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CONNECTIONS BETWEEN AXIOMS OF SET THEORY AND BASIC THEOREMS OF UNIVERSAL ALGEBRA

H. ANDRÉKA. Á. KURUCZ. AND I. NÉMETI

Abstract. One of the basic theorems in universal algebra is Birkhoff's variety theorem: the smallest equationally axiomatizable class containing a class $\mathbf{K}$ of algebras coincides with the class obtained by taking homomorphic images of subalgebras of direct products of elements of $\mathbf{K}$. G. Grätzer asked whether the variety theorem is equivalent to the Axiom of Choice. In 1980. two of the present authors proved that Birkhoff's theorem can already be derived in ZF. Surprisingly. the Axiom of Foundation plays a crucial role here: we show that Birkhoff's theorem cannot be derived in $ZF + AC\backslash\{\text{Foundation}\}$. even if we add Foundation for Finite Sets. We also prove that the variety theorem is equivalent to a purely set-theoretical statement. the Collection Principle. This principle is independent of $ZF\backslash\{\text{Foundation}\}$. The second part of the paper deals with further connections between axioms of ZF-set theory and theorems of universal algebra.

§**1. Introduction.** The problems investigated here fit into the field which is called after S. G. Simpson "reverse mathematics". In this field (also called "inverse set theory") one tries to determine what is the exact fragment of set theory truly needed to establish the core theorems of certain mathematical disciplines. In universal algebra the first reverse questions were formulated by G. Grätzer. Problem 31 in [7] asks whether Birkhoff's variety theorem (see below for an exact formulation) is equivalent to the Axiom of Choice. As shown in [3], the answer is no: Birkhoff's theorem can be derived already in ZF. In §2 we show (Theorem 1) that the Axiom of Foundation (Regularity) is necessary in that derivation. Even the extension of $ZF\backslash\{\text{Foundation}\}$ with Foundation for Finite Sets is not enough to derive Birkhoff's theorem. Moreover, we prove (Theorem 2) that on the basis of $ZF\backslash\{\text{Foundation}\}$, Birkhoff's theorem is equivalent to a purely set-theoretical statement, the Collection Principle. This principle is implied by (but not equivalent to) the Axiom of Foundation. The main technical means in proving the results above are Theorems 3 and 4 below.

In §3 we discuss similar questions, namely, connections between properties of operators on classes of algebras and axioms of ZF-set theory. Theorem 5–7 also contain partial answers to Problem 28 in [7]: what the semigroup generated by the operators (on classes of algebras) $\mathbf{I}$, $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$ (see below), etc., is like without AC.

Notation. Our set-theoretical usage follows [10]. In particular, ZF denotes Zermelo-Fraenkel Set Theory (which includes AF, the *Axiom of Foundation*),

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ZF^- is ZF without AF, and AC denotes the *Axiom of Choice*. The *Axiom of Foundation for Finite Sets* (AF_ω) is the following statement

$$(AF_\omega) \qquad \forall x(0 < |x| < \omega \to \exists y \in x(y \cap x = \emptyset)),$$

where $|x| < \omega$ abbreviates the formula ($\exists$ function f) ($\mathrm{Dom}\, f \in \omega \wedge \mathrm{Rng}\, f = x$).

For universal algebraic notions we generally follow [9]. If $\mathbf{K}$ is a class of algebras, we let

$\mathbf{IK} \overset{\mathrm{def}}{=} \{\mathfrak{A} \colon \mathfrak{A}$ is isomorphic to some $\mathfrak{B} \in \mathbf{K}\}$,

$\mathbf{HK} \overset{\mathrm{def}}{=} \{\mathfrak{A} \colon \mathfrak{A}$ is a homomorphic image of some $\mathfrak{B} \in \mathbf{K}\}$,

$\mathbf{SK} \overset{\mathrm{def}}{=} \{\mathfrak{A} \colon \mathfrak{A}$ is a subalgebra of some $\mathfrak{B} \in \mathbf{K}\}$,

$\mathbf{P'K} \overset{\mathrm{def}}{=} \{\mathfrak{A} \colon \mathfrak{A} = P_{i \in I}\mathfrak{A}_i$, $\langle \mathfrak{A}_i \colon i \in I\rangle$ is a system of members of $\mathbf{K}$ for some set I, the universe A of $\mathfrak{A}$ is nonempty$\}$,

$\mathbf{PK} \overset{\mathrm{def}}{=} \mathbf{IP'K}$,

$\mathbf{P^rK} \overset{\mathrm{def}}{=} \{\mathfrak{A} \colon \mathfrak{A}$ is isomorphic to a reduced product of members of $\mathbf{K}\}$.

A *similarity type* is a function t mapping some set into ω; we take as understood when an algebra is of type t. Two algebras are called *similar* if they are of the same type. If $\mathbf{Q}$ and $\mathbf{Q}'$ are operators on classes of algebras, then $\mathbf{Q} \le \mathbf{Q}'$ denotes the schema

$$\forall \text{ class } \mathbf{K} \text{ of similar algebras } (\mathbf{QK} \subseteq \mathbf{Q'K}).$$

An operator $\mathbf{Q}$ on classes of algebras is a *closure operator* iff $\mathbf{QQ} = \mathbf{Q}$.

Equations of type t are formulated in a first-order language of type t having a countable sequence $v_0, v_1, \ldots$ of variables. $\mathbf{EqK}$ is the set of all equations holding in every member of $\mathbf{K}$. For any set Γ of formulas of our language, $\mathrm{Mod}\,\Gamma$ is the class of all models of Γ.

Birkhoff's variety theorem is the schema

$$\forall \text{ class } \mathbf{K} \text{ of similar algebras } (\mathrm{Mod}\,\mathbf{EqK} = \mathbf{HSP\ K}).$$

§2. A set theoretical equivalent to Birkhoff's variety theorem.

THEOREM 1. $ZF^- + AC + AF_\omega \nvdash \mathrm{Mod}\,\mathbf{EqK} = \mathbf{HSP\ K}$.

REMARK. Recall that in ZF-set theory, a *class* is just an informal version of a formula. Thus, the formal counterpart of Theorem 1 is as follows:

There is a formula $\varphi(v_0)$ of the language of set theory such that

$$ZF^- + AC + AF_\omega \nvdash \forall v_0(\varphi(v_0) \to \text{“}v_0 \text{ is an algebra”})$$
$$\to \forall v_0(\varphi^{\mathbf{HSP}}(v_0) \leftrightarrow (\forall \text{ equation } e)[\forall \mathfrak{A}(\varphi(\mathfrak{A}) \to \text{“}\mathfrak{A} \vDash e\text{”}) \to \text{“}v_0 \vDash e\text{”}]),$$

where $\varphi^{\mathbf{HSP}}(v_0)$ expresses $v_0 \in \mathbf{HSP}\{v_1 \colon \varphi(v_1)\}$.

PROOF OF THEOREM 1.

LEMMA (Fraenkel-Mostowski). *Let $\mathfrak{V} = \langle V, \in\rangle$ be a class model of $ZF^- + AC$. Let $F \colon V \to V$ be a permutation (i.e., bijection) of V definable in $\mathfrak{V}$, and let*

$$\forall x \forall y (x \in^F y \overset{\mathrm{def}}{\Leftrightarrow} F(x) \in y).$$

Then $\mathfrak{V}^F \overset{\mathrm{def}}{=} \langle V, \in^F\rangle \vDash ZF^- + AC$.

The proof of this lemma can be found e.g., in [4, pp. 48–50].

From now on we use the epsilon symbol $\in$ to denote the element relation $\in^F$ of $\mathfrak{V}^F$. The usual operation symbols, such as $\{\ \}$, $\mathscr{P}$ (power set), etc., also denote the corresponding operations of $\mathfrak{V}^F$. We apply the Fraenkel-Mostowski lemma to the following permutation F. For each countable successor ordinal α, let F interchange α and $\{\alpha+1\}$, and let $F(x) \stackrel{\text{def}}{=} x$ for all other elements of V. Now there is a set $A = \{a_{i,j}: i, j \in \omega\}$ in $\mathfrak{V}^F$ with the following properties:

$$\begin{aligned} a_{i,j} &\neq a_{n,m} \quad \text{for all } \langle i,j\rangle \neq \langle n,m\rangle,\ i,j,n,m \in \omega;\\ a_{i,j} &= \{a_{i,j+1}\} \quad \text{for all } i,j \in \omega. \end{aligned}$$

Since $\mathfrak{V}^F$ is a ZF^--model, we can define the cumulative hierarchy "built on" the set A in the usual way. Let On be the class of ordinals in $\mathfrak{V}^F$, and let

$$W_\emptyset \stackrel{\text{def}}{=} A, \qquad W_{\alpha+1} \stackrel{\text{def}}{=} \mathscr{P}(W_\alpha),$$

$$W_\beta \stackrel{\text{def}}{=} \bigcup_{\alpha<\beta} W_\alpha \quad \text{for } \beta \in On, \beta \text{ limit},$$

and let

$$W \stackrel{\text{def}}{=} \bigcup_{\alpha \in On} W_\alpha,$$

$$\mathfrak{W} \stackrel{\text{def}}{=} \langle W, \in\rangle, \quad \text{i.e., let } \mathfrak{W} \text{ be the submodel of } \mathfrak{V}^F \text{ with universe } W.$$

We note that the *rank* function is also definable in $\mathfrak{W}$ as follows. For every $x \in W$ we let

$$\mathrm{rk}(x) \stackrel{\text{def}}{=} \min\{\alpha \in On: x \in W_\alpha\}.$$

Claim 1. $\mathfrak{W} \vDash ZF^- + AC$.

PROOF OF CLAIM 1. The claim will be proved in a way parallel to the well-known consistency proof of the Axiom of Foundation (see, e.g., [10, pp. 83–85]).

Since A is transitive, each W_α and W itself are transitive; hence, Extensionality holds in $\mathfrak{W}$.

For Pairing: if $x \in W_\alpha$, $y \in W_\beta$ for some $\alpha \leq \beta$, then $\{x,y\} \in W_{\beta+1}$, and it is the "real" pair of x and y, because "$z = \{x,y\}$" is a restricted formula (i.e., a formula containing restricted quantifiers only) and W is a transitive class.

For Union: if $x \in W_\alpha$ for some α, then $\bigcup x \in W_{\alpha+1}$ and "$y = \bigcup x$" is a restricted formula.

For Power set: if $x \in W_\alpha$ for some α, then $\mathscr{P}(x) \in W_{\alpha+2}$. However, "$y = \mathscr{P}(x)$" is not a restricted formula, but we can argue as follows. Let φ be the restricted formula $[u \in y \leftrightarrow (\forall z \in u) z \in x]$. Then for every $u \in W$ $\mathfrak{W} \vDash \varphi$, since $\mathfrak{V} \vDash \varphi$ for all u.

For Infinity: $\omega \in W_{\omega+1}$, and ω can be defined by a restricted formula.

For Replacement: let f be a partial function defined by the formula φ with parameter $p \in W$; that is, let $f \stackrel{\text{def}}{=} \{\langle x,y\rangle \in W: \mathfrak{W} \vDash \varphi(x,y,p)\}$. For an $X \in W$ let $Y \stackrel{\text{def}}{=} f''X$. Then Y is a set and $Y \subseteq W$, hence $\bigcup\{\mathrm{rk}(y): y \in Y\}$ is an ordinal, say β, and $Y \in W_{\beta+1}$. Since for every $y \in W$, $[y \in Y \leftrightarrow (\exists x \in X)\varphi(x, y, p)]$ is a restricted formula, we are done.

For the Axiom of Choice: Let $\emptyset \neq R \in W$, and let f be a choice function on R; that is, for every $\emptyset \neq X \in R$ $f(X) \in X$. Then one can check that if $R \in W_\alpha$ for some α, then $f \in W_{\alpha+3}$. Since "f is a choice function on R" is a restricted formula, the proof is completed. □

Claim 2. $\mathfrak{W} \models AF_\omega$.

Proof of Claim 2. We prove by induction that for every $\alpha \in On$

$$(\sharp) \qquad (\forall X \subseteq W_\alpha)[0 < |X| < \omega \rightarrow (\exists y \in X) y \cap X = \emptyset].$$

For $\alpha = \emptyset$ ($\sharp$) obviously holds.

Assume that ($\sharp$) holds for α, and let $X \subseteq W_{\alpha+1}$ be a finite nonempty set. Assume that X is not well founded. Then there is cycle in X, i.e., there are $x_0, \dots, x_n \in X$ with $x_0 \in \cdots \in x_n \in x_0$. Then $x_0, \dots, x_n \in W_{\alpha+1}$, thus $x_0, \dots, x_n \subseteq W_\alpha$. Namely, by $x_0 \subseteq W_\alpha$, $x_n \in W_\alpha$. Hence,by the transitivity of W_α, $x_0, \dots, x_n \in W_\alpha$. But then there is a non-well-founded set $\{x_0, \dots, x_n\} \subseteq W_\alpha$, which contradicts our assumption.

Let β be a limit ordinal, and assume that for every $\alpha < \beta$ ($\sharp$) holds. Let $X \subseteq W_\beta$ be a finite nonvoid set. Let $\gamma \stackrel{\text{def}}{=} \bigcup\{\text{rk}(x) : x \in X\}$. Then $X \subseteq W_\gamma$ and since $\gamma < \beta$, ($\sharp$) holds for X by the induction hypothesis. □

We recall that $W_\emptyset = A = \{a_{i,j} : i, j \in \omega\}$. We let $A_i \stackrel{\text{def}}{=} \{a_{i,j} : j \in \omega\}$ for each $i \in \omega$.

Claim 3. *Let $X \in W$ be a descending chain of singletons; that is, let $X \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$ with $x_i = \{x_{i+1}\}$ $(i \in \omega)$. Then there exist $n, m \in \omega$ such that $\{x_i : i \in \omega \setminus n\} \subseteq A_m$; that is,*

$$|X \setminus A_m| < \omega.$$

Proof of Claim 3. We prove by induction on α that the statement holds for every $X \subseteq W_\alpha$.

For $X \subseteq W_\emptyset$ the statement obviously holds.

Assume that the statement holds for every $X \subseteq W_\alpha$ for some α. Let $Y \subseteq W_{\alpha+1}$, and assume $Y = \{y_i : i \in \omega\}$ with $y_i = \{y_{i+1}\}$ $(i \in \omega)$. Then $\{y_1\} = y_0 \in W_{\alpha+1}$, hence $y_1 \in W_\alpha$. Then by the transitivity of W_α, $y_i \in W_\alpha$ for every $i > 0$; that is, $Y \setminus \{y_0\} \subseteq W_\alpha$. Then by our assumption there is an $m \in \omega$ such that $|(Y \setminus \{y_0\}) \setminus A_m| < \omega$. Hence, $Y \setminus A_m$ is finite too, which was required.

Let β be a limit ordinal. Assume that for every $\alpha < \beta$ and $X \subseteq W_\alpha$ the statement holds. Assume $Y = \{y_i : i \in \omega\}$ with $y_i = \{y_{i+1}\}$ $(i \in \omega)$. Then $y_0 \in W_\alpha$ for some $\alpha < \beta$ and, by the transitivity of W_α, $Y \subseteq W_\alpha$. Thus, the statement holds by the induction hypothesis. □

We let $\mathbf{TC}(X)$ denote the *transitive closure* of an $X \in W$. $\mathbf{TC}(X)$ exists in every model satisfying the Union, Infinity, and Replacement axioms; hence, it exists in our $\mathfrak{W}$ too for all $X \in W$.

Now we will define a new model structure. Let

$$U \stackrel{\text{def}}{=} \{X \in W : (\exists n \in \omega)(\forall m \in \omega)(\mathbf{TC}(X) \cap A_m \neq \emptyset \rightarrow m < n)\}.$$

That is, $X \in U$ iff the transitive closure of X intersects only finitely many of the sets A_m. This U is definable in $\mathfrak{W}$. Namely, the following formula defines U (by Claim 3):

$$(\exists \text{ function } f)[\operatorname{Dom} f \in \omega \\ \wedge (\forall g \in {}^{\omega}\mathbf{TC}(x))[(\forall n \in \omega) g_n = \{g_{n+1}\} \to \operatorname{Rng} g \cap \operatorname{Rng} f \neq \emptyset]].$$

Let $\mathfrak{U} \stackrel{\text{def}}{=} \langle U, \in \rangle \subseteq \mathfrak{W}$ be the submodel of $\mathfrak{W}$ with universe U.

Claim 4. $\mathfrak{U} \models ZF^- + AC + AF_\omega$.

Proof of Claim 4. Since U is a transitive class by definition, Extensionality holds.

Since $\mathbf{TC}(\{x, y\}) = \mathbf{TC}(x) \cup \mathbf{TC}(y)$, $\mathbf{TC}(\bigcup x) \subseteq \mathbf{TC}(x)$ and $\mathbf{TC}(\mathscr{P}(x)) = \mathbf{TC}(x)$ (and the corresponding formulas are restricted), Pairing, Union, and Power set hold in $\mathfrak{U}$, respectively.

$\mathbf{TC}(\omega) \cap A = \emptyset$, hence $\omega \in U$ and, since ω can be defined by a restricted formula, thus Infinity holds.

Now consider Replacement. Let f_p be a partial function defined by the formula φ with parameter $p \in U$; that is, let $f_p \stackrel{\text{def}}{=} \{\langle x, y \rangle \in U : \mathfrak{U} \models \varphi(x, y, p)\}$. For an $X \in U$ let $Y \stackrel{\text{def}}{=} f_p''X$. Then $Y \in W$ and $Y \subseteq U$. We want to prove that $Y \in U$. By the definition of U there is an $n \in \omega$ such that $(\forall m \in \omega)$ $(\mathbf{TC}(X \cup p) \cap A_m \neq \emptyset \to m < n)$. We will show that for this n

$$(+) \qquad (\forall m \in \omega)(\mathbf{TC}(Y) \cap A_m \neq \emptyset \to m < n).$$

Let $z \in X$, and assume that there is a $k \geq n$ with $\mathbf{TC}(f_p(z)) \cap A_k \neq \emptyset$. Since $f_p(z) \in U$, there is an $l > k$ such that $\mathbf{TC}(f_p(z)) \cap A_l = \emptyset$. Let $g : U \to U$ be an automorphism interchanging A_l and A_k and leaving all the other elements of U fixed. That is, for every $j \in \omega$ let $g(a_{l,j}) \stackrel{\text{def}}{=} a_{k,j}$, $g(a_{k,j}) \stackrel{\text{def}}{=} a_{l,j}$, and for every $u \notin A_k \cup A_l$ $g(u) \stackrel{\text{def}}{=} u$. Now since $z \in X$, $g(z) = z$ and $g(p) = p$. Since g is an isomorphism, $g(f_p(z)) = f_{g(p)}(g(z)) = f_p(g(z)) = f_p(z)$. Therefore,

$$\emptyset = g''\emptyset = g''[\mathbf{TC}(f_p(z)) \cap A_l] = \mathbf{TC}(f_p(z)) \cap A_k \neq \emptyset,$$

that is a contradiction, proving that (+) holds. Hence, $Y \in U$ as desired.

For AC: Let $R \in U$, and let f be a choice function on R. Then $f \subseteq R \times \bigcup R$; thus, $f \in U$. Since "f is a choice function on R" is a restricted formula, AC holds in $\mathfrak{U}$.

For AF_ω: Since U is transitive, the $\in$-least element of a set belonging to U also belongs to U. □

Claim 5. $\mathfrak{U} \not\models \operatorname{Mod} \mathbf{EqK} = \mathbf{HSP}\,\mathbf{K}$.

Proof of Claim 5. We let

$$\begin{aligned} \mathbf{K} \stackrel{\text{def}}{=} \{\langle B, f \rangle\ & |B| = n \text{ for some } n \in \omega; \\ & (\forall b \in B)(b \text{ is a descending chain of singletons}); \\ & (\forall b \neq c \in B)\mathbf{TC}(b) \cap \mathbf{TC}(c) = \emptyset; \\ & \text{there is an enumeration } b_0, \dots, b_{n-1} \text{ of the elements of } B \\ & \qquad \text{such that } (\forall 0 < m < n) f(b_m) = b_{m-1} \text{ and } f(b_0) = b_0\}. \end{aligned}$$

Then $\mathbf{K}$ is a class in $\mathfrak{U}$, i.e., one can give a formula χ such that $\mathfrak{B} \in \mathbf{K}$ iff $\mathfrak{U} \vDash \chi(\mathfrak{B})$. Let ϱ be the ω-ary formula

$$\bigvee_{n\in\omega} \forall x(f^{n+1}(x) = f^n(x)).$$

Then by the definition of $\mathbf{K}$, $\mathbf{K} \vDash \varrho$.

We claim that $\mathbf{PK} \vDash \varrho$. To see this let $\mathfrak{A} \in {}^I\mathbf{K}$ for some set I. Since $\mathfrak{A} \in U$, $\mathbf{TC}(\mathfrak{A})$ intersects only finitely many of the sets A_m; hence, only finitely many elements of $\mathbf{K}$ can occur in $\operatorname{Rng}\mathfrak{A}$. Let $N \in \omega$ be the maximum of the cardinalities of the members of $\operatorname{Rng}\mathfrak{A}$. Thus, $P_{i\in I}\mathfrak{U}_i \vDash \forall x(f^{N+1}(x) = f^N(x))$, which was required.

Since ϱ is preserved under $\mathbf{HS}$, we have $\mathbf{HSP\ K} \vDash \varrho$.

But $\{\langle n, \bigcup\rangle : n \in \omega\} \subseteq \mathbf{IK}$ (where $\bigcup m = m - 1$ if $m > 0$ and $\bigcup 0 = 0$). Therefore, $\mathbf{PI\ K} \nvDash \varrho$. We proved that for our $\mathbf{K}$, or $\mathfrak{U} \vDash \mathbf{PI\ K} \nsubseteq \mathbf{HSP\ K}$, which proves $\mathfrak{U} \nvDash \operatorname{Mod} \mathbf{EqK} = \mathbf{HSP\ K}$. □

Now the proof of Theorem 1 is completed. By Claims 4 and 5, the model $\mathfrak{U}$ proves that $ZF^- + AC + AF_\omega \nvDash \operatorname{Mod}\mathbf{EqK} = \mathbf{HSP\ K}$. □

In fact, Theorem 1 is a consequence of the following result. We show that on the basis of ZF^-, Birkhoff's theorem is equivalent to a purely set-theoretical statement. This statement, the so-called Collection Principle, is implied by (but not equivalent to) the Axiom of Foundation.

The *Collection Principle* (CP) is the schema

$$(CP) \quad \forall \text{ class } R\,(\operatorname{Dom} R \text{ is a set } \rightarrow \exists r \subseteq R\,(r \text{ is a set } \wedge \operatorname{Dom} r = \operatorname{Dom} R))$$

$(CP$ can be defined formally as in [10, pp. 72–73]).

Theorem 2. *On the basis of* ZF^-, "Mod $\mathbf{EqK} = \mathbf{HSP\ K}$" *is equivalent to CP.*

The two directions of Theorem 2 are proved as Theorem 3 and Corollary 1 below. For a careful formalization of these statements (and the other theorems below) in the language of ZF-set theory cf. the remarks following Theorem 1 and Theorem 4.

Theorem 3. $ZF^- + CP \vdash \operatorname{Mod}\ \mathbf{EqK} = \mathbf{HSP\ K}$.

Proof. We only prove the nontrivial direction, that is, Mod $\mathbf{EqK} \subseteq \mathbf{HSP\ K}$.

Suppose that $\mathfrak{A} \in \operatorname{Mod}\mathbf{EqK}$. We show in two steps that $\mathfrak{A} \in \mathbf{HSP\ K}$, and only the second step involves CP.

Let $\mathfrak{F}_A$ be the word-algebra (absolutely free algebra) generated by the set A, and let $f : \mathfrak{F}_A \to \mathfrak{A}$ be the surjective homomorphism induced by the inclusion map of the generator set A. Let for all $\sigma, \tau \in F_A$

$$\sigma \equiv \tau \overset{\text{def}}{\Leftrightarrow} g\sigma = g\tau \text{ for every homomorphism } g \text{ mapping } \mathfrak{F}_A \text{ into some } \mathfrak{B} \in \mathbf{K}.$$

Then there is a homomorphism f' mapping $\mathfrak{F}_A/_{\equiv}$ onto $\mathfrak{A}$ such that $f'[\sigma] = f\sigma$ for every $\sigma \in F_A$.

Now it remains to prove that $\mathfrak{F}_A/_{\equiv} \in \mathbf{SP\ K}$. To prove this, let

$$I \overset{\text{def}}{=} \{\langle \sigma, \tau\rangle : \sigma, \tau \in F_A, \sigma \not\equiv \tau\},$$

$$R \overset{\text{def}}{=} \{\langle\langle \sigma, \tau\rangle, \langle \mathfrak{B}, h\rangle\rangle : \langle \sigma, \tau\rangle \in I, \mathfrak{B} \in \mathbf{K}, h \text{ is a homomorphism mapping } \mathfrak{F}_A \text{ into } \mathfrak{B}, \text{ and } h\sigma \neq h\tau\}.$$

Then $\operatorname{Dom} R = I$ and I is a set. By CP there is a subset r of R with $\operatorname{Dom} r = I$. Let $\mathfrak{C} \stackrel{\text{def}}{=} P_{\langle\langle\sigma,\tau\rangle,\langle\mathfrak{B},h\rangle\rangle\in r}\mathfrak{B}$, and let $c_{\langle\langle\sigma,\tau\rangle,\langle\mathfrak{B},h\rangle\rangle} \stackrel{\text{def}}{=} h\sigma$ for each $\langle\langle\sigma,\tau\rangle,\langle\mathfrak{B},h\rangle\rangle \in r$. Then $c \in C$, i.e., C is nonempty; thus $\mathfrak{C} \in \mathbf{PK}$. Define the function

$$g : F_A/_{\equiv} \to C \text{ by } (g[\varrho])_{\langle\langle\sigma,\tau\rangle,\langle\mathfrak{B},h\rangle\rangle} \stackrel{\text{def}}{=} h\varrho.$$

Then one can easily check that g is an injective homomorphism from $\mathfrak{F}_A/_{\equiv}$ into $\mathfrak{C}$, so $\mathfrak{F}_A/_{\equiv} \in \mathbf{SP}\ \mathbf{K}$ as desired. □

The proof of the following theorem originates from J. D. Monk.

THEOREM 4. *Let* $\mathbf{Q}$ *be an operator on classes of algebras such that for any class* $\mathbf{K}$ *of similar algebras the following two properties hold. For every* $\mathfrak{A}$, $\mathfrak{A} \in \mathbf{QK}$ *implies that* $\mathfrak{A} \in \mathbf{QK}_0$ *for some subset* $\mathbf{K}_0$ *of* $\mathbf{K}$, *and* $\mathbf{K} \models e$ *implies* $\mathbf{QK} \models e$ *for any equation* e. *Then*

(1) $ZF^- + \mathbf{PI} \leq \mathbf{Q} \vdash CP$;

(2) $ZF^- + \mathbf{PS} \leq \mathbf{Q} \vdash CP$.

REMARK. The formal counterpart of Theorem 4 is as follows. Suppose that with every formula $\varphi(v_0)$ of the language of set theory we associate another set-theoretical formula $\varphi^{\mathbf{Q}}(v_0)$ such that the next three conditions hold:

(a)
$$\begin{aligned} ZF^- \models &\forall v_0(\varphi(v_0) \to \text{“}v_0 \text{ is an algebra”}) \\ &\to \forall v_0(\varphi^{\mathbf{Q}}(v_0) \to \text{“}v_0 \text{ is an algebra”}); \end{aligned}$$

(b)
$$\begin{aligned} ZF^- \models &\forall v_0(\varphi(v_0) \to \text{“}v_0 \text{ is an algebra”}) \wedge \varphi^{\mathbf{Q}}(\mathfrak{A}) \\ &\to \exists v_1[(\forall v_2 \in v_1)(\varphi(v_2) \wedge (v_0 \in v_1)^{Q}(v_0/\mathfrak{A}))]; \end{aligned}$$

(c)
$$\begin{aligned} ZF^- \models &\forall v_0(\varphi(v_0) \to \text{“}v_0 \text{ is an algebra”}) \\ &\to (\forall \text{ equation } e)[\forall\mathfrak{A}(\varphi(\mathfrak{A}) \to \text{“}\mathfrak{A} \models e\text{”}) \to \forall\mathfrak{A}(\varphi^{\mathbf{Q}}(\mathfrak{A}) \to \text{“}\mathfrak{A} \models e\text{”})]. \end{aligned}$$

Now let $\mathbf{ALG}$ be the set of all formulas φ with $ZF^- \models \forall v_0(\varphi(v_0) \to$ “v_0 is an algebra”). Then we claim e.g. (1):

$$ZF^- + \{\forall v_0(\varphi^{\mathbf{PI}}(v_0) \to \varphi^{\mathbf{Q}}(v_0)) : \varphi \in \mathbf{ALG}\} \vdash CP.$$

where $\varphi^{\mathbf{PI}}(v_0)$ expresses $v_0 \in \mathbf{PI}\ \{v_1 : \varphi(v_1)\}$.

PROOF OF (1) OF THEOREM 4. Let R be a class such that $d \stackrel{\text{def}}{=} \operatorname{Dom} R$ is a set. Since in ZF^- there is no set of all sets, there is a set $z \notin d$. Let t be the similarity type $\{\{c_z\} \cup \{c_x : x \in d\}\} \times \{0\}$ (i.e., all symbols are constants). For each $x \in d$ we define an algebra $\mathfrak{A}_x$ of type t: the universe A_x of $\mathfrak{A}_x$ is 3, and all constants denote 0 except for c_x which denotes 1. For each $\langle x, y\rangle \in R$ we define a t-type algebra $\mathfrak{B}_{xy}$: the universe B_{xy} of $\mathfrak{B}_{xy}$ is $2 \cup \{\langle x, y\rangle\}$, and all constants denote 0 except for c_x which denotes 1. Let $\mathbf{K} \stackrel{\text{def}}{=} \{\mathfrak{B}_{xy} : \langle x, y\rangle \in R\}$. Note that $\langle 0 : x \in d\rangle \in P_{x\in d}A_x$; hence, $\mathfrak{A} \stackrel{\text{def}}{=} P_{x\in d}\mathfrak{A}_x$ is a t-type algebra with nonempty universe. Clearly, $\mathfrak{A}_x \cong \mathfrak{B}_{xy}$

whenever $\langle x.y\rangle \in R$, so $\mathfrak{A} \in \mathbf{PI\ K}$. Hence, by assumption $\mathfrak{A} \in \mathbf{QK}$. Let $\mathbf{K}_0$ be a subset of $\mathbf{K}$ such that $\mathfrak{A} \in \mathbf{QK}_0$ holds. Now we claim that

$$(*) \qquad (\forall x \in d)\exists y\ \mathfrak{B}_{xy} \in \mathbf{K}_0.$$

To see this, let $x \in d$. Since $\mathfrak{A}_x \nvDash c_x = c_z$, we have $\mathfrak{A} \nvDash c_x = c_z$. Hence, $\mathbf{QK}_0 \nvDash c_x = c_z$, and therefore $\mathbf{K}_0 \nvDash c_x = c_z$. But for every $\langle u.v\rangle \in R$ if $u \neq x$, then $\mathfrak{B}_{uv} \vDash c_x = c_z$. Hence, there is some y with $\mathfrak{B}_{xy} \in \mathbf{K}_0$, as desired in $(*)$.

Now let $r \stackrel{\text{def}}{=} \bigcup\{B_{xy} : \mathfrak{B}_{xy} \in \mathbf{K}_0\}\backslash 2$. Thus, r is a subset of the class R. Since r contains no ordinals, by $(*)$ $\operatorname{Dom} R \subseteq \operatorname{Dom} r$, which completes the proof. □

Proof of (2) of Theorem 4. We have to change the proof of (1) only by letting $\mathfrak{A}_x$ be the subalgebra of $\mathfrak{B}_{xy}$ with universe 2, for each $x \in d$. □

Corollary 1. $ZF^- + (\operatorname{Mod} \mathbf{EqK} = \mathbf{HSP\ K}) \vdash CP$.

Proof. The operator **HSP** has the two properties required from **Q**, and for every class **K** of similar algebras, e.g., $\mathbf{PS\ K} \subseteq \operatorname{Mod} \mathbf{EqK} \subseteq \mathbf{HSP\ K}$ holds. □

Corollary 2.

(1) *CP is independent of ZF^-;*

(2) $ZF^- + AF_\omega + CP \nvdash AF$.

Proof of (1). First, $ZF^- \nvdash CP$ by Theorems 1 and 3. Second, $ZF^- \nvdash \neg CP$, since CP is implied by the Axiom of Foundation (see [10, pp. 73–74]). □

Proof of (2). Recall the set A of descending chains of singletons and the permutation model $\mathfrak{V}^F$ from the proof of Theorem 1. Foundation obviously fails in $\mathfrak{V}^F$, since e.g., the set A is not well-founded. But $\mathfrak{V}^F \vDash CP$ can easily be checked in a way similar to the proof of $\mathfrak{V}^F \vDash$ Replacement (see, e.g., [4, p. 49]). □

§3. Further connections between the axioms of set theory and the behaviour of operators on classes of algebras.

In this section we give several other statements concerning operators on classes of algebras which are equivalent to the Collection Principle above. There are some further statements which are equivalent to $AC + CP$. Hence, none of them are derivable from $ZF^- + AC + AF_\omega$.

Theorem 5. *Each of the following statements holds in ZF^-:*

(1) **I**, **S**, **H**, *and* **HS** *are closure operators.*

(2) $\mathbf{IS} = \mathbf{SI}$, $\mathbf{IP}^r = \mathbf{P}^r$, $\mathbf{IP} = \mathbf{P}$, $\mathbf{IH} = \mathbf{HI} = \mathbf{H}$.

Proof. The proofs are straightforward. $\mathbf{IS} = \mathbf{SI}$ and "**HS** is a closure operator" are proved as 0.2.15 of [9, Part I, p. 72], and there it is emphasized that AC is not used in the proof. It is easy to check that AF is not used either. □

Theorem 6. *In ZF^- each of the following statements is equivalent to CP:*

(1) **HSP** *is a closure operator.*

(2) $\mathbf{PP} \leq \mathbf{SP}$.

(3) $\mathbf{PI} \leq \mathbf{SP}$.

(4) $\mathbf{P}^r\mathbf{I} \leq \mathbf{HSP}^r$.

(5) $\mathbf{P}^r\mathbf{S} \leq \mathbf{SP}^r$.

(6) $\mathbf{PS} \leq \mathbf{SP}$.

(7) $\mathbf{P}^r\mathbf{S} \leq \mathbf{HSP}^r$.

(8) **SP** *is a closure operator.*

PROOF. Each of the statements (1)–(4) implies CP on the basis of ZF^-, because of (1) of Theorem 4 with $\mathbf{Q} = \mathbf{HSP}^r$. To the other statements, apply (2) of Theorem 4 with $\mathbf{Q} = \mathbf{HSP}^r$ too.

To the reverse direction:

(1) follows from Theorem 3: $\mathbf{HSPHSP\ K} = \mathrm{Mod}\,\mathbf{EqHSP\ K} \subseteq \mathrm{Mod}\,\mathbf{EqK} = \mathbf{HSP\ K}$.

To prove (2), let $\mathfrak{A} \in \mathbf{PP\ K}$. Say $\mathfrak{A}$ is isomorphic to $P_{i\in I}\mathfrak{B}_i$, where $\mathfrak{B}_i \in \mathbf{PK}$ for each $i \in I$. Hence, the relation

$$R \overset{\text{def}}{=} \{\langle i, \langle \mathfrak{C}, f, k\rangle\rangle : i \in I, \mathfrak{C} = \langle \mathfrak{C}_j : j \in J\rangle \in^J \mathbf{K} \text{ for some set } J,$$
$$f \text{ is an isomorphism of } \mathfrak{B}_i \text{ into } P_{j\in J}\mathfrak{C}_j, k \in J\}$$

has domain I. By CP there is a subset r of R with $\mathrm{Dom}\, r = I$. Let

$$\mathfrak{D} \overset{\text{def}}{=} P_{\langle i,\langle \mathfrak{C}, f, k\rangle\rangle \in r}\mathfrak{C}_k;$$

thus, $\mathfrak{D} \in \mathbf{PK}$. Now define a function $g : P_{i\in I}B_i \to P_{\langle i,\langle \mathfrak{C}, f, k\rangle\rangle\in r} C_k$ with $(gb)_{\langle i,\langle \mathfrak{C}, f, k\rangle\rangle} \overset{\text{def}}{=} (fb_i)_k$ for some $b \in P_{i\in I}B_i$. Then it is easy to check that g isomorphically embeds $P_{i\in I}\mathfrak{B}_i$ into $\mathfrak{D}$, i.e., $\mathfrak{A} \in \mathbf{IISP\ K}$. Hence, by Theorem 5 or, $\mathfrak{A} \in \mathbf{SP\ K}$, as desired.

(3) and (4) follows from (2): $\mathbf{P}^r\mathbf{I} \leq \mathbf{PI} \leq \mathbf{PP} \leq \mathbf{SP} \leq \mathbf{HSP}^r$.

To prove (5), let $\mathfrak{A} \in \mathbf{P}^r\mathbf{S\ K}$. Say $\mathfrak{A}$ is isomorphic to $P_{i\in I}\mathfrak{B}_{i/F}$, where $\mathfrak{B}_i \in \mathbf{SK}$ for each $i \in I$ and F is a filter over I. Hence, the relation

$$R \overset{\text{def}}{=} \{\langle i, \mathfrak{C}\rangle : i \in I, \mathfrak{C} \in \mathbf{K}, \mathfrak{B}_i \subseteq \mathfrak{C}\}$$

has domain I. By CP there is a subset r of R with $\mathrm{Dom}\, r = I$. Let

$$E \overset{\text{def}}{=} \{y \subseteq r : \exists x \in F \text{ with } \{\langle i, \mathfrak{C}\rangle \in r : i \in x\} \subseteq y\}.$$

Then $E \subseteq \mathscr{P}(r)$ is a filter over r because F is a filter over I. Now define a function $g : P_{i\in I}B_i \to P_{\langle i,\mathfrak{C}\rangle\in r}C$ with $(gb)_{\langle i,\mathfrak{C}\rangle} \overset{\text{def}}{=} b_i$ for each $b \in P_{i\in I}B_i$, and let $f[b]_F \overset{\text{def}}{=} [gb]_E$. Then one can check that f isomorphically embeds $P_{i\in I}\mathfrak{B}_{i/F}$ into $P_{\langle i,\mathfrak{C}\rangle\in r}\mathfrak{C}/_E$; that is, $\mathfrak{A} \in \mathbf{IISP}^r\,\mathbf{K}$. hence, by Theorem 5 or, $\mathfrak{A} \in \mathbf{SP}^r\,\mathbf{K}$, as desired.

To prove (6), repeat the proof of (5) (using its notation) with filter $F \overset{\text{def}}{=} \{I\}$. Then $E = \{r\}$ by its definition. Hence, $\mathfrak{A} \cong P_{i\in I}\mathfrak{B}_i \cong P_{i\in I}\mathfrak{B}_i/_{\{I\}}$ which can be isomorphically embedded into $P_{\langle i,\mathfrak{C}\rangle\in r}\mathfrak{C}/_{\{r\}} \cong P_{\langle i,\mathfrak{C}\rangle\in r}\mathfrak{C}$. By Theorem 5 the proof is complete. We note that [2] contains a proof of "$ZF \vdash$ (5) and (6)".

(7) follows from (6): $\mathbf{P}^r\mathbf{S} \leq \mathbf{PS} \leq \mathbf{SP} \leq \mathbf{HSP}^r$.

Finally, (8) follows from (2), (6), and Theorem 5: $\mathbf{SPSP} \leq \mathbf{SSPP} \leq \mathbf{SSSP} = \mathbf{SP}$. □

THEOREM 7. *In ZF^- each of the following statements is equivalent to $AC + CP$:*

(1) $\mathbf{P}$ *is a closure operator.*

(2) $\mathbf{HP}$ *is a closure operator.*

(3) $\mathbf{PI} \leq \mathbf{P}$.

(4) $\mathbf{P}^r\mathbf{I} \leq \mathbf{HP}^r$.

(5) $\mathbf{PP} \le \mathbf{HP}^r$.

PROOF. That each statement is a consequence of $ZF^- + AC + CP$ can be seen in a way similar to the proof of, e.g., (2) of Theorem 6. The only difference is that here it is not enough to "reduce" a set of classes to a set of sets (with the help of CP) but we must choose exactly one element from each class, which is possible with the help of CP and AC together only.

For the other direction: each statement implies CP on the basis of ZF^-, by (1) of Theorem 4 with $\mathbf{Q} = \mathbf{HP}^r$. To prove that each of the statements implies AC first we show that each of them implies $\mathbf{P'I} \le \mathbf{HP}^r$.

For (1): $\mathbf{P'I} \le \mathbf{IP'IP'} = \mathbf{PP} = \mathbf{P} \le \mathbf{HP}^r$.

For (2): $\mathbf{P'I} \le \mathbf{HIP'HIP'} = \mathbf{HPHP} \le \mathbf{HP}^r$ (by Theorem 5).

For (3): $\mathbf{P'I} \le \mathbf{IP'I} = \mathbf{PI} \le \mathbf{P} \le \mathbf{HP}^r$.

For (5): $\mathbf{P'I} \le \mathbf{IP'IP'} = \mathbf{PP} \le \mathbf{HP}^r$.

Now it remains to prove that

$$ZF^- + \mathbf{P'I} \le \mathbf{HP}^r \vdash AC.$$

Let $X \stackrel{\text{def}}{=} \{X_i : i \in I\}$ be a set of nonempty sets. We want to give a choice function for X. Let $t \stackrel{\text{def}}{=} \{\langle g, 1\rangle\}$ be a similarity type; i.e., let t consist of one unary function symbol. Define an algebra $\mathfrak{C}$ of type t as follows. Let $C \stackrel{\text{def}}{=} \{\langle x, i\rangle : i \in I, x \in X_i\} \cup \{0, 1, 2\}$, and let

$$g^{\mathfrak{C}} c \stackrel{\text{def}}{=} \begin{cases} c & \text{if } c = 0 \text{ or } c = \langle x, i\rangle \text{ for some } i \in I, x \in X_i, \\ 1 & \text{if } c = 2, \\ 2 & \text{if } c = 1. \end{cases}$$

Since no ordered pair is a member $\{0, 1; 2\}$, $g^{\mathfrak{C}}$ is well defined. Now let

$$\mathbf{K} \stackrel{\text{def}}{=} \{\mathfrak{B} : \mathfrak{B} \subseteq \mathfrak{C} \text{ and } \{0, 1, 2\} \subseteq B\};$$

thus, $\mathbf{K}$ is a class of t-type algebras. For every $i \in I$ define an algebra $\mathfrak{A}_i$ with

$$A_i \stackrel{\text{def}}{=} \{\langle x, i\rangle : x \in X_i\} \cup \{1, 2\} \quad \text{and} \quad g^{\mathfrak{A}_i} \stackrel{\text{def}}{=} g^{\mathfrak{C}} \restriction_{A_i}.$$

We will show that $\mathfrak{A}_i \in \mathbf{IK}$. Fix some $z \in X_i$, and let $C_i \stackrel{\text{def}}{=} (A_i \setminus \{\langle z, i\rangle\}) \cup \{0\}$ and $g^{\mathfrak{C}_i} \stackrel{\text{def}}{=} g^{\mathfrak{C}} \restriction_{C_i}$; thus, $\mathfrak{C}_i \in \mathbf{K}$. Define the isomorphism $k_i : \mathfrak{A}_i \to \mathfrak{C}_i$ needed by

$$k_i a \stackrel{\text{def}}{=} \begin{cases} a & \text{if } a \neq \langle z, i\rangle, \\ 0 & \text{if } a = \langle z, i\rangle. \end{cases}$$

$P_{i \in I} A_i \neq \emptyset$, since $\langle 1 : i \in I\rangle \in P_{i \in I} A_i$. Therefore, $\mathfrak{D} \stackrel{\text{def}}{=} P_{i \in I} \mathfrak{A}_i \in \mathbf{P'I\,K}$; hence,by our assumption $\mathfrak{D} \in \mathbf{HP}^r\,\mathbf{K}$. Then there exist algebras $\mathfrak{B}_j \in \mathbf{K}$ ($j \in J$ for some set J), a filter F over J, and a homomorphism h from $\mathfrak{B} \stackrel{\text{def}}{=} P_{j \in J} \mathfrak{B}_j / F$ onto $\mathfrak{D}$. Let $b \stackrel{\text{def}}{=} [\langle 0 : j \in J\rangle]_F$. Then $b \in B$ and $g^{\mathfrak{B}} b = b$, since for each $\mathfrak{B}_j$ ($j \in J$) $g^{\mathfrak{B}_j} 0 = 0$ and $J \in F$. Hence, $hb \in D$ and $g^{\mathfrak{D}} hb = hg^{\mathfrak{B}} b = hb$, since h is a homomorphism. Recall that $\mathfrak{D} = P_{i \in I} \mathfrak{A}_i$ and $A_i = \{\langle x, i\rangle : x \in X_i\} \cup \{1, 2\}$ for each $i \in I$. Thus, for each $i \in I$ $(hb)_i = \langle x_i, i \rangle$ for some $x_i \in X_i$. Now let the function m be defined with

$$m(X_i) \stackrel{\text{def}}{=} \mathrm{pr}_0(hb)_i \quad \text{for each } X_i \in X,$$

where pr_0 is the usual first projection defined on $(\bigcup_{i \in I} X_i) \times I$. Then m is a choice function for the set X. □

PROBLEMS.

(1) Is $ZF^- \vdash \mathbf{PSP} = \mathbf{SPIS}$ true?
Is $ZF^- \vdash \mathbf{PSPS} = \mathbf{SPIS}$ true?

(2) $(\exists n \in \omega) ZF \vdash \mathbf{P}^n = \mathbf{P}^{n+1}$?
$(\exists n \in \omega) ZF \vdash (\mathbf{HP})^n = (\mathbf{HP})^{n+1}$?

(If $\mathbf{Q}$ is an operator on classes of algebras, then $\mathbf{Q}^1 \stackrel{\text{def}}{=} \mathbf{Q}$ and $\mathbf{Q}^{n+1} \stackrel{\text{def}}{=} \mathbf{QQ}^n$ for every $0 \neq n \in \omega$.) Solving these last two problems would complete the solution of Problem 28 in [7, p. 161]: whether without AC the semigroup generated by the operators **I**, **H**, **S**, **P** is finite.

Connections between algebraic theorems and further axioms of ZF-set theory will be discussed in a future paper. Other similar investigations on "reverse mathematics" are e.g., in [1], [5], [6], [8], and [11].

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MATHEMATICAL INSTITUTE OF THE HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, H-1364, HUNGARY

E-mail: andreka@rmk530.rmki.kfki.hu

E-mail: kurucz@rmk530.rmki.kfki.hu

E-mail: h1469nem@huella.bitnet