Towards a Natural Language Semantics without Functors and Operands

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Abstract. The paper sets out to offer an alternative to the function/argument approach to the most essential aspects of natural language meanings. That is, we question the assumption that semantic completeness (of, e.g., propositions) or incompleteness (of, e.g., predicates) exactly replicate the corresponding grammatical concepts (of, e.g., sentences and verbs, respectively). We argue that even if one gives up this assumption, it is still possible to keep the compositionality of the semantic interpretation of simple predicate/argument structures. In our opinion, compositionality presupposes that we are able to compare arbitrary meanings in term of information content. This is why our proposal relies on an 'intrinsically' type free algebraic semantic theory. The basic entities in our models are neither individuals, nor eventualities, nor their properties, but 'pieces of evidence' for believing in the 'truth' or 'existence' or 'identity' of any kind of phenomenon. Our formal language contains a single binary non-associative constructor used for creating structured complex terms representing arbitrary phenomena. We give a finite Hilbert-style axiomatisation and a decision algorithm for the entailment problem of the suggested system.

Keywords: completeness, compositionality, decision algorithm, finite axiomatisability, finite entailment problem, function/argument metaphor, measurements, natural language semantics, pieces of evidence

1. Introduction

The cornerstones of the Fregean approach (Frege, 1984) to linguistic and semantic structure are the distinction between 'complete' and 'incomplete' expressions and meanings, on the one hand, and the assumption that these two are entirely parallel, on the other. Whatever is linguistically complete (incomplete) is also semantically complete (incomplete). The linguistic insight behind this distinction is age-long: Predicates were always seen as 'requiring' subjects, transitive expressions 'require' objects, modifiers 'require' something to modify, and so on. Because of the fully general and cross-linguistic character of such 'requirements', it has been assumed that they are an ingredient of our 'language of thought' rather than a superficial property of the natural languages we speak.

The inadequacy of the treatment of natural-language predicates as n-ary predicates (because of both their fixed arity and the fixed order of arguments) has been extensively argued by (Davidson, 1967) and his followers. If semantic 'incompleteness' only partially matches linguistic incompleteness, then the alleged parallelism between syntactic and semantic types (as assumed by, e.g., (Montague, 1974)) is difficult to sustain. Puzzles related to this problem include the facts that

- many verbs (e.g., *eat*) can be used both transitively and intransitively in the same meaning;
- nominalisations (e.g., *investment*) do not require the presence of the obligatory arguments of the corresponding verbs (e.g., *invest*);
- even semantically empty expressions, such as pronouns, can make an expression linguistically complete (e.g., #*I borrowed* vs. ^{OK}*I borrowed it*), and that in many languages (in the so-called 'pro-drop' languages) such devices are not required for linguistic completeness;
- some parameters that are always understood (e.g., the time and place where an event takes place) are not obligatorily expressed linguistically;
- in many cases there is a mismatch between the category of a word and its possible uses, e.g., a noun like *storm* can be used as referring to events, places, time intervals etc.

Our proposal is the following: Let linguistic analysis account for linguistic incompleteness (this is motivated by its largely language specific character), and let semantic analysis not rely on the completeness/ incompleteness distinction. For example, let linguistic analysis explain the linguistic behaviour of a verb like English *eat,* and let the semantics assign it a meaning that explains how the meaning of a subject or direct object argument (or a time/place adverbial), *when present,* can combine with it.

This is in line with (Davidson, 1967), but our solution departs from Davidsonian approaches in various ways. In a Davidsonian model, there are entities corresponding to events (like, say, an eating event) and entities corresponding to individuals (say, the eater and the thing eaten), plus relations between such entities (e.g., the eater stands in the AGENT relation with the event, whereas the thing eaten stands in the THEME relation with it). According to our modelling, all these different entities and relations are conceived of as a single type of entities, which we will call **phenomena.** In the semantics, a phenomenon is modelled with a set containing possible **constellations** (or 'observations,' or 'measurements') indicating or justifying the presence of the phenomenon. (Constellations play the same role with respect to phenomena as the 'truthmakers' of (Mulligan et al., 1984) with respect to propositions. The existence of a 'truthmaker' for a proposition is a necessary and sufficient condition for the proposition to be true; the existence of a constellation for a phenomenon is a necessary and sufficient condition for the phenomenon to exist.)

This way we obtain an essentially **type free** account of meanings: in addition to the completeness/incompleteness distinction, we also dispense with the strict parallelism between linguistic and semantic type distinctions. For example, consider the example of *storm* mentioned above. In a type free semantics, we do not have to decide whether storms are individuals, regions of space, temporal intervals or eventualities: they are simply phenomena the existence of which can be proven by constellations (for example, by a constellation containing meteorological measurements or visual pieces of information). Below we concentrate on the most essential aspects of meanings, such as predicates and their arguments, and leave the treatment of further, more complex features like adverbials, quantifiers etc. to a subsequent paper.

The paper is structured as follows. First, in Section 2, we illustrate the basics of our representation language and its intended semantics by using some simple examples. Then, in Section 3, we present the precise formalism, discuss the possible reasoning tasks, and give a finite Hilbert-style axiomatisation and a decision algorithm for the entailment problem of the suggested system. Finally, in Section 4, we turn to theoretical implications. In particular, we discuss the problem that abandoning the strict parallelism between linguistic and semantic structure apparently contradicts the principle of **compositionality.** We argue that, as a matter of fact, not only can compositionally be maintained in our system, but it even takes a more severe form than its usual interpretation. There are certain issues that any theory of natural language semantics has to face sooner or later, but which are not directly relevant to the ideas put forward here. We briefly discuss our plans concerning such issues in the last subsection.

2. Towards a non-Fregean natural-language semantics

2.1. 'Predicates' and 'arguments'

As we have discussed in the Introduction, semantics may not be the right place to account for the linguistically complete or incomplete character of expressions. For example, the fact that English *eat* can be used both transitively and intransitively may not be a fact about semantics (although it may have to do with semantic properties). The fact that eating involves an 'eater' and a 'thing eaten' need not be captured in terms of the 'incompleteness' of the meaning of *eat;* it can simply be considered as part of the complexity of that meaning, which is usually reflected in natural languages, although not necessarily in the same way in all of them.

Instead of going through the various arguments against the strict arity of natural-language predicates by Davidson and his followers, let us turn immediately to the analysis we propose. We will follow the Davidsonian tradition in that we conceive of eventualities as properties of spatio-temporal locations. So we think of the meaning of 'eat' (or 'eating') as a set of (possibly very different) **constellations,** namely, those in which there is evidence that eating takes place. In contrast to the standard view, however, we claim that this kind of semantics can be extended to other types of expression, those that do not refer to eventualities. For example, individuals can be seen as contiguous spatio-temporal regions, therefore, they can be modelled with the same type of constellation sets as eventualities. Properties and abstract entities have much more complex semantics, but the principle can be extended to them. Clearly, one needs some kind of evidence for admitting the presence of a property or an abstract entity, and there may be very different types of evidence for it, say an 'observation' or some kind of 'measurement:' each such piece of evidence can be conceived of as a constellation.

According to this general perspective, we propose that the meaningful entities of natural-language expressions like *Joe is eating* or *Joe is eating bread* should all be considered of the same type, phenomena, and interpreted as sets of constellations. Let us assume that the meaningful entities in question are Joe, eat, agent, bread and theme. (Following the usual practice, we leave out certain details, such as the fine points of treating proper names, the progressive etc.) The point is that Joe corresponds to the various possible ways of verifying Joe's presence/identity, eat to the eventualities that can be characterized as eating, bread to constellations in which some bread can be detected, agent to constellations proving that some animate entity performs an activity on purpose, and THEME to constellations in which an entity undergoes some change of state or location.

As we want to leave open the exact ontological status of constellations, we would like to assume only a minimal structure on them. A possible choice is to compare constellations according to their **information content**, say, a picture of a corner of a room is clearly less informative than a picture showing a larger part of the same room. Similarly, a picture can be more informative than another by virtue of its better resolution. It is natural to assume that this kind of informativeness relation is a **partial order**.

Though we propose every meaning to be a set of constellations, we do not think that every such set is appropriate as the meaning of some phenomenon. Clearly, if a constellation is sufficient for proving the presence of a phenomenon, then all more informative constellations are also sufficient for proving it. So a sensible choice would be to consider only upward-closed sets of constellations as meanings. Another possibility (which is equivalent, at least in well-founded partial orders) is to collect the **minimal** constellations only from each such set. This would correspond to the intuition that the meaning of a phenomenon contains the **necessary and sufficient** proofs for its presence, etc.

Let us give another reason for why we would like to choose this latter option. Since all the meaningful expressions are of the same semantic type (sets of constellations), we cannot combine them in the ways familiar from pre-Davidsonian or Davidsonian semantics. For example, the fact that Joe is the agent of eating cannot be expressed as $\textsc{agent}(\textsc{Joe}, \textsc{eat})$, because we do not interpret AGENT as a relation. Instead, we must produce a set of constellations proving that 'Joe is the agent of eating' from the constellations for JOE, AGENT and EAT. Let $A \circ B$ stand for the **combination** of A and B; we suggest that, in order to produce a meaning for such a combination, we must look at the **overlapping** constellations in the meaning $|A|$ of A and the meaning $|B|$ of B. For example, let $j \in |\textsc{Joe}|$ be a constellation proving Joe's presence, and $a \in |\textsc{agent}|$ be a constellation proving the presence of an agent. Then, by saying that j and a overlap, we would like to mean that there exists a common lower bound x of j and a according to their information content. This constellation x should show that our evidence to the effect of Joe's presence is not independent of our evidence for agenthood. This must be because Joe is the intentional agent in question. Now suppose that, say, j shows the picture of a room where both Joe and Pam are in, and j is not minimal in $|\textsc{Joe}|$ (as we also have there a smaller picture of the same room showing only the part where Joe is). Suppose also that a is an observation that is related to Pam, who is performing a purposeful activity. Then the overlap of j and a does not have much to do with Joe being our agent, but rather it is some constellation related to Pam.

Now let us return to the problem of producing constellations proving both Joe's presence and his agenthood. We would like to collect those constellations that are more informative than some overlapping pairs of constellations proving Joe's presence, on the one hand, and his agenthood, on the other. Further, to comply with the uniformity of all meanings, we should only keep the minimal ones among such constellations. In terms of the above, we propose the following translations for *Joe is eating* and *Joe is eating bread:*

$$\begin{array}{l} \textit{Joe is eating} \leadsto (\textsc{Joe} \circ \textsc{agent}) \circ (\textsc{agent} \circ \textsc{eat}); \\ \textit{Joe is eating bread} \leadsto \\ \quad ((\textsc{Joe} \circ \textsc{agent}) \circ (\textsc{agent} \circ \textsc{eat})) \circ \\ \qquad ((\textsc{bread} \circ \textsc{theme}) \circ (\textsc{theme} \circ \textsc{eat})). \end{array}$$

In these translations, the order in which the terms are combined is irrelevant: JOE $\circ$ AGENT is the same as AGENT $\circ$ JOE. So "$\circ$" denotes a **commutative** operation; however, it is important that it should be **non-associative** since, for example, in the second translation it is crucial that 'Joe' is the AGENT and 'bread' is the THEME rather than the other way round. It is easy to infer from what we said about combinations of meanings that "$\circ$" is an **idempotent** operation.

To sum up, we can translate *eating*, *Joe is eating* and *Joe is eating bread* in a uniform manner, without considering any of these expressions 'incomplete'. In addition, the fact that *Joe is eating* means, roughly, 'Joe is eating something', comes for free: this is exactly what our translation expresses.

2.2. 'Metonymical' interpretation

The case of argument-taking verbs is not the only one in which we find a mismatch between the semantic character and the syntactic behaviour of natural-language expressions. Take the English word *storm.* Since its grammatical category is 'noun', the type of its denotation is traditionally a predicate, namely, the one true for all and only the individual storms in a model of the world. To what extent one can consider a storm an individual is an interesting ontological question which will play some role in what follows, but it is not our main concern here. What is more intriguing is that *storm,* together with a legion of other nouns (mainly, nominalisations), can refer to places, time intervals and eventualities just as easily as 'individuals'. For example, in addition to *the storm moved West* (in which the storm is seen as an individual), we can say *in the storm* (location), *after the storm* (time interval) or *because of the storm* (eventuality).

The usual explanation of such systematic ambiguities relies on some concept of 'metonymy', i.e., on conceiving of such expressions as somehow elliptical. In particular, under this view, *in the storm* would 'stand for' *in [the place where] the storm [was],* whereas *after the storm* should be interpreted as *after [the time interval of the existence of] the storm,* and *because of the storm* as *because of [the event of] the storm [taking place].*

This may well be a legitimate treatment, but it does not explain why just words denoting 'individuals' like storms can function in these ways. A treatment not relying on a metonymy mechanism, but accounting for the behaviour of, say, *storm* on the basis of its meaning alone would be clearly preferable. We believe a type free treatment can do this job.

Take *The storm moved West* first. The meaning of this sentence can be produced in a way similar to that of *Joe is eating* (setting aside the entirely independent problem of how we treat definite articles), except that the grammatical subject here is the THEME argument (the storm is not a purposeful agent, but an individual undergoing change of location):

$$\textit{The storm moved West} \leadsto (\textsc{the-storm} \circ \textsc{theme}) \circ (\textsc{theme} \circ \textsc{moved-West}).$$

In this case, *the storm* is conceived of as an individual (assuming that, if we do not consider figurative meanings, only individuals can undergo change of location). That is, only those constellations in $|\textsc{the-storm}|$ will overlap with constellations in $|\textsc{theme}|$ which serve as evidence for storms **as individuals,** in the sense of contiguous time-space regions, i.e., entities which come into existence and then die, and which are delimited by more or less clear boundaries throughout their lifetime. Clearly, there must be ways of seeing storms in this way, and there will be constellations supporting such a view.

The other uses of *storm* can be explained in an analogous manner. For example, *in the storm* can be translated as

$$\textit{in the storm} \leadsto (\textsc{in} \circ \textsc{ground-area}) \circ (\textsc{ground-area} \circ \textsc{the-storm}).$$

Note that *in* has a relational meaning, so we treat it analogously to *eat* above. Namely, it means that a (somehow delimited) spatial area, the so-called **figure,** is compared to (namely, is included in) another delimited area, the so-called **ground.** In this case, 'the storm' plays the role of ground (GROUND-AREA). The translation above will be meaningful (i.e., it will not denote the empty set) only if some constellations in $|\textsc{the-storm}|$ have a non-empty overlap with $|\textsc{ground-area}|$, i.e., if they make reference to the spatial area occupied by the storm. That is, instead of recurring to metonymy, we assume that *storm* is inherently capable of being conceived of as a delimited area; at the same time, we make it explicit what it means for something to be **seen as** something else. The assumption that *storm* can refer to an area is not a stipulation: it involves the substantive claim that some (minimal) sets of observations proving the presence of a storm make reference to the its spatial boundaries.

3. Formalism

In this section, we give the precise definitions of the syntax and semantics of the suggested formalism. As in this paper we intend to take only the first steps of a rather unorthodox approach, we have chosen a very basic formal representation language. Our **terms** (representing phenomena) are built up (freely) from (phenomenon) **variables** with the help of the binary connective $\circ$:

$$t \;=\; p \mid t_1 \circ t_2 \; .$$

Our **formulas** represent the questions we want to ask about such phenomena. We introduce the simplest possible formulas only, expressing the equality of two phenomena:

$$\varphi \;=\; t_1 \equiv t_2 \; .$$

Our formal expressions are evaluated in **models**. Each model consists of a set of **constellations,** and can be considered as the current 'snapshot of the world,' or the 'aspects' we are interested in. We assume only a minimal structure on the constellations: they are 'arranged' according to their **informativeness**: $x \leq x'$ intends to mean that x' is at least as informative as x. So it is natural to assume that $\leq$ is (at least) a **partial order**. (In this paper we do not make any further assumptions, but we intend to investigate other possibilities in future work.) In addition, a model should represent the information we could collect about the simplest phenomena we are talking about at a given moment, that is, a valuation for the variables.

We define a **model** to be a triple $\mathcal{M} = \langle U, \leq, I \rangle$, where

- U is a non-empty set,
- $\leq$ is a partial order on U, and
- I is a function from the set of variables to the powerset of U such that for every variable p, $I(p)$ is an **antichain** (i.e., for all $x, x' \in I(p)$, if $x \leq x'$ then $x = x'$).

Given a model $\mathcal{M} = \langle U, \leq, I \rangle$, we define the **meaning** $|t|^{\mathcal{M}}$ for each term t as follows:

$$\begin{aligned} |p|^{\mathcal{M}} &= I(p) \\ |t_1 \circ t_2|^{\mathcal{M}} &= \min(\{u \in U : \exists u_1 \in |t_1|^{\mathcal{M}}, \exists u_2 \in |t_2|^{\mathcal{M}} \text{ such that } \\ &\qquad u_1 \text{ and } u_2 \text{ have a common } \leq\text{-lower bound,} \\ &\qquad \text{and } u \text{ is a common } \leq\text{-upper bound of} \\ &\qquad u_1 \text{ and } u_2\}) \end{aligned}$$

(Here $\min(X) = \{x \in X : \text{for all } x' \in X, \text{ if } x' \leq x \text{ then } x = x'\}$.) Note that this way $|t|^{\mathcal{M}}$ is always an antichain, for each term t. In what follows, we will omit the superscript from $|t|^{\mathcal{M}}$ and use simply $|t|$ when $\mathcal{M}$ is clear from the context.

Next, for each formula φ, we define the relation $\mathcal{M} \models \varphi$ (φ is **true in** $\mathcal{M}$) as follows:

$$\mathcal{M} \models t_1 \equiv t_2 \qquad \text{iff} \qquad |t_1|^{\mathcal{M}} = |t_2|^{\mathcal{M}} \ .$$

REMARK 1. Another possible simple way of comparing concepts is more permissive than equality. We might want to say something similar to material implication: A formula of the form $t_1 \to t_2$ would mean, roughly, that one's evidence for t_2 adds nothing to one's evidence for t_1, it is already included in it. In terms of information content, this means that t_1 is a phenomenon that is either the same as, or a **refinement** of, t_2. The following can be a corresponding truth relation:

$$\mathcal{M} \models t_1 \to t_2 \qquad \text{iff} \qquad \forall u_1 \in |t_1|^{\mathcal{M}}\ \exists u_2 \in |t_2|^{\mathcal{M}}\ u_2 \leq u_1 \ .$$

However, it is not hard to see that $\to$ can be defined with the help of $\circ$ and $\equiv$:

$$\mathcal{M} \models t_1 \to t_2 \qquad \text{iff} \qquad \mathcal{M} \models (t_1 \circ t_2) \equiv t_1 \ .$$

Indeed, suppose first $\mathcal{M} \models t_1 \to t_2$. Take some $x \in |t_1 \circ t_2|$. Then there is $y \leq x$ with $y \in |t_1|$, so there is $z \leq y$ with $z \in |t_2|$. So y is a common $\leq$-upper bound of y and z. Then $y = x$ follows, since $x \in |t_1 \circ t_2|$. So we have $|t_1 \circ t_2| \subseteq |t_1|$. Conversely, take some $x \in |t_1|$. Then there is $y \leq x$ with $y \in |t_2|$. So x is a common $\leq$-upper bound of x and y. Now $x \in |t_1 \circ t_2|$ follows because $|t_1|$ is an antichain. Now suppose that $|t_1 \circ t_2| = |t_1|$. Take some $x \in |t_1|$. Then $x \in |t_1 \circ t_2|$, so there is $y \leq x$ with $y \in |t_2|$.

Note that the relation $\sqsubseteq^{\mathcal{M}}$ defined on term-meanings by

$$|t_1|^{\mathcal{M}} \sqsubseteq^{\mathcal{M}} |t_2|^{\mathcal{M}} \qquad \text{iff} \qquad \mathcal{M} \models t_1 \to t_2$$

is always a partial order, so $\to$ is indeed similar to material implication.

3.1. Reasoning tasks

What kind of reasoning tasks should we investigate about the suggested formal system? The **satisfiability** or **consistency problem** would be a natural candidate:

- Given a (finite or infinite) set Σ of formulas (i.e., equalities), is there a model where every formula in Σ is true?

It is easy to see that our formalism is not strong enough to meaningfully address this question, as every set of formulas in satisfiable in some (probably intuitively weird) model. What is sensible in our context is the dual **validity problem**:

- Given a set Σ of formulas, is it the case that every formula in Σ is true in every model?

This doesn't really sound as a particularly interesting question in connection to phenomenon-equalities. However, it is a special case of the more relevant **entailment problem**:

- Given a set Σ of formulas and a formula φ, is φ true in all those models where every formula of Σ is true (in symbols: $\Sigma \models \varphi$)?

Below we show that the entailment problem is **finitely axiomatisable**, and the entailment problem is **decidable** and has the **finite model property**, whenever the set Σ of 'assumptions' is **finite**. (see Theorems 9 and 10 below.)

3.2. Hilbert-style calculus

$$p \equiv p \tag{1}$$
$$\text{given } p \equiv q, \text{ derive } q \equiv p, \tag{2}$$
$$\text{given } p \equiv q \text{ and } q \equiv r, \text{ derive } p \equiv r, \tag{3}$$
$$\text{given } p \equiv p' \text{ and } q \equiv q', \text{ derive } p \circ q \equiv p' \circ q', \tag{4}$$
$$p \circ q \equiv q \circ p, \tag{5}$$
$$p \circ p \equiv p, \tag{6}$$
$$(p \circ (q \circ r)) \circ r \equiv p \circ (q \circ r). \tag{7}$$

We say that

$$\Sigma \vdash s \equiv t$$

if there is a finite sequence of formulas ending with $s \equiv t$ and such that each formula in the sequence is either a substitution instance of an axiom above, or in Σ, or obtained from earlier formulas in the sequence by applying a substitution instance of one of the rules above.

Observe that (1)–(4) just say that $\equiv$ obeys the axioms and rules of the equational calculus. While (5) and (6) express that $\circ$ is a commutative and idempotent operation, one can regard (7) as some kind of 'weak associativity' (as an associative and idempotent $\circ$ clearly would have this property).

Interesting consequences of (1)–(7) are:

$$(p \circ q) \circ p \equiv p \circ q, \tag{8}$$

$$\text{given } p \circ q \equiv p \text{ and } q \circ r \equiv q, \text{ derive } p \circ r \equiv p. \tag{9}$$

Indeed, for (8):

$$(p \circ q) \circ p \overset{(5)}{\equiv} (q \circ p) \circ p \overset{(6)}{\equiv} (q \circ (p \circ p)) \circ p \overset{(7)}{\equiv} q \circ (p \circ p) \overset{(6)}{\equiv} q \circ p \overset{(5)}{\equiv} p \circ q.$$

For (9): Suppose we have $p \circ q \equiv p$ and $q \circ r \equiv q$. Then

$$p \circ (q \circ r) \equiv p \circ q \equiv p,$$

and so

$$p \circ r \equiv (p \circ (q \circ r)) \circ r \overset{(7)}{\equiv} p \circ (q \circ r) \equiv p.$$

Note that in fact the calculus defined by (1)–(6), (8) and (9) is equivalent to the above one, as (7) can be derived in it. Note also that using $\rightarrow$ instead of $\equiv$ (cf. Remark 1), (6) and (8) are equivalent to $p \rightarrow p$ and $(p \circ q) \rightarrow p$, respectively, while (9) is equivalent to the rule

$$\text{given } p \rightarrow q \text{ and } q \rightarrow r, \text{ derive } p \rightarrow r.$$

The proof of the following lemma is straightforward:

LEMMA 2. **(Soundness.)**
For all Σ, s, t, if $\Sigma \vdash s \equiv t$ then $\Sigma \models s \equiv t$.

3.3. Normal forms

Fix some linear order on the terms. Then say that a term t is in **pre-normal form** if, whenever $t_1 \circ t_2$ is a subterm of t, then t_1 is not later in the order than t_2. Clearly, every term t can be turned into an equivalent term $\tilde{t}$ in pre-normal form (by equivalent we mean both $\models t \equiv \tilde{t}$ and $\vdash t \equiv \tilde{t}$). Moreover, we can use the following algorithm: we start 'inside out' (that is, bottom up in the parsing tree), and when we find a $\circ$-term in the wrong order, swap the components. In what follows we don't bother with pre-normal forms, that is, with a slight abuse of notation, when we write $s \circ t$, we might mean $t \circ s$.

Now, given a term t in pre-normal form, we define inductively the normal form $\bar{t}$ of t by taking

$$\bar{p} = p \quad \text{for variables } p;$$
$$\overline{t \circ s} = \begin{cases} \bar{t} & \text{if } \bar{s} \text{ is a subterm of } \bar{t}; \\ \bar{s} & \text{if } \bar{t} \text{ is a subterm of } \bar{s}; \\ \bar{t} \circ \bar{s} & \text{otherwise.} \end{cases}$$

Clearly, this definition is also an algorithm: again we start 'inside out'.

Claim 3. (i) *For each term t, we have $\bar{\bar{t}} = \bar{t}$.*

(ii) *For all terms t, s, if s is a subterm of t then $\vdash t \circ s \equiv t$.*

(iii) *For each term t, we have $\vdash t \equiv \bar{t}$.*

Proof. Each statement follows by induction on the $\circ$-rank (the number of nested $\circ$s) of t. We give some details on the proof of (ii) and (iii).

(ii): If $t = p$ then $s = p$ should hold and $p \circ p \equiv p$ is (a substitution instance of) axiom (6). Suppose that $t = t_1 \circ t_2$ and s is a subterm of, say, t_1. Then by the induction hypothesis, we have $\vdash t_1 \circ s \equiv t_1$. On the other hand, $(t_1 \circ t_2) \circ t_1 \equiv t_1 \circ t_2$ is a substitution instance of (8), so we also have $\vdash (t_1 \circ t_2) \circ t_1 \equiv t_1 \circ t_2$. Now by (9) we obtain $\vdash (t_1 \circ t_2) \circ s \equiv t_1 \circ t_2$.

(iii): If $t = p$ then the statement is an instance of axiom (1). Suppose that $t = t_1 \circ t_2$ and we know that $\vdash t_1 \equiv \bar{t}_1$ and $\vdash t_2 \equiv \bar{t}_2$. By rule (4), we have $\vdash t \equiv \bar{t}_1 \circ \bar{t}_2$. So only the first two cases in the definition of $\overline{t_1 \circ t_2}$ are problematic. Suppose, say, that $\bar{t}_2$ is a subterm of $\bar{t}_1$, that is, $\overline{t_1 \circ t_2} = \bar{t}_1$. But then we have $\vdash \bar{t}_1 \circ \bar{t}_2 \equiv \bar{t}_1$ by (ii). $\square$

It follows from Claim 3 and rule (3) that if $\bar{t}$ and $\bar{s}$ are the same then $\vdash t \equiv s$. As we shall see (cf. Corollary 11), the converse statement also holds. In other words, normal forms are suitable tools for dealing with (unconditional) term-equalities.

3.4. Decision algorithm

Given a set Σ of formulas and a formula φ, we will define an infinite sequence $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_n \subseteq \ldots$ of sets of normal form terms, and an infinite sequence $\sim_0 \subseteq \sim_1 \subseteq \ldots \subseteq \sim_n \subseteq \ldots$ of equivalence relations ($\sim_i$ will be an equivalence relation on T_i) as follows. Let

$$T_0 = \{\bar{t} \mid t \text{ is a subterm of some term in } \Sigma \text{ or } \varphi\},$$

and for all $s, t \in T_0$, let

$$s \sim_0 t \;\Leftrightarrow\; s = t \text{ or } \exists u, v\,(s = \bar{u},\ t = \bar{v} \text{ and either } (u \equiv v) \in \Sigma \text{ or } (v \equiv u) \in \Sigma).$$

Observe that T_0 is closed under taking subterms, and it is *finite* if Σ is finite.

Now suppose that T_n and $\sim_n$ have already been defined (such that T_n is closed under taking subterms). Let T_{n+1} be the smallest set containing T_n and $\sim_{n+1}$ the smallest equivalence relation containing $\sim_n$ such that

$(*)$ for all $s, t, s \circ u \in T_n$: if $\boxed{s \sim_n t \text{ and } s \circ u \not\sim_n s,}$

then $\boxed{\overline{t \circ u} \in T_{n+1} \text{ and } \overline{t \circ u} \sim_{n+1} s \circ u.}$

In other words, we obtain T_{n+1} by closing T_n under applications of 'rule $(*)$' above.

Note that T_{n+1} is closed under taking subterms: if v is a proper subterm of $\overline{t \circ u}$ then it is a subterm of either $\bar{t}$ or $\bar{u}$. Since u is a subterm of an element of T_n, it is also in T_n and so, since $\bar{t} = t$ and $\bar{u} = u$ by Claim 3(i), v is in $T_n \subseteq T_{n+1}$.

Observe that each application of this rule either

(R1) adds a new element to an (existing) equivalence class (if $\overline{t \circ u} \notin T_n$, and so $\overline{t \circ u} = t \circ u$), or

(R2) unites two equivalence classes (if $\overline{t \circ u} \in T_n$, and $\overline{t \circ u} \not\sim_n s \circ u$), or

(R3) just has no effect (if $\overline{t \circ u} \in T_n$, and $\overline{t \circ u} \sim_n s \circ u$).

In particular, the number of equivalence classes does not increase as n grows. Note that if Σ is *finite* then each T_n $(n < \omega)$ is finite as well.

EXAMPLE 4. $\Sigma = \{x \equiv (y \circ z) \circ (v \circ z),\ y \equiv x \circ w\}$ $\quad \varphi = x \equiv y$.

$\sim_0$ classes:

$\{x,\ (y \circ z) \circ (v \circ z)\},\ \{y,\ x \circ w\},\ \{y \circ z\},\ \{v \circ z\},\ \{v\},\ \{w\},\ \{z\}$

$\sim_1$ classes:

$\{x,\ (y \circ z) \circ (v \circ z)\},\ \{y,\ x \circ w,\ ((y \circ z) \circ (v \circ z)) \circ w\},$
$\{y \circ z,\ (x \circ w) \circ z\},\ \{v \circ z\},\ \{v\},\ \{w\},\ \{z\}$

$\sim_2$ classes:

$\{x,\ (y \circ z) \circ (v \circ z),\ ((x \circ w) \circ z) \circ (v \circ z)\},$
$\{y,\ x \circ w,\ ((y \circ z) \circ (v \circ z)) \circ w,\ y \circ z,\ (x \circ w) \circ z\},$
$\{v \circ z\},\ \{v\},\ \{w\},\ \{z\}$

$\sim_n$ classes, for $n \geq 3$:

$$\begin{aligned}&\{x,\ (y\circ z)\circ(v\circ z),\ ((x\circ w)\circ z)\circ(v\circ z),\ (x\circ w)\circ(v\circ z),\\ &\quad y\circ(v\circ z),\ y,\ ((x\circ w)\circ z)\circ(v\circ z),\ x\circ w,\ ((y\circ z)\circ(v\circ z))\circ w,\\ &\quad y\circ z,\ (x\circ w)\circ z\},\\ &\{v\circ z\},\ \{v\},\ \{w\},\ \{z\}\end{aligned}$$

Lemma 5. *If Σ is finite then the 'algorithm always stops:' Given Σ and φ, there is a number $N_{\Sigma,\varphi}$ such that for all $m \geq N_{\Sigma,\varphi}$, we have $T_m = T_{N_{\Sigma,\varphi}}$ and $\sim_m = \sim_{N_{\Sigma,\varphi}}$.*

Proof. For every n and $s \in T_n$, we denote the $\sim_n$-class of s by $[s]_n$. We define two relations $\prec_n^-$ and $\preceq_n$ on $\sim_n$-classes by taking, for all $s, t \in T_n$,

$$\begin{aligned}[s]_n \prec_n^- [t]_n \quad &\Leftrightarrow \quad \exists s' \sim_n s\colon \exists u \in T_n\colon (\overline{s'\circ u} = s'\circ u \text{ and } s'\circ u \sim_n t);\\ \preceq_n \quad &= \quad \text{the reflexive and transitive closure of } \prec_n^- .\end{aligned}$$

Then the relation $\preceq_n$ is 'non-decreasing' as n grows: if $s, t \in T_n$ and $[s]_n \preceq_n [t]_n$, then $[s]_m \preceq_m [t]_m$ as well, for all $m \geq n$.

Claim 6. *If $[s]_n \preceq_n [t]_n$ and $s \not\sim_n t$, then there exist $m \geq n$ and $t' \in T_m$ such that $t' \sim_m t$ and s is a subterm of t'.*

Proof. As $[s]_n \preceq_n [t]_n$ and $s \not\sim_n t$, there exist $k > 0$ and $u_0, \ldots, u_k$, $a_0, \ldots, a_{k-1} \in T_n$ such that

$$\begin{aligned}&s \sim_n u_0, \quad u_k \sim_n t,\\ &u_i \not\sim_n u_{i+1} \text{ for } i < k,\\ &\overline{u_i\circ a_i} = u_i\circ a_i, \text{ for } i < k,\\ &u_{i+1} \sim_n u_i\circ a_i, \text{ for } i < k.\end{aligned}$$

Now we will apply rule $(*)$ several times. With a slight abuse of notation, we use $\sim$ to denote the obtained extensions of $\sim_n$.

First, as $s \sim u_0 \not\sim u_1 \sim (u_0\circ a_0)$, an application of rule $(*)$ yields $\overline{s\circ a_0} \sim u_1$. Then, either $u_1 \sim u_2$ or $u_1 \not\sim u_2$ at this point. In the latter case, another application of rule $(*)$ yields $\overline{\overline{(s\circ a_0)}\circ a_1} \sim u_2$. And so on, finally we obtain a term $t' \sim t$ of the form

$$t' = \overline{(\ldots\overline{(\overline{\overline{(s\circ a_0)}}\circ a_{i_1})}\circ\ldots)\circ a_{i_c}}, \tag{10}$$

where $1 \leq i_1 \leq \ldots \leq i_c < k$ are such that $u_i \sim u_{i+1}$ whenever $i \in \{1, \ldots, k-1\} - \{i_1, \ldots, i_c\}$. $\square$

CLAIM 7. *$\preceq_n$ is 'eventually antisymmetric:' If $[s]_n \preceq_n [t]_n$ and $[t]_n \preceq_n [s]_n$, then there is an $m \geq n$ such that $s \sim_m t$.*

Proof. Suppose $[s]_n \preceq_n [t]_n$ and $[t]_n \preceq_n [s]_n$. If $s \sim_n t$, then $m = n$ is a good choice.

So suppose $s \not\sim_n t$. Throughout, we will use the terms $u_0, \ldots, u_k$, $a_0, \ldots, a_{k-1}$ and the $\sim$-notation, as introduced in the proof of Claim 6. As $[s]_n \preceq_n [t]_n$, after some applications of rule $(*)$, we obtain a term $t' \sim t$ of the form (10). Now either $t' \sim s$ at this point or, by $[t]_n \preceq_n [s]_n$ and Claim 6, t' is a subterm of some $s' \sim s$. Next, either $s' \sim t$ at this point and we are ready, or let $i + 1 \leq k$ be the smallest number such that $s' \not\sim u_{i+1}$, that is,

$$s \sim s' \sim u_0 \sim \ldots \sim u_i \not\sim u_{i+1}.$$

Then, by (10), a_i is a subterm of t', and so it is a subterm of s' as well. Therefore, an application of rule $(*)$ yields

$$s' = \overline{s' \circ a_i} \sim (u_i \circ a_i) \sim u_{i+1}.$$

And so on, finally uniting the classes of s and t, as required. □

Since the number of $\sim_0$-classes is finite, and the number of equivalence classes does not increase as n grows, there is a smallest number K such that the number of $\sim_m$-classes remains the same, for all $m \geq K$. If the number c_K of $\sim_K$-classes is 1, then rule $(*)$ cannot be applied any more to extend T_K and $\sim_K$, so we can choose $N_{\Sigma,\varphi}$ to be K.

So suppose that $c_K > 1$. Since the number of equivalence classes does not change after step K, by Claim 7 we obtain that $\preceq_K$ is antisymmetric, so it is a partial order. Moreover, since '$\preceq_n$ is non-decreasing as n grows', there is a smallest $M \geq K$ such that $\preceq_M = \preceq_m$ for all $m \geq M$ (in the sense that for all $s, t \in T_M$, $[s]_M \preceq_M [t]_M$ iff $[s]_m \preceq_m [t]_m$). With a slight abuse of notation, we will denote this 'stable' partial order on (possibly growing in size) equivalence classes by $\preceq$.

By the choice of K, we know that after step K each application of rule $(*)$ is either an (R1) or an (R3). It remains to show that there are only finitely many (R1)s after step M. In other words, we need to show that after step M each class can be extended only by finitely many new terms. To this end, we claim that each time we extend the class of a term $s \circ u$ by an application of rule (R1) at a step $n \geq M$, we need to have a term t such that $[t]_n \not\preceq [s \circ u]_n$. This is because, on the one hand, it is a precondition of rule $(*)$ that $[t]_n \neq [s \circ u]_n$. And, on the other hand, as a result of applying (R1), $[t]_{n+1} \preceq [s \circ u]_{n+1}$ (since $t \circ u$ gets into $[s \circ u]_{n+1}$), and $\preceq$ doesn't grow after step M, so in fact we should already have had $[t]_n \preceq [s \circ u]_n$.

In particular, after step M, $\preceq$-minimal classes cannot be extended. Classes of '$\preceq$-degree' 1 can be extended only by terms having a 't-component' from a $\preceq$-minimal class. Further, the 'u-component' of a new term is always such that $[u]_n \preceq [s \circ u]_n$. So (at first sight), for each t-component, there can be an infinite supply of u-components (out of the newly added terms) as the class grows. But we cannot use a t as t-component if it is already a subterm of the u-component (otherwise it is not an (R1)-type application). So a t-component cannot be reused with a u-component created using the t-component in question, thus classes of '$\preceq$-degree' 1 can be extended only with finitely many new terms. Then we extend classes of '$\preceq$-degree' 2, and so on. Clearly, this way each class can be extended only with finitely many new terms, completing the proof of Lemma 5. □

REMARK 8. If $\Sigma = \emptyset$, that is, we want to decide whether a formula φ is valid, then all the $\sim_0$-classes are singletons by definition. Therefore, all applications of rule $(*)$ are of type (R3), that is, we can always choose $N_{\emptyset,\varphi}$ to be 0.

3.5. Main results

Given a set Σ of formulas and a formula φ, take the infinite sequences T_n and $\sim_n$ $(n < \omega)$ defined above, and let

$$T = \bigcup_{n<\omega} T_n \quad \text{and} \quad \sim = \bigcup_{n<\omega} \sim_n .$$

It is easy to see that $\sim$ is an equivalence relation on T. We call a pair (Σ, φ) (where φ is $s \equiv t$) a **YES-instance** iff $\bar{s} \sim \bar{t}$ holds.

THEOREM 9. *Let Σ be a set of formulas and φ a formula. Then the following are equivalent:*

(1) $\Sigma \vdash \varphi$

(2) $\Sigma \models \varphi$

(3) (Σ, φ) *is a YES-instance.*

THEOREM 10. *The finite entailment problem is decidable and has the finite model property.*

COROLLARY 11. *The validity problem is decidable. In particular,*

$$\models s \equiv t \qquad \textit{iff} \qquad \overline{s} \textit{ and } \overline{t} \textit{ are the same.}$$

Proof. If $\overline{s}$ and $\overline{t}$ are the same then $\models s \equiv t$ follows by Claim 3 and Lemma 2. Conversely, if $\models s \equiv t$ then, by Theorem 9, $(\emptyset, s \equiv t)$ is a YES-instance. By Remark 8, the decision algorithm stops in step 0, meaning that $\overline{s}$ and $\overline{t}$ are the same. $\square$

Proof of Theorem 9. (1) $\Rightarrow$ (2): It is Lemma 2.

(2) $\Rightarrow$ (3): Suppose that (Σ, φ) is not a YES-instance. Then we will give a model $\mathcal{M}$ such that all formulas in Σ are true in $\mathcal{M}$, but φ is not.

To this end, we denote the $\sim$-class of a term s by $[s]$. We define a relation $\preceq$ on $\sim$-classes by taking, for all $s, t \in T$,

$$[s] \preceq [t] \qquad \text{iff} \qquad \exists n < \omega\colon ([s]_n \preceq_n [t]_n)$$

(cf. the proof of Lemma 5 for notation). By Claim 7, we have that

$$\preceq \text{ is antisymmetric and so it is partial order.} \tag{11}$$

We will also use the following property of $\preceq$:

CLAIM 12. *For all terms $s, t \in T$, if $[s] \preceq [t]$ and $s \circ t \in T$, then $s \circ t \sim t$.*

Proof. If $[t] = [s]$ but $s \circ t \not\sim t$ then, by rule $(*)$, $\overline{s \circ s} = s$ belongs to $[s \circ t]$, so $s \circ t \sim t$, a contradiction.

So suppose that $[s] \preceq [t]$, $s \not\sim t$ and $s \circ t \not\sim t$. Then, by Claim 6, s is a subterm of some $t' \in [t]$. So, by rule $(*)$, $\overline{s \circ t'} = t'$ belongs to $[s \circ t]$, a contradiction again. $\square$

Now we define a (non-empty) set U, a (irreflexive) binary relation $<^-$ on U, and a labelling function $\ell : U \to \{[t] \mid t \in T\} \cup \{\emptyset\}$ as follows:

(i) For each class C, put a fresh x_C into U, and define $\ell(x_C) = C$.

(ii) Then, for every such x_C and every $s \circ t \in C$ such that $s, t \in C$ do not hold:

- if $s \in C$, but $t \notin C$, then put a new point y into U, and define $\ell(y) = [t]$ and $y <^- x_C$;
- if $t \in C$, but $s \notin C$, then put a new point y into U, and define $\ell(y) = [s]$ and $y <^- x_C$;
- if $s, t \notin C$ (by Claim 12, $s \not\sim t$ follows), then put three new points y_1, y_2, y into U, and define $\ell(y_1) = [s]$, $\ell(y_2) = [t]$, $\ell(y) = \emptyset$ (we call such points **dummy**), and $y_1 <^- x_C$, $y_2 <^- x_C$, $y <^- y_1$, $y <^- y_2$.

(iii) Then continue 'unfolding the terms' in the labels of the newly created points like in (ii). Then again, and so on.

Now let $\leq$ be the reflexive and transitive closure of $<^-$. Since each dummy point can be $\geq$ than itself only, it is straightforward to see that, for all $x, y \in U$ such that $\ell(x) \neq \emptyset$, $\ell(y) \neq \emptyset$,

$$\text{if } x \leq y \text{ then } \ell(x) \preceq \ell(y). \tag{12}$$

Now it follows from (11) that $\leq$ is antisymmetric, so it is a partial order on U. Moreover,

$$\text{if } x \leq y \text{ and } x \neq y \text{ then } \ell(x) \neq \ell(y). \tag{13}$$

Now, for each propositional variable p, we let

$$I(p) = \{x \in U \mid p \in \ell(x)\}.$$

Then $I(p)$ is a $\leq$-antichain: take $x, y \in I(p)$ with $x \leq y$. Then $p \in \ell(x) \cap \ell(y)$, so $\ell(x) = \ell(y)$. Now $x = y$ follows by (13). So $\mathcal{M} = \langle U, \leq, I \rangle$ is a model.

CLAIM 13. *For every $t \in T$, $|t| = \{x \in U \mid t \in \ell(x)\}$.*

Proof. It is by induction on the structure of term t. If t is a propositional variable then the claim follows by the definition of I.

Suppose that the claim is true for terms s and t such that $s \circ t \in T$. Suppose first that $x \in |s \circ t|$. There are four cases:

1: $x \in |s|$, $x \in |t|$. By IH, $s, t \in \ell(x)$, so $[t] \preceq [s]$. By Claim 12, we have $s \circ t \in \ell(x)$.

2: $x \in |s|$, $x \notin |t|$. Then there is a $y \in U$ such that $y \leq x$, $y \neq x$ and $y \in |t|$. By IH, $s \in \ell(x)$ and $t \in \ell(y)$. So by (12), we have $[t] \preceq [s]$. Again by Claim 12, we have $s \circ t \in \ell(x)$.

3: $x \in |t|$, $x \notin |s|$. This is similar to Case 2.

4: $x \notin |s|$, $x \notin |t|$. Then there are $y, y_1, y_2 \in U$, all different from each other and from x such that $y \leq y_1$, $y \leq y_2$, $y_1 \leq x$, $y_2 \leq x$, $y_1 \in |s|$, $y_2 \in |t|$. By the definition of U and $\leq$, this implies that $y <^- y_1$, $y <^- y_2$, $\ell(y) = \emptyset$, and there are $z \in U$ and $u_1 \circ u_2 \in T$ such that $y_1 <^- z$, $y_2 <^- z$, $z \leq x$, $u_1 \in \ell(y_1)$, $u_2 \in \ell(y_2)$, $u_1 \circ u_2 \in \ell(z)$. Since z is a $\leq$-upper bound for y_1 and y_2, $x \in |s \circ t|$ implies that $x \leq z$, and so $x = z$, from which $x \neq y_1$ and $x \neq y_2$ follow. By IH, we have $s \in \ell(y_1)$ and $t \in \ell(y_2)$. By (13), we have $\ell(x) \neq \ell(y_1)$ and $\ell(x) \neq \ell(y_2)$. So, by rule $(*)$, $\overline{s \circ u_2} \in \ell(x)$, and we also have $\overline{s \circ u_2} \neq s$ and $\overline{s \circ u_2} \neq u_2$. Then, again by rule $(*)$, we obtain that $\overline{s \circ t} = s \circ t$ belongs to $\ell(x)$.

Conversely, suppose $s \circ t \in \ell(x)$. By the definition of U and $\leq$, x is a $\leq$-upper bound of some y_1, y_2 such that y_1 and y_2 have a common $\leq$-lower bound, and $s \in \ell(y_1)$, $t \in \ell(y_2)$. By IH, $y_1 \in |s|$ and $y_2 \in |t|$, so x is a common $\leq$-upper bound as needed. Let $z \leq x$ be such that $z \in |s \circ t|$. By the direction already proven, $s \circ t \in \ell(z)$ follows. So $\ell(x) = \ell(z)$. Then we have $x = z$ by (13). This completes the proof of Claim 13. $\square$

Now it follows from Claims 13, 3(iii) and Lemma 2 that all formulas in Σ are true in $\mathcal{M}$, but φ is not true in $\mathcal{M}$.

(3) $\Rightarrow$ (1): We show by induction on n that, for all terms s, t, if $\bar{s} \sim_n \bar{t}$ then $\Sigma \vdash s \equiv t$. First, let $n = 0$ and suppose $\bar{s} \sim_0 \bar{t}$. Then either $\bar{s} = \bar{t}$ and then $\Sigma \vdash s \equiv t$ by Claim 3(iii) and rule (3), or $(s \equiv t) \in \Sigma$ or $(t \equiv s) \in \Sigma$ and then $\Sigma \vdash s \equiv t$ by rule (2).

Now suppose $\bar{s} \sim_{n+1} \bar{t}$. If $\bar{s} \sim_n \bar{t}$ then we have $\Sigma \vdash s \equiv t$ by IH. Otherwise, there are $s', t', u \in T_n$ such that $s' \sim_n t'$, $\bar{t} = \overline{t' \circ u}$ and $\bar{s} = s' \circ u$. By IH, we have $\Sigma \vdash s' \equiv t'$. So, by rules (4), (3) and Claim 3(iii) we obtain $\Sigma \vdash s' \circ u \equiv \overline{t' \circ u}$, that is, $\Sigma \vdash \bar{s} \equiv \bar{t}$. Now $\Sigma \vdash s \equiv t$ follows again by Claim 3(iii). $\square$

Proof of Theorem 10. By Lemma 5 and Theorem 9, the procedure described in subsection 3.4 is a decision procedure for the finite entailment problem.

As concerns the finite model property, we claim that if Σ is finite then the model $\mathcal{M} = \langle U, \leq, I \rangle$ defined in the proof of the '(2) $\Rightarrow$ (3)' part of Theorem 9 is finite. Indeed, by definition, every point in U has finitely many $<^-$-predecessors, so the finiteness of U follows from (11), (12) and (13). $\square$

Note that in an implementation of the decision algorithm we may stop in a step n much smaller than $N_{\Sigma, s\equiv t}$, in case we detect that $\bar{s}$ and $\bar{t}$ have become $\sim_n$-equivalent.

4. Conclusion

4.1. What about compositionality?

Under the view proposed in this paper, linguistic and semantic completeness/incompleteness need not be directly related. This seemingly contradicts one of the most important methodological principles of modern semantics, namely, the **compositionality principle,** which emphasises the parallelism of semantic and linguistic structure:

> '*The meaning of a complex expression is a function of its structure and the meanings of its constituents.*'

In fact, in some of its formulations, e.g., Montague's ((Montague, 1974)), this principle requires a total parallelism between syntactic and semantic types; more precisely, it posits a homomorphism between the semantic (function application based) algebra and the syntactic (constituency based) algebra. So the question naturally arises whether we are willing to reject the compositionality principle itself.

To the contrary, we are convinced that compositionality does not impose anything on us with respect to 'completeness.' In our view, the principle of compositionality is indeed of utmost importance for natural language semantics in that it carves out that aspect of natural language use that can and is to be dealt with successfully by semantics at all. Compositionality means that the only phenomena that legitimately can be termed semantic must exhibit a systematic correspondence between form and meaning. That is why, for example, the eventual motivatedness of more or less idiomatic expressions (e.g., of *mouse* 'computer pointing device'), although discernible to the speakers to some extent, falls outside the scope of compositionality, hence, of semantics in general. Compositionality (fortunately) contains no stipulation to the effect that all aspects of linguistic form must be explained by a parallelism with meaning (e.g., one could hardly claim that stems with similar phonological shapes are also semantically similar), therefore it allows linguistic 'completeness' to be a phenomenon independent of or only loosely related to meaning. (As a matter of course, the converse is also true: compositionality also does not stipulate that all aspects of meanings must be reflected by linguistic form.)

On the other hand, we think compositionality should be **strengthened** from another point of view. The principle says nothing on what a 'function' can be. In general, there is nothing a 'function' cannot do; therefore, under the traditional, weak definition of compositionality one would expect very unusual ways of combining meanings. For example, a function that combines two predicates and yields as a value the one that has greater cardinality than the other would be a perfectly compositional way of combination; or, in principle, a combination function would be allowed to behave in a wildly non-uniform manner, in the sense of performing totally different operations depending on the meanings of its operands. (Similar arguments were made by (Zadrozny, 1994; Kálmán, 1996).) Obviously, what one would expect from a compositional combination function is that it should be uniform in the above sense, and that it should preserve the meanings that it combines (i.e., it must not be destructive). These two requirements together suggest an **essentially additive** operation, in very much the same vein as in traditional linguistics since the Antiquity, which thought of combining meanings as 'adding them together'. (Note that others, such as (Vermeulen and Visser, 1996; Visser, 2003) have also proposed semantic formalisms that, although with different motivations, share the feature of additiveness.) Technically speaking, an additive combination operation means that the value that it yields is

- is richer than either one of its operands in terms of information content;
- contains only information originating from one of the operands.

This, in turn, presupposes that we are able to compare arbitrary meanings (both the ones to be combined and the resulting value) in terms of information content. A sufficient (and under certain reasonable assumptions, necessary) condition for this to be feasible is that semantics must be **type free,** just like the semantics we have proposed in the present paper.

4.2. Open issues

In this paper we have been concerned with the basic ingredients of natural language semantics, without touching upon notorious problems such as adjectival modification, modalities and intensionality, propositional attitudes, quantification, information structure, and so on. Some of these problems obviously require the enrichment of the apparatus described above, but we believe that the problems of such enrichments **can and should be separated** from the essentials discussed here.

For example, on the one hand it would be easy to construct a first-order (or even higher-order) language the atomic formulas of which are exactly the formulas of our suggested language. But this would amount to begging the question whether a type free approach to natural language semantics can be pursued in general. On the other hand, natural languages are able to express propositions involving **collections** and second-order predicates (e.g., **quantification**). Clearly, such meanings cannot be treated with a machinery that cannot express arithmetic. But it is arguable whether arithmetic is indeed part of natural language semantics. In a subsequent paper we are planning to complement our framework with a quantificational component which can treat at least certain restricted types of quantification. We also intend to investigate the possibilities of extending the stance that we have taken above to some other features, such as modalities and various notions of incompatibility among phenomena (which can lead to different kinds of negation).

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