable by showing that these classes all have the *finite algebra property*. This property says that if an equation fails in an algebra of the class in question then it must already fail in a finite algebra.

Now let us try to increase the expressive power of Arrow Logics by adding new logical connectives, like the difference operator "D", universal modality "$\langle u\rangle$", the "stratified" or "graded" modalities $\langle n\text{-times}\rangle$ $(n \in \omega)$,* and the Kleene-star "$*$". The truth definitions of these new connectives in a frame $\langle W, \mathbf{C}, \ldots\rangle$ are as follows.

$w \Vdash D\varphi$ iff $(\exists v \in W)[w \neq v$ and $v \Vdash \varphi]$,

$w \Vdash \langle n\text{-times}\rangle\varphi$ iff $\{v \in W : v \Vdash \varphi\}$ has at least n elements.

$w \Vdash \varphi^*$ iff w can be "C-decomposed" into some finite sequence of worlds satisfying φ.

* See e.g. Sain (1984, 1988), Gargov *et al.* (1987), Ohlbach (1993), van der Hoek (1992) and the references therein.

The *universal modality* $\langle u \rangle$ is defined by $\langle u \rangle \varphi \stackrel{\text{def}}{=} \varphi \vee D\varphi$. Sain (1988) pointed out that for $n < 3$, "$\langle n$-times$\rangle$" is expressible from "D" (but not vica versa).

THEOREM 6. *Any logic in Theorem 5 remains decidable if we add "D", or "$\langle u \rangle$", or "$\langle n$-times$\rangle$", or "$*$" (or all of them) to it.*

The statements concerning "D" and "$\langle n$-times$\rangle$" are proved in Andréka *et al.* (1994e), see also Marx *et al.* (1995). For the statements including Kleene-star but without (A^-) see van Benthem (1994), for the weakly associative cases see Mikulás *et al.* (1995).

The significance of Theorem 6 is that the logic $\mathcal{AL}_{\text{times}}$, which is obtained from $\mathcal{AL}_{1\text{-}7}$ + (A^-) by adding connectives "D" and "$\langle n$-times$\rangle$" ($n \in \omega$), is very expressive. It is decidable, but it is more expressive than the undecidable associative logic $\mathcal{AL}_{1\text{-}7}$ + (A) (the logical equivalent of Relation Algebras), since already "$\langle$4-times$\rangle$" is not expressible in the latter.

Finally we discuss another direction, in which one can obtain an associative but decidable logic, namely by omitting axiom (K) (for connective "$\bullet$"). Recall that now our language includes the Booleans and a binary "$\circ$" (and perhaps other connectives as well). We say that $\mathcal{L}$ is a *classical logic with a binary modality* iff $\mathcal{L}$ contains (pt) as axiom and the set of validities of $\mathcal{L}$ is closed under substitution, modus ponens and replacement of equivalents.*

THEOREM 7. *The minimal associative classical logic with a binary modality is decidable.*

This theorem is a consequence of a general result of Pigozzi (1974), saying that the "join" of two disjoint decidable equational theories is decidable.

5. Some more advanced results

In Sections 2–3 above we outlined a proof method for undecidability results. The emphasis was on the method itself, and not on the results. One of the key points was a "sub-method" for avoiding the assumption that our algebras have a discriminator term (which on the logic side amounts to expressibility of a universal modality), and still be able to code quasiequations by equations. One of the reasons for presenting the method in Sections 2–3 was that this method can be pushed further and yields stronger results. Below we illustrate this by stating a few results obtained this way. Several (but not all) results below will have algebraic conditions in their formulations; we hope that having studied the proof in Section 2, the reader will be prepared for this.

Recall that a Boolean-ordered semigroup $\mathbf{A}$ is called *normal* if $x \circ 0 = 0 = 0 \circ x$ is valid in $\mathbf{A}$.

* Such logics are called "classical" in the usual mono-modal setting, cf. Segerberg (1971).

A semigroup $\mathbf{S} = \langle S, \cdot \rangle$ is called *eventually zero* iff there is some $n \in \omega$ with

$$\mathbf{S} \models \exists z (\forall x \, (x \cdot z = z \cdot x = z) \; \& \; \forall x_1 \ldots x_n \, (x_1 \cdot x_2 \cdot \ldots \cdot x_n = z)).$$

We note that this is a quite strong property, and that it is not true that every finite semigroup would be embeddable into an eventually zero semigroup.

Items (1, 2) of Theorem 8 below are joint results with Hajnal Andréka and Steven Givant.

THEOREM 8. *Let $\mathcal{L}$ be a logic with a binary modality "$\circ$". Assume "$\circ$" is associative. Then any one of conditions (1–6) below is sufficient for undecidability of $\mathcal{L}$.*

1. *$\mathcal{L}$ is a logic with a binary conjugated modality. Further, there is a normal infinite $\mathbf{A} \in \mathsf{Alg}(\mathcal{L})$ such that the reduct $\langle A, \circ, \triangleright, \triangleleft \rangle$ of $\mathbf{A}$ contains a subalgebra whose universe B is an anti-chain in $\mathbf{A}$ (that is, $x \wedge y = 0$ if $x, y \in B$ are different).*
2. *The same as 1 but with an infinite collection $\mathbf{A}_i$ and (not necessarily infinite) B_i ($i \in \omega$) in place of $\mathbf{A}$ and B such that if $i < j \in \omega$ then $\langle B_i, \circ \rangle \not\cong \langle B_j, \circ \rangle$.*
3. *Every finite group is embeddable into the $\circ$-reduct of some normal member of $\mathsf{Alg}(\mathcal{L})$.*
4. *Every finite two-generated eventually zero semigroup is embeddable into the $\circ$-reduct of some normal member of $\mathsf{Alg}(\mathcal{L})$.*
5. *To each $n \in \omega$, there are nontrivial finite groups $\mathbf{G}_1, \ldots, \mathbf{G}_n$ and a normal $\mathbf{A} \in \mathsf{Alg}(\mathcal{L})$ such that $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{G}_1 \times \ldots \times \mathbf{G}_n$ is a subalgebra of $\langle A, \circ \rangle$ and G is an anti-chain in $\mathbf{A}$.*

Next we state the solutions of some open problems from Jipsen (1992) (see Andréka *et al.* (1994c), Németi *et al.* (1995)). For this, we use the notation of Jipsen (1992) without recalling it.

COROLLARY 8.1. All the classes listed in Problem 3.61 of Jipsen (1992) have undecidable equational theories. Namely, EUR, ERM, IERM, CERM, RM, IRM, CRM, ICRM, ARM, SERM and ISERM; further EUR, IEUR, SEUR, ISEUR, CEUR, ICEUR have undecidable equational theories.

The minimal Euclidean logic was introduced above Theorem 4. The class RA of relation algebras is defined e.g. in Henkin *et al.* (1985), Jónsson and Tarski (1948), Venema (1992), Andréka *et al.* (1994b). Theorem 9 below is a generalization of the corresponding theorem for RA, stated in Andréka *et al.* (1994a, 1994b).

THEOREM 9. *Let $\mathcal{L}$ be an extension of minimal Euclidean logic. Assume there is a simple $\mathbf{A} \in \mathsf{Alg}(\mathcal{L}) \cap \mathsf{RA}$ in which there are infinitely many elements below Id. Then $\mathcal{L}$ is undecidable.*

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