

# Algebra Universalis 

# Comparing decision problems for various paradigms of algebraic logic 

Agnes Kurucz


#### Abstract

We show that in many cases the decision problems for varieties of cylindric algebras are much harder than those for the corresponding relation algebra reducts. We also give examples of varieties of cylindric and relation algebras which are algorithmically more complicated than the subvarieties of their representable algebras.


## 1. Introduction

The theories of various classes of Tarskian algebraic logic, such as relation algebras (RA) and $n$-dimensional cylindric algebras $\left(\mathrm{CA}_{n}\right)$, were developed in a parallel fashion, see e.g. Henkin et al. [6], Maddux [8], Monk [9]. There is a standard translation, due to Henkin and Tarski, connecting the two paradigms. As the equational theories of both RA and $\mathrm{CA}_{n}$ (for $n>2$ ) are recursively enumerable but not decidable, and the same holds for the subclasses of representable algebras, one might conjecture that this translation preserves the degree of unsolvability.

Our first result (Theorem 1) shows that this conjecture does not hold. For instance, we prove that there exist many undecidable subvarieties of $\mathrm{CA}_{3}$ such that the corresponding relation algebra varieties are decidable.

Our next aim is to compare the recursion theoretic behaviour of varieties of relation algebras (or cylindric algebras) and the corresponding subvarieties generated by their representables. Again, the most common examples might suggest an analogous behaviour: the varieties of all relation algebras and of all representable relation algebras are both recursively enumerable but not decidable; the same holds for all cylindric algebras and all representable cylindric algebras of dimension greater than 2 ; while in dimension 2 both the representable and axiomatic varieties of

[^0]
cylindric algebras are decidable. Here we show that one can find highly unsolvable abstract varieties such that their representables have less complicated equational theories (Theorems 2 and 3).

The paper is organised as follows. The next section contains the definitions of the various algebras, our results, and related open problems. Sections $3-5$ are devoted to the proofs of Theorems 1-3, respectively.

## 2. Background and results

We assume as known such basic concepts of universal algebra as varieties, homomorphisms, subalgebras, and direct products. Some basic knowledge about Boolean algebras is also required. The reader may find the recursion theoretic notions not defined here (such as degree of unsolvability) e.g. in [3]. However, in order to make the paper more or less self-contained, below we give a short summary of the basic definitions and properties concerning relation algebras and cylindric algebras. For more details, consult Henkin et al. [6] and Maddux [8].

Notation. For a set $U,|U|$ denotes the cardinality of $U$, and $\mathcal{P}(U)$ stands for the set of all subsets of $U$. The usual Boolean operations on subsets of $U$ are denoted by $\cup, \cap$, and $-{ }^{U}$. $\omega$ denotes the set of natural numbers, and for every $n \in \omega$ we assume that $n=\{k \in \omega: k<n\}$. For $n \in \omega,{ }^{n} U$ is the set of all $n$-tuples of elements in $U$. We use notation $u=\left\langle u_{0}, u_{1}, \ldots, u_{n-1}\right\rangle$ for $n$-tuples. Algebras are denoted by gothic letters with the corresponding roman letters denoting their universes. Given algebras $\mathfrak{A}$ and $\mathfrak{B}, \mathfrak{A} \subseteq \mathfrak{B}$ denotes that $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$, that is, $\mathfrak{A}$ is embeddable into $\mathfrak{B}$. Given some class $K$ of algebras, IK, HK, SK, and PK denote, respectively, the classes of all isomorphic copies, homomorphic images, isomorphic copies of subalgebras, and isomorphic copies of direct products of members of $\mathrm{K} . \mathrm{Eq}(\mathrm{K})$ denotes the equational theory of K .

Relation algebras. A relation algebra is an algebra of the form

$$
\mathfrak{A}=\left\langle A,+, \cdot,-, 1,0, ;,,^{\prime}, 1^{\prime}\right\rangle
$$

such that
(R0) $\langle A,+, \cdot,-, 1,0\rangle$ is a Boolean algebra (called the Boolean reduct of $\mathfrak{A}$ ), $1^{\prime} \in A$, ; and ${ }^{`}$ are binary and unary operations on $A$, respectively, satisfying the following properties, for all $x, y, z \in A$ :
(R1) $\quad x ;(y ; z)=(x ; y) ; z$
(R2) $\quad(x+y) ; z=(x ; z)+(y ; z)$
(R3) $\quad x ; 1^{\prime}=x$

(R4) $\quad x^{\leadsto}=x$
(R5) $\quad(x+y)^{\breve{ }}=x^{\breve{ }}+y^{\breve{ }}$
(R6) $\quad(x ; y)^{\breve{ }}=y^{\breve{ }} ; x^{\smile}$
(R7) $\quad x^{\sim} ;-(x ; y) \leq-y$
RA denotes the class of all relation algebras. An element $x$ of a relation algebra $\mathfrak{A}$ is called an atom of $\mathfrak{A}$, if $x$ is an atom of the Boolean reduct of $\mathfrak{A}$, that is, for all $y \in A$, if $y \leq x$ then either $y=0$ or $y=x$. A relation algebra is called complete atomic if its Boolean reduct is complete atomic.

It is well-known that a relation algebra $\mathfrak{A}$ is simple (i.e., it has no non-trivial homomorphic images) iff the quantifier free formula

$$
0 \neq 1 \wedge(x \neq 0 \rightarrow 1 ; x ; 1=1)
$$

is valid in $\mathfrak{A}$. Thus for any variety V of relation algebras, the class $\mathrm{Si}(\mathrm{V})$ of simple algebras in V is a universal class. Further, for every quantifier free formula $\varphi$ in the language of relation algebras, one can find in an effective way an equation $e_{\varphi}$ with the same free variables such that the formula $\varphi \leftrightarrow e_{\varphi}$ is valid in all simple relation algebras. Thus for any universal class K of simple relation algebras, SPK is a variety, and $\mathrm{Si}(\mathbf{S P K})=\mathrm{K}$ holds. And, for any variety V of relation algebras, we have $\mathbf{S P S i}(\mathrm{V})=\mathrm{V}$.

An example for a simple relation algebra is the full relation set algebra with base $U$ :

$$
\mathcal{R} s(U)=\left\langle\mathcal{P}\left({ }^{2} U\right), \cup, \cap,-{ }^{2} U,{ }^{2} U, \emptyset, \mid,{ }^{-1}, I d_{U}\right\rangle
$$

where for all $X, Y \subseteq{ }^{2} U$,

$$
\begin{aligned}
X \mid Y & =\left\{\langle u, v\rangle \in{ }^{2} U:(\exists z \in U)\langle u, z\rangle \in X \text { and }\langle z, v\rangle \in Y\right\} \\
X^{-1} & =\left\{\langle u, v\rangle \in{ }^{2} U:\langle v, u\rangle \in X\right\} \\
I d_{U} & =\left\{\langle u, u\rangle \in{ }^{2} U: u \in U\right\} .
\end{aligned}
$$

The variety RRA of representable relation algebras is

$$
\operatorname{RRA}=\mathbf{S P}\{\mathcal{R} s(U): U \text { is a set }\}
$$

Cylindric algebras. For every $n \in \omega$, a cylindric algebra of dimension $n$ is an algebra of the form

$$
\mathfrak{A}=\left\langle A,+, \cdot,-, 1,0, c_{i}, d_{i j}\right\rangle_{i, j<n}
$$

such that
(C0) $\langle A,+, \cdot,-, 1,0\rangle$ is a Boolean algebra (called the Boolean reduct of $\mathfrak{A}$ ),
$d_{i j} \in A$ for all $i, j<n$, and $c_{i}$ are unary operations on $A$ for all $i<n$ satisfying the following properties, for all $x, y \in A$ :
(C1) $c_{i} 0=0$

(C2) $\quad x \leq c_{i} x$
(C3) $\quad c_{i}\left(x \cdot c_{i} y\right)=c_{i} x \cdot c_{i} y$
(C4) $\quad c_{i} c_{j} x=c_{j} c_{i} x$
(C5) $\quad d_{i i}=1$
(C6) $\quad d_{j k}=c_{i}\left(d_{j i} \cdot d_{i k}\right) \quad($ for all $i \neq j, k)$
(C7) $\quad c_{i}\left(d_{i j} \cdot x\right) \cdot c_{i}\left(d_{i j} \cdot-x\right)=0 \quad($ for all $i \neq j)$
$\mathrm{CA}_{n}$ denotes the class of all cylindric algebras of dimension $n$. An element $x$ of an $n$-dimensional cylindric algebra $\mathfrak{A}$ is called an atom of $\mathfrak{A}$, if $x$ is an atom of the Boolean reduct of $\mathfrak{A}$. An $n$-dimensional cylindric algebra is called atomic if its Boolean reduct is atomic. The following term definable substitution operators will play an important role in the paper: For any $i, j<n$, put

$$
s_{j}^{i} x=c_{i}\left(d_{i j} \cdot x\right)
$$

It is well-known that an $n$-dimensional cylindric algebra $\mathfrak{A}$ is simple iff the quantifier free formula

$$
0 \neq 1 \wedge\left(x \neq 0 \rightarrow c_{0} c_{1} \cdots c_{n-1} x=1\right)
$$

is valid in $\mathfrak{A}$. Similarly to relation algebras, for every quantifier free formula $\varphi$ in the language of $n$-dimensional cylindric algebras, one can find in an effective way an equation $e_{\varphi}$ with the same free variables such that the formula $\varphi \leftrightarrow e_{\varphi}$ is valid in all simple $n$-dimensional cylindric algebras.

An example for a simple cylindric algebra is the full cylindric set algebra of dimension $n$ with base $U$ :

$$
\mathcal{C} s_{n}(U)=\left\langle\mathcal{P}\left({ }^{n} U\right), \cup, \cap,-{ }^{n} U,{ }^{n} U, \emptyset, C_{i}, D_{i j}\right\rangle_{i, j<n}
$$

where for all $i, j<n$ and $X \subseteq{ }^{n} U$,

$$
\begin{aligned}
C_{i}(X) & =\left\{u \in{ }^{n} U:(\exists v \in X)(\forall j<n) \text { if } j \neq i \text { then } u_{j}=v_{j}\right\}, \\
D_{i j} & =\left\{u \in{ }^{n} U: u_{i}=u_{j}\right\} .
\end{aligned}
$$

The variety $\mathrm{RCA}_{n}$ of representable cylindric algebras of dimension $n$ is

$$
\operatorname{RCA}_{n}=\mathbf{S P}\left\{\mathcal{C} s_{n}(U): U \text { is a set }\right\}
$$

Constructing relation algebras from cylindric algebras. The following translation which associates relation algebras with cylindric algebras is due to Henkin and Tarski, see $[6,5.3 .7]$. For an algebra $\mathfrak{A} \in \mathrm{CA}_{n}(n \geq 3)$, let

$$
N r_{2} \mathfrak{A}=\left\{x \in A: c_{i} x=x, \text { for all } 2 \leq i<n\right\} .
$$


$N r_{2} \mathfrak{A}$ is called the set of 2 -dimensional elements of $\mathfrak{A}$. Define a binary operation ; and a unary operation ${ }^{`}$ on $A$ by taking, for all $x, y \in A$,

$$
\begin{aligned}
x ; y & =c_{2}\left(s_{2}^{1} x \cdot s_{2}^{0} y\right) \\
x^{\breve{2}} & =s_{1}^{2} s_{0}^{1} s_{2}^{0} x .
\end{aligned}
$$

It is straightforward to check, using the properties (C0)-(C7), that $N r_{2} \mathfrak{A}$ is closed under the Booleans, contains $d_{01}$, and is closed under ; and ${ }^{〔}$. Define the algebra

$$
\mathfrak{R a} \mathfrak{A}=\left\langle N r_{2} \mathfrak{A},+, \cdot,-, 1,0, ;,,^{\smile}, d_{01}\right\rangle,
$$

which is called the relation algebra type reduct of $\mathfrak{A}$. For a class $\mathrm{K} \subseteq \mathrm{CA}_{n}, n \geq 3$, let

$$
\mathfrak{R a} \mathfrak{a}^{*} \mathrm{~K}=\{\mathfrak{R a} \mathfrak{A}: \mathfrak{A} \in \mathrm{K}\} .
$$

It is shown in $[6,5.3 .8]$ that $\mathfrak{R a} \mathfrak{A} \in R A$, whenever $n \geq 4$ and $\mathfrak{A} \in \mathrm{CA}_{n}$. Note that $\mathfrak{R a} \mathfrak{A}$ is not always a relation algebra if $\mathfrak{A} \in \mathrm{CA}_{3}$, see Simon [11] for a discussion. Nevertheless, the 3-dimensional cylindric algebras we introduce in Section 3 are such that their relation algebra type reducts are in fact relation algebras (see (Ca2) below).

Since the universe and the operations of $\mathfrak{R a} \mathfrak{A}$ are all term definable in $\mathfrak{A}$, it is easy to see that

$$
\begin{equation*}
\text { if } \mathfrak{A} \subseteq \mathfrak{B} \text { then } \mathfrak{R a} \mathfrak{A} \subseteq \mathfrak{R a} \mathfrak{B}, \tag{1}
\end{equation*}
$$

and similar statements hold for homomorphic images and direct products as well. Thus for all $\mathrm{K} \subseteq \mathrm{CA}_{n}(n \geq 3)$ we have

$$
\mathfrak{R a ^ { * }} \mathbf{H S P} K \subseteq \mathbf{H S P} \mathfrak{R} \mathfrak{a}^{*} K .
$$

Moreover, since if $\mathrm{K}_{1} \subseteq \mathrm{~K}_{2}$ then $\mathbf{H S P} \mathrm{K}_{1} \subseteq \mathbf{H S P} \mathrm{~K}_{2}$ and $\mathfrak{R a} \mathfrak{a}^{*} \mathrm{~K}_{1} \subseteq \mathfrak{R} \mathfrak{a}^{*} \mathrm{~K}_{2}$, we obtain:

$$
\begin{equation*}
\text { For every class } \mathrm{K} \subseteq \mathrm{CA}_{n}(n \geq 3), \mathbf{H S P} \mathfrak{R} \mathfrak{a}^{*} \mathbf{H S P} \mathrm{~K}=\mathbf{H S P} \mathfrak{R} \mathfrak{a}^{*} \mathrm{~K} . \tag{2}
\end{equation*}
$$

Results. Our results are the following:
Theorem 1. For any two degrees $\theta_{1}$ and $\theta_{2}$ of unsolvability with $\theta_{1}>\theta_{2}$, there are continuum many subvarieties V of $\mathrm{CA}_{3}$ such that the degree of unsolvability of $\mathrm{Eq}(\mathrm{V})$ is $\geq \theta_{1}$, while the degree of unsolvability of $\mathrm{Eq}\left(\mathfrak{R} \mathfrak{a}^{*} \mathrm{~V}\right)$ is equal to $\theta_{2}$. In particular, there are continuum many subvarieties V of $\mathrm{CA}_{3}$ such that $\mathrm{Eq}(\mathrm{V})$ is undecidable, while $\mathrm{Eq}\left(\mathfrak{R a}{ }^{*} \mathrm{~V}\right)$ is decidable.

Theorem 2. For each degree $\theta$ of unsolvability, there is a subvariety V of RA such that the degree of unsolvability of $\mathrm{Eq}(\mathrm{V})$ is equal to $\theta$, while $\mathrm{Eq}(\mathrm{V} \cap \mathrm{RRA})$ is decidable.


Theorem 3. For every natural number $n \geq 2$, and every degree $\theta$ of unsolvability, there are continuum many subvarieties V of $\mathrm{CA}_{n}$ such that the degree of unsolvability of $\mathrm{Eq}(\mathrm{V})$ is $\geq \theta$, while $\mathrm{Eq}\left(\mathrm{V} \cap \mathrm{RCA}_{n}\right.$ ) is recursively enumerable (in fact, $\mathrm{V} \cap \mathrm{RCA}_{n}=$ $\mathrm{RCA}_{n}$ holds). In particular, there are continuum many subvarieties V of $\mathrm{CA}_{2}$ such that $\mathrm{Eq}(\mathrm{V})$ is undecidable, while $\mathrm{Eq}\left(\mathrm{V} \cap \mathrm{RCA}_{2}\right)$ is decidable.

Finally, we mention some related open problems:
(Q1) Give some subvariety V of $\mathrm{CA}_{n}$, for $n \geq 3$, such that $\mathrm{Eq}(\mathrm{V})$ is undecidable but $\mathrm{Eq}\left(\mathrm{V} \cap \mathrm{RCA}_{n}\right)$ is decidable.
(Q2) Give some subvariety of either RA or $\mathrm{CA}_{n}$ such that its degree of unsolvability is lower than that of its representables.

## 3. Cylindric algebras vs. relation algebras

In this section we prove Theorem 1. We will use splittings of 3 -dimensional cylindric algebras which are constructed from Lyndon relation algebras. So to begin with, let us introduce these notions.

Constructing 3-dimensional cylindric algebras from relation algebras. Such a construction was first published in Monk [9], see also Maddux [8, Ch.10].

Given a complete atomic relation algebra $\mathfrak{A}$, let

$$
\begin{gathered}
B=\left\{a: a=\left\langle a_{0}, a_{1}, a_{2}\right\rangle, a_{0}, a_{1}, a_{2} \text { are atoms of } \mathfrak{A}, \text { and } a_{0} \leq a_{1} ; a_{2}\right\}, \\
d_{01}^{\mathfrak{A}}=\left\{a \in B: a_{0} \leq 1^{\prime}\right\}, \\
d_{02}^{\mathfrak{2}}=\left\{a \in B: a_{1} \leq 1^{\prime}\right\}, \\
d_{12}^{\mathfrak{2}}=\left\{a \in B: a_{2} \leq 1^{\prime}\right\},
\end{gathered}
$$

and for all $X \subseteq B$,

$$
\begin{aligned}
& c_{0}^{\mathfrak{A}}(X)=\left\{a \in B:(\exists b \in X) a_{2}=b_{2}\right\}, \\
& c_{1}^{\mathfrak{A}}(X)=\left\{a \in B:(\exists b \in X) a_{1}=b_{1}\right\}, \\
& c_{2}^{\mathfrak{A}}(X)=\left\{a \in B:(\exists b \in X) a_{0}=b_{0}\right\} .
\end{aligned}
$$

Define the 3-dimensional cylindric algebra type algebra

$$
\mathfrak{C a}_{3} \mathfrak{A}=\left\langle\mathcal{P}(B), \cup, \cap,-{ }^{B}, B, \emptyset, c_{i}^{\mathfrak{A}}, d_{i j}^{\mathfrak{A}}\right\rangle_{i, j<3} .
$$

And, for any class K of complete atomic relation algebras, let

$$
\mathfrak{C a}_{3}^{*} \mathrm{~K}=\left\{\mathfrak{C a}_{3} \mathfrak{A}: \mathfrak{A} \in \mathrm{K}\right\} .
$$

Then we have (see [9], [8]):
(Ca1) $\quad \mathfrak{C a}_{3} \mathfrak{A} \in \mathrm{CA}_{3}$. If $\mathfrak{A}$ is simple then $\mathfrak{C a}_{3} \mathfrak{A}$ is simple as well.

(Ca2) $\mathfrak{R a} \mathfrak{C a}_{3} \mathfrak{A} \in \mathrm{RA}$, and the function $h$ defined by $h(x)=\left\{a \in B: a_{0} \leq x\right\}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{R a} \mathfrak{C a}_{3} \mathfrak{A}$.
In particular, for all atoms $a_{0}$ of $\mathfrak{A}$, we have

$$
h\left(a_{0}\right)=\left\{\left\langle a_{0}, b, c\right\rangle: b, c \text { are atoms of } \mathfrak{A}\right\} .
$$

Let $s_{j}^{i \mathfrak{A}}$ denote the substitution operations of $\mathfrak{C a} \mathfrak{a}_{3} \mathfrak{A}$, for $i, j<3$. Then it is not hard to see that, for all atoms $a_{0}, a_{1}, a_{2}$ of $\mathfrak{A}$, we have

$$
\begin{aligned}
& s_{2}^{1 \mathfrak{A}} h\left(a_{1}\right)=\left\{\left\langle b, a_{1}, c\right\rangle: b, c \text { are atoms of } \mathfrak{A}\right\}, \\
& s_{2}^{0 \mathfrak{A}} h\left(a_{2}\right)=\left\{\left\langle b, c, a_{2}\right\rangle: b, c \text { are atoms of } \mathfrak{A}\right\} .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
h\left(a_{0}\right) \cap s_{2}^{1 \mathfrak{A}} h\left(a_{1}\right) \cap s_{2}^{0 \mathfrak{A}} h\left(a_{2}\right)=\left\{\left\langle a_{0}, a_{1}, a_{2}\right\rangle\right\}, \tag{3}
\end{equation*}
$$

and so

$$
\begin{equation*}
h\left(a_{0}\right) \cap s_{2}^{1 \mathfrak{A}} h\left(a_{1}\right) \cap s_{2}^{0 \mathfrak{A}} h\left(a_{2}\right) \text { is an atom of } \mathfrak{C} \mathfrak{a}_{3} \mathfrak{A} . \tag{4}
\end{equation*}
$$

Lyndon algebras. Lyndon [7] constructed relation algebras from projective geometries of arbitrary dimension. Here we define those Lyndon algebras which can be obtained from projective lines.

Let $U$ be a set with $|U| \geq 3$ and let $e$ be such that $e \notin U$. We define the Lyndon algebra on $U$ as

$$
\mathcal{L}(U)=\left\langle\mathcal{P}(U \cup\{e\}), \cup, \cap,-U \cup\{e\}, U \cup\{e\}, \emptyset, ;,^{\smile}, 1^{\prime}\right\rangle
$$

where $1^{\prime}=\{e\}, X^{\llcorner }=X$, for all $X \subseteq U \cup\{e\}$, and ; is the completely additive binary operation on $\mathcal{P}(U \cup\{e\})$ defined between singletons of $\mathcal{P}(U \cup\{e\})$ as follows. For any $u \neq v \in U$,

$$
\begin{aligned}
& \{u\} ;\{e\}=\{e\} ;\{u\}=\{u\} \\
& \{u\} ;\{u\}=\{u\} \cup 1^{\prime}=\{u, e\} \\
& \{u\} ;\{v\}=1-\left(\{u\} \cup\{v\} \cup 1^{\prime}\right)=U-\{u, v\} \\
& \{e\} ;\{e\}=\{e\} .
\end{aligned}
$$

Then clearly the singletons $\{e\}=1^{\prime}$ and $\{u\}$ for $u \in U$ are the atoms of $\mathcal{L}(U)$. Moreover, we have (see [7]):
(Ly1) For any set $U$ with $|U| \geq 3, \mathcal{L}(U)$ is a simple and complete atomic relation algebra.
Observe that for all non-zero elements $X$ of $\mathcal{L}(U)$ with $e \notin X$, we have $X ; X \neq U$ iff $X$ is a (non-identity) atom of $\mathcal{L}(U)$. Therefore the following holds:


Claim 3.1. Suppose $\mathfrak{A} \subseteq \mathcal{L}(U)$ for some set $U$ with $|U| \geq 3$. Then for every element $x$ in $\mathfrak{A}, x$ is a non-identity atom of $\mathcal{L}(U)$ iff $x$ satisfies the formula

$$
L a(x)=\left(0<x<-1^{\prime}\right) \wedge(x ; x \neq 1)
$$

in $\mathfrak{A}$.
Now let

$$
\mathrm{L}_{\emptyset}=\{\mathfrak{A}: \mathfrak{A} \subseteq \mathcal{L}(U), U \text { is a set with }|U| \geq 3, \text { and } \mathfrak{A} \text { is not }
$$

isomorphic to $\mathcal{L}(n)$, for any $n \in \omega-3\}$,
and for any $W \subseteq \omega-3$, let

$$
\mathrm{L}_{W}=\mathrm{L}_{\emptyset} \cup \mathbf{I}\{\mathcal{L}(n): n \in W\}
$$

The following statements are proved in Andréka et al. [2], for every $W \subseteq \omega-3$ :
(Ly2) [2, Thm.3.1(i)] $\mathrm{L}_{W}$ is a universal class of simple relation algebras. Therefore $\mathbf{S P} \mathrm{L}_{W}$ is a subvariety of RA.
(Ly3) $\quad\left[2\right.$, Thm.3.4(iii)] $\mathrm{Eq}\left(\mathbf{S P} \mathrm{L}_{W}\right)$ and $W$ have the same degree of unsolvability. In particular, $\mathrm{Eq}\left(\mathbf{S P} \mathrm{L}_{W}\right)$ is decidable iff $W$ is decidable.
For each $W \subseteq \omega-3$, define the subclass $\mathrm{L}_{W}^{-}$of $\mathrm{L}_{W}$ by taking

$$
\begin{aligned}
& \mathrm{L}_{W}^{-}=\left\{\mathfrak{A} \in \mathrm{L}_{\emptyset}: \mathfrak{A} \subseteq \mathcal{L}(n), \text { for some } n \in \omega-3\right\} \cup \mathbf{I}\{\mathcal{L}(U): U \text { is an infinite set }\} \\
& \cup \mathbf{I}\{\mathcal{L}(n): n \in W\} .
\end{aligned}
$$

Since the only algebras in $\mathrm{L}_{W}$ which are missing from $\mathrm{L}_{W}^{-}$are those which are isomorphic to subalgebras of some infinite Lyndon algebra $\mathcal{L}(U)$, we have

$$
\mathrm{L}_{W} \subseteq \mathrm{SL}_{W}^{-}
$$

Therefore $\mathrm{L}_{W}$ and $\mathrm{L}_{W}^{-}$generate the same variety, that is, by (Ly2) we have

$$
\begin{equation*}
\mathbf{H S P} \mathrm{L}_{W}^{-}=\mathbf{S P} \mathrm{L}_{W} \tag{5}
\end{equation*}
$$

Splitting. This is a technique for obtaining new cylindric algebras from cylindric algebras, and is a special case of dilation, see Henkin et al. [6, 3.2.69], Andréka [1]. Here we define "splitting to two" for finite algebras only.

Suppose $\mathfrak{A}$ is a finite (thus atomic) $n$-dimensional cylindric algebra, and $a$ is an atom of $\mathfrak{A}$ such that

$$
a \leq \prod_{i, j<n, i \neq j}-d_{i j}
$$

We call such an $a$ a subdiversity atom of $\mathfrak{A}$. We will 'split' $a$ into two new atoms $a^{\prime}$ and $a^{\prime \prime}$ such that "the $c_{j}$ and $d_{j k}$ behaviour" of both $a^{\prime}$ and $a^{\prime \prime}$ will "imitate" that of $a$.

To this end, we define a finite atomic algebra $\operatorname{split}(\mathfrak{A}, a)$ of the similarity type of $n$-dimensional cylindric algebras as follows: The atoms of $\operatorname{split}(\mathfrak{A}, a)$ are $a^{\prime}, a^{\prime \prime}$, and all those atoms of $\mathfrak{A}$ which are different from $a$. For all $i, j<n$, and atoms $x, y$ of $\mathfrak{A}$ different from $a$, let

$$
\begin{array}{lll}
x \leq d_{i j} \text { in } \operatorname{split}(\mathfrak{A}, a) & \text { iff } & x \leq d_{i j} \text { in } \mathfrak{A}, \\
x \leq c_{i} y \text { in } \operatorname{split}(\mathfrak{A}, a) & \text { iff } & x \leq c_{i} y \text { in } \mathfrak{A}, \\
x \leq c_{i} a^{\prime} \text { in } \operatorname{split}(\mathfrak{A}, a) & \text { iff } & x \leq c_{i} a \text { in } \mathfrak{A}, \\
x \leq c_{i} a^{\prime \prime} \text { in } \operatorname{split}(\mathfrak{A}, a) & \text { iff } & x \leq c_{i} a \text { in } \mathfrak{A},
\end{array}
$$

and if $i \neq j$ then

$$
\begin{aligned}
& a^{\prime} \leq-d_{i j}, \quad a^{\prime \prime} \leq-d_{i j} \\
& a^{\prime} \leq c_{i} a^{\prime}, \quad a^{\prime} \leq c_{i} a^{\prime \prime}, \quad a^{\prime \prime} \leq c_{i} a^{\prime}, \quad a^{\prime \prime} \leq c_{i} a^{\prime \prime}
\end{aligned}
$$

Finally, for every element $z$ of $\operatorname{split}(\mathfrak{A}, a)$, define

$$
c_{i} z=\sum\left\{c_{i} x: x \text { is an atom of } \operatorname{split}(\mathfrak{A}, a), \text { and } x \leq z\right\} .
$$

Then we have (see [6], [1]):
(Sp1) $\quad \operatorname{split}(\mathfrak{A}, a) \in \mathrm{CA}_{n}$, and if $\mathfrak{A}$ is simple then $\operatorname{split}(\mathfrak{A}, a)$ is simple as well.
(Sp2) $\mathfrak{A}$ is isomorphic to a subalgebra of $\operatorname{split}(\mathfrak{A}, a)$.
(Sp3) All the elements of $\operatorname{split}(\mathfrak{A}, a)$ of the form $c_{i} z(i<n)$ belong to (the isomorphic image of) $\mathfrak{A}$.
(Sp4) The isomorphic image of $a$ (that is, $\left.a^{\prime}+a^{\prime \prime}\right)$ is not an atom of $\operatorname{split}(\mathfrak{A}, a)$.
Thus by (Sp2) and (Sp3), we obtain:
Claim 3.2. If $n \geq 3$ then $\mathfrak{R a} \mathfrak{A}$ is isomorphic to $\mathfrak{R a} \operatorname{split}(\mathfrak{A}, a)$.
Proof of Theorem 1. For any $m \in \omega-3$, take the Lyndon algebra $\mathcal{L}(m)$ on $m$. Then $\{0\},\{1\}$ and $\{2\}$ are non-identity atoms of $\mathcal{L}(m)$ such that $\{0\} \leq\{1\} ;\{2\}$ holds. Therefore, $\langle\{0\},\{1\},\{2\}\rangle$ is a subdiversity atom of $\mathfrak{C a}{ }_{3} \mathcal{L}(m)$. Define

$$
\mathfrak{B}_{m}=\operatorname{split}\left(\mathfrak{C a}_{3} \mathcal{L}(m),\langle\{0\},\{1\},\{2\}\rangle\right) .
$$

Now suppose $\theta_{1}$ and $\theta_{2}$ are degrees of unsolvability with $\theta_{1}>\theta_{2}$. Take some sets $D \subseteq \omega-3$ and $H \subseteq D$ such that the degree of unsolvability of $D$ is $\theta_{2}$, and the degree of unsolvability of $H$ is $\theta_{1}$. Let

$$
\mathrm{K}_{D H}={\mathfrak{C} \mathfrak{a}_{3}^{*} \mathrm{~L}_{D}^{-} \cup\left\{\mathfrak{B}_{m}: m \in D-H\right\} . . . . ~}_{\text {. }}
$$

We are about to find a quantifier free formula $\varphi_{m}$ which "singles $\mathfrak{B}_{m}$ out of $\mathrm{K}_{D H}$ ", that is, for all $\mathfrak{A} \in \mathrm{K}_{D H}, \varphi_{m}$ is valid in $\mathfrak{A}$ iff $\mathfrak{A} \neq \mathfrak{B}_{m}$. Roughly, $\varphi_{m}$ will say the following: "If $x_{0}, \ldots, x_{m-1}$ is the list of all distinct non-identity atoms of the

'underlying' Lyndon algebra then $\left\langle x_{i}, x_{j}, x_{k}\right\rangle$ are cylindric algebra atoms, for all distinct $i, j, k<m$."

To this end, let $L a^{c a}(x)$ be the formula in the language of 3-dimensional cylindric algebras which is obtained from the relation algebraic formula $L a(x)$ of Claim 3.1 by replacing ; and $1^{\prime}$ with the corresponding cylindric algebraic terms:

$$
L a^{c a}(x)=\left(0<x<-d_{01}\right) \wedge\left(c_{2}\left(s_{2}^{1} x \cdot s_{2}^{0} x\right) \neq 1\right)
$$

For every $m \in \omega-3$, let $\psi_{m}\left(x_{0}, \ldots, x_{m-1}\right)$ be the following formula:

$$
\bigwedge_{i<m}\left(c_{2} x_{i}=x_{i} \wedge L a^{c a}\left(x_{i}\right)\right) \wedge \bigwedge_{i<j<m}\left(x_{i} \neq x_{j}\right) \wedge\left(d_{01}+\sum_{i<m} x_{i}=1\right)
$$

Claim 3.3. For all $\mathfrak{A} \in \mathrm{K}_{D H}$, if the elements $x_{0}, \ldots, x_{m-1}$ of $\mathfrak{A}$ are such that $\psi_{m}\left(x_{0}, \ldots, x_{m-1}\right)$ holds in $\mathfrak{A}$ then either
(i) $\mathfrak{A}$ is isomorphic to $\mathcal{L}(m)$ and $x_{i} \cdot s_{2}^{1} x_{j} \cdot s_{2}^{0} x_{k}$ are atoms of $\mathfrak{A}$, for all distinct $i, j, k<m$; or
(ii) $\mathfrak{A}=\mathfrak{B}_{m}$ and $x_{i} \cdot s_{2}^{1} x_{j} \cdot s_{2}^{0} x_{k}$ is not an atom of $\mathfrak{A}$, for some $i, j, k<m$.

Proof. Suppose first that $\mathfrak{A}=\mathfrak{C a}_{3} \mathfrak{B}$ for some $\mathfrak{B} \in \mathrm{L}_{D}^{-}$. By the definition of $\mathrm{L}_{D}^{-}$, there is a set $U$ such that either $\mathfrak{B}$ is isomorphic to $\mathcal{L}(U)$, or $\mathfrak{B} \subseteq \mathcal{L}(U)$ and $U$ is finite. Therefore, by (Ly1), $\mathfrak{B}$ is a complete atomic relation algebra, so by (Ca2) $\mathfrak{B}$ is isomorphic to $\mathfrak{R a} \mathfrak{A}$. Let $h$ denote the isomorphism from $\mathfrak{B}$ to $\mathfrak{R a} \mathfrak{A}$. Suppose $\psi_{m}\left(x_{0}, \ldots, x_{m-1}\right)$ holds in $\mathfrak{A}$ for some $x_{0}, \ldots, x_{m-1}$. Then there are $y_{0}, \ldots, y_{m-1}$ in $\mathfrak{B}$ such that $h\left(y_{i}\right)=x_{i}, L a\left(y_{i}\right)$ holds in $\mathfrak{B}$ for all $i<m$, and $1^{\prime}+\sum_{i<m} y_{i}=1$ holds in $\mathfrak{B}$. Therefore, by Claim 3.1, $y_{0}, \ldots, y_{m-1}$ are all the non-identity atoms of the Lyndon algebra $\mathcal{L}(U)$, which implies that $|U|=m$ and $\mathfrak{A}$ is isomorphic to $\mathcal{L}(m)$. Further, for all distinct $i, j, k<m$, we have $y_{i} \leq y_{j} ; y_{k}$ in $\mathcal{L}(m)$, thus $x_{i} \cdot s_{2}^{1} x_{j} \cdot s_{2}^{0} x_{k}$ are atoms of $\mathfrak{A}$ by (4).

Next, suppose $\mathfrak{A}=\mathfrak{B}_{\ell}$, for some $\ell \in D-H$. Then by Claim 3.2 and (Ca2), $\mathfrak{R a} \mathfrak{A}$ is isomorphic to $\mathcal{L}(\ell)$. Now by repeating the previous argument we obtain that $\ell=m$ must hold, thus $\mathfrak{A}=\mathfrak{B}_{m}$. But then $y_{i}=\{0\}, y_{j}=\{1\}$, and $y_{k}=\{2\}$, for some $i, j, k<m$. Thus, by (3) and ( Sp 4 ), we have that

$$
x_{i} \cdot s_{2}^{1} x_{j} \cdot s_{2}^{0} x_{k}=\langle\{0\},\{1\},\{2\}\rangle
$$

is not an atom of $\mathfrak{A}$.
Now define the formula $\varphi_{m}\left(x_{0}, \ldots, x_{m-1}, y\right)$ as

$$
\psi_{m}\left(x_{0}, \ldots, x_{m-1}\right) \wedge\left(y \leq x_{0} \cdot s_{2}^{1} x_{1} \cdot s_{2}^{0} x_{2}\right) \rightarrow\left((y=0) \vee\left(y=x_{0} \cdot s_{2}^{1} x_{1} \cdot s_{2}^{0} x_{2}\right)\right) .
$$

By (Ca1), (Ly1-2), and ( Sp 1 ), $\mathrm{K}_{D H}$ is a class of simple 3-dimensional cylindric algebras. Thus there is an equation $e_{m}$ with the same free variables as $\varphi_{m}$ such that $e_{m} \leftrightarrow \varphi_{m}$ is valid in $\mathrm{K}_{D H}$.


Let

$$
\mathbf{V}_{D H}=\mathbf{H S P} \mathrm{K}_{D H}
$$

As a consequence of Claim 3.3 we obtain that for all $m \in \omega-3$,

$$
e_{m} \text { is valid in } \mathrm{V}_{D H} \quad \text { iff } \quad \mathfrak{B}_{m} \notin \mathrm{~K}_{D H} \quad \text { iff } \quad m \in H \quad \text { or } \quad m \in(\omega-3)-D,
$$

thus
the degree of unsolvability of $\mathrm{Eq}\left(\mathrm{V}_{H}\right)$ is $\geq$ the degree of unsolvability of $H$.
Moreover, if $H_{1} \nsubseteq H_{2}$ then the equation $e_{m}$ for any $m \in H_{1}-H_{2}$ distinguishes between $V_{D H_{1}}$ and $V_{D H_{2}}$. Thus

$$
\begin{equation*}
\text { if } H_{1} \neq H_{2} \text { then } \vee_{D H_{1}} \neq \mathrm{V}_{D H_{2}} \tag{7}
\end{equation*}
$$

On the other hand, observe that by (Ca2) and Claim 3.2, $\mathfrak{R a} \mathfrak{a}^{*} \bigvee_{D H}$ is a variety of relation algebras, and we have

$$
\begin{gathered}
\mathbf{H S P} \mathfrak{R \mathfrak { a } ^ { * }} \mathrm{V}_{D H} \stackrel{(2)}{=} \mathbf{H S P} \mathfrak{R \mathfrak { a } ^ { * }} \mathrm{K}_{D H} \stackrel{\text { Claim }}{=} 3.2 \mathbf{H S P} \mathfrak{R} \mathfrak{a}^{*} \mathfrak{C a}_{3}^{*} \mathrm{~L}_{D}^{-} \\
\stackrel{(\mathrm{Ca} 2)}{=} \mathbf{H S P} \mathrm{L}_{D}^{-} \stackrel{(5)}{=} \mathbf{S P} \mathrm{L}_{D} .
\end{gathered}
$$

Therefore by (Ly3),
$\mathrm{Eq}\left(\mathfrak{R a}^{*} \mathrm{~V}_{D H}\right)$ and $D$ have the same degree of unsolvability.
Finally, (6)-(8) clearly prove Theorem 1.

## 4. Representable vs. axiomatic relation algebras

In this section we prove Theorem 2. To this end, we discuss some connections between representability of Lyndon algebras and the existence of projective planes.

A projective plane is a collection of lines and points, satisfying certain properties, see e.g. Coxeter [5] for more details. Monk [10] proves the following:
(Ly4) For any $n \in \omega-3$, if $\mathcal{L}(n) \in$ RRA then there is some projective plane $P$ such that $P$ contains a line $\ell$ with $n$ points, and $\mathcal{L}(n)$ is embeddable into the full relation set algebra whose base consists of those points of $P$ which are not on $\ell$.

Claim 4.1. For every $n \in \omega-3, \mathcal{L}(n) \in \operatorname{RRA}$ iff $\mathcal{L}(n)$ is embeddable into the full relation set algebra with base $n^{2}+1$.

Proof. The right-to-left direction is obvious. For the other, it is well-known (see [5]) that if $P$ is a projective plane having a line with $n$ points then in fact every line of $P$ has $n$ points, $P$ has $n$ lines as well, and thus the number of points of

$P$ is $n^{2}+n+1$. Therefore by (Ly4), if $\mathcal{L}(n)$ is representable then $\mathcal{L}(n)$ must be embeddable into the full relation set algebra with base $n^{2}+1$.

Since for each $n$ there are only finitely many possibilities for such an embedding, we obtain that
it is decidable whether a finite Lyndon algebra $\mathcal{L}(n)$ is representable.
Monk [10] shows that for any set $U$ with $|U| \geq 3$, if there is a projective plane containing a line with $|U|$ points then $\mathcal{L}(U) \in$ RRA. If $U$ is an infinite set then there is always some projective plane containing a line with $|U|$ points, and if $n=p^{k}+1$ for some $k>0$ and prime $p$ then there exists a projective plane containing a line with $n$ points, see also [10]. Further, Bruck and Ryser [4] proved that there are infinitely many natural numbers $n$ such that there is no projective plane containing a line with $n$ points. Thus, we have the following properties concerning representability of Lyndon algebras:
(Ly5) If $U$ is an infinite set then $\mathcal{L}(U) \in \operatorname{RRA}$.
(Ly6) There are infinitely many $n \in \omega$ such that $\mathcal{L}(n) \in$ RRA.
(Ly7) There are infinitely many $n \in \omega$ such that $\mathcal{L}(n) \notin$ RRA.
Next, recall the classes $\mathrm{L}_{W}(W \subseteq \omega-3)$ from the previous section. The following is shown in Andréka et al. [2, Thm.3.1(iv),(v)]:
(Ly8) For every $n \in \omega-3$ and every $\mathfrak{A} \subseteq \mathcal{L}(n)$ such that $\mathfrak{A}$ is not isomorphic to $\mathcal{L}(n)$, there is an $m_{\mathfrak{A}} \in \omega$ such that for all $k \geq m_{\mathfrak{A}}$ we have $\mathfrak{A} \subseteq \mathcal{L}(k)$.
As a consequence we have:
Claim 4.2. $\mathrm{L}_{\emptyset} \subseteq$ RRA.
Proof. Suppose $\mathfrak{A} \in \mathrm{L}_{\emptyset}$ and $\mathfrak{A} \subseteq \mathcal{L}(U)$ for some set $U$ with $|U| \geq 3$. If $U$ is infinite then $\mathfrak{A} \in$ RRA by (Ly5). If $|U|=n$ then, by the definition of $\mathrm{L}_{\emptyset}, \mathfrak{A}$ is not isomorphic to $\mathcal{L}(n)$. Thus, by (Ly6) and (Ly8), $\mathfrak{A} \subseteq \mathcal{L}(k)$ for some $\mathcal{L}(k) \in$ RRA, so we have $\mathfrak{A} \in$ RRA as well.

Proof of Theorem 2. Let $R=\{n \in \omega: \mathcal{L}(n) \in$ RRA $\}$. By (Ly7), $(\omega-3)-R$ is infinite. Given a degree $\theta$ of unsolvability, choose $H \subseteq(\omega-3)-R$ such that its degree of unsolvability is $\theta$. Then by (9), $R \cup H$ and $H$ have the same degree of unsolvability. Let

$$
\mathrm{V}_{H}=\mathbf{S P} \mathrm{L}_{R \cup H}
$$

By (Ly2), $\mathrm{V}_{H}$ is a variety of relation algebras, and by (Ly3),
$\mathrm{Eq}\left(\mathrm{V}_{H}\right)$ and $H$ have the same the degree of unsolvability.

On the other hand, Claim 4.2 implies that $\mathrm{L}_{R \cup H} \cap \mathrm{RRA}=\mathrm{L}_{R}$. Then by (Ly2),

$$
\mathrm{Si}\left(\mathrm{~V}_{H} \cap \mathrm{RRA}\right)=\mathrm{L}_{R \cup H} \cap \mathrm{RRA}=\mathrm{L}_{R}
$$

thus $\mathrm{V}_{H} \cap \operatorname{RRA}=\mathbf{S P} \operatorname{Si}\left(\mathrm{V}_{H} \cap \mathrm{RRA}\right)=\mathbf{S P} \mathrm{L}_{R}$ holds. Therefore, by (Ly3) and (9) we have that

$$
\begin{equation*}
\mathrm{Eq}\left(\mathrm{~V}_{H} \cap \mathrm{RRA}\right) \text { is decidable. } \tag{11}
\end{equation*}
$$

Finally, (10) and (11) prove Theorem 2.

## 5. Representable vs. axiomatic cylindric algebras

In this section we prove Theorem 3. Fix a natural number $n \geq 2$ (this $n$ is the dimension of the cylindric algebras of this section). For each $m \in \omega-n$, take the full $n$-dimensional cylindric set algebra $\mathcal{C} s_{n}(m)$ with base $m$. It is not hard to see that $\{\langle 0,1, \ldots, n-1\rangle\}$ is a subdiversity atom of $\mathcal{C} s_{n}(m)$. Put

$$
\mathfrak{C}_{m}=\operatorname{split}\left(\mathcal{C} s_{n}(m),\{\langle 0,1, \ldots, n-1\rangle\}\right) .
$$

Given a degree $\theta$ of unsolvability, choose a set $H \subseteq \omega-n$ such that the degree of unsolvability of $H$ is $\theta$, and let

$$
\mathrm{K}_{H}=\left\{\mathcal{C} s_{n}(U): U \text { is a set }\right\} \cup\left\{\mathfrak{C}_{m}: m \in \omega-n \text { and } m \notin H\right\}
$$

We are going to define a quantifier free formula $\varphi_{m}$ which "singles $\mathfrak{C}_{m}$ out of $\mathrm{K}_{H}$ ", that is, for all $\mathfrak{A} \in \mathrm{K}_{H}, \varphi_{m}$ is valid in $\mathfrak{A}$ iff $\mathfrak{A} \neq \mathfrak{C}_{m}$.

To this end, for each $i<n$, define the term

$$
c_{(i)} x=c_{0} \cdots c_{i-1} c_{i+1} \cdots c_{n-1} x
$$

and take the following formula $\delta_{i}^{n}(x)$ in the language of $n$-dimensional cylindric algebras:

$$
\delta_{i}^{n}(x)= \begin{cases}c_{(0)} x \cdot s_{1}^{0} c_{(0)} x \leq d_{01}, & \text { if } i=0 \\ c_{(i)} x \cdot s_{0}^{i} c_{(i)} x \leq d_{0 i}, & \text { otherwise }\end{cases}
$$

Now it is straightforward to check that for every $X \subseteq{ }^{n} U$,

$$
\begin{align*}
& \delta_{i}^{n}(X) \text { holds in } \mathcal{C} s_{n}(U) \text { iff }  \tag{12}\\
& \quad \mid\left\{u \in U: \text { there is some }\left\langle x_{0}, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n-1}\right\rangle \in X\right\} \mid \leq 1
\end{align*}
$$

We call a one-element subset $\left\{\left\langle x_{0}, \ldots, x_{n-1}\right\rangle\right\}$ of ${ }^{n} U$ a permutational singleton of $\mathcal{C} s_{n}(U)$, if all $x_{i}$ are distinct, for $i<n$. Let $\operatorname{psingl}^{n}(x)$ be the following formula:

$$
(x \neq 0) \wedge\left(x \leq \prod_{i, j<n, i \neq j}-d_{i j}\right) \wedge \bigwedge_{i<n} \delta_{i}^{n}(x)
$$

Now (12) implies that for all elements $x$ of $\mathcal{C} s_{n}(U)$,

$$
\begin{equation*}
\operatorname{psingl}^{n}(x) \text { holds in } \mathcal{C} s_{n}(U) \text { iff } x \text { is a permutational singleton of } \mathcal{C} s_{n}(U) . \tag{13}
\end{equation*}
$$



For every $m \in \omega-n$, let

$$
f(m)=m(m-1) \cdots(m-n+1)
$$

(Note that if $m=n$ then $f(m)=n!$.) Let $\sharp(U)$ denote the number of elements satisfying $p \operatorname{sing} l^{n}(x)$ in $\mathcal{C} s_{n}(U)$. Then by (13),

$$
\sharp(U)= \begin{cases}0, & \text { if }|U|<n, \\ f(m), & \text { if }|U|=m, m \in \omega-n, \\ \text { infinite, } & \text { if } U \text { is infinite } .\end{cases}
$$

Now let $m \in \omega-n$, and take an element $x$ of $\mathfrak{C}_{m}$ such that $x$ does not belong to (the isomorphic image of) $\mathcal{C} s_{n}(m)$. Let $a=\{\langle 0,1, \ldots, n-1\rangle\}$, and $a^{\prime}$ and $a^{\prime \prime}$ the two 'split-atoms.' Then there is an element $y$ of $\mathcal{C} s_{n}(m)$ such that $a \cdot y=0$, and either $x=y+a^{\prime}$ or $x=y+a^{\prime \prime}$. Suppose $\operatorname{psingl}^{n}(x)$ holds in $\mathfrak{C}_{m}$. Then by the definition of splitting, $\operatorname{psingl}^{n}(y+a)$ holds in $\mathfrak{C}_{m}$, thus by ( $\left.\operatorname{Sp} 2\right)$ psingl ${ }^{n}(y+a)$ holds in $\mathcal{C} s_{n}(m)$ as well. Now (13) implies that $y=0$ must hold. Thus by (Sp2), we obtain that

$$
\begin{align*}
& \operatorname{psingl}^{n}(x) \text { holds in } \mathfrak{C}_{m} \text { iff } \\
& \text { either }  \tag{14}\\
& \text { or } \quad x \text { is a permutational singleton of } \mathcal{C} s_{n}(U) \\
& \text { of the two new 'split-atoms.' }
\end{align*}
$$

Now for each $m \in \omega-n$, let $\psi_{m}\left(x_{0}, \ldots, x_{f(m)-1}\right)$ be the following formula:

$$
\bigwedge_{i<f(m)} p \operatorname{sing} l^{n}\left(x_{i}\right) \wedge \bigwedge_{i<j<f(m)}\left(x_{i} \neq x_{j}\right) \wedge\left(\sum_{i<f(m)} x_{i}=\prod_{i, j<n, i \neq j}-d_{i j}\right) .
$$

Then define the formula $\varphi_{m}\left(x_{0}, \ldots, x_{f(m)-1}, y\right)$ as

$$
\psi_{m}\left(x_{0}, \ldots, x_{f(m)-1}\right) \wedge\left(y \leq x_{0}\right) \rightarrow\left((y=0) \vee\left(y=x_{0}\right)\right)
$$

Claim 5.1. For all $\mathfrak{A} \in \mathrm{K}_{H}, \varphi_{m}$ is valid in $\mathfrak{A}$ iff $\mathfrak{A} \neq \mathfrak{C}_{m}$.
Proof. Suppose first that $\mathfrak{A}=\mathcal{C} s_{n}(U)$ for some set $U$. Since all the permutational singletons of $\mathcal{C} s_{n}(U)$ are in fact atoms of $\mathcal{C} s_{n}(U)$, the above computation on $\sharp(U)$ shows there are no elements $x_{0}, \ldots, x_{f(m)-1}$ satisfying $\psi_{m}$ in $\mathfrak{A}$, unless $|U|=m$. Therefore $\varphi_{m}$ is valid in $\mathcal{C} s_{n}(U)$ for $|U| \neq m$ because $\psi_{m}$ is always false, and $\varphi_{m}$ is valid in $\mathcal{C} s_{n}(m)$ because all permutational singletons of $\mathcal{C} s_{n}(m)$ are atoms of $\mathcal{C} s_{n}(m)$.

Next, assume that $\mathfrak{A}=\mathfrak{C}_{k}$ for some $k \in \omega-n$, and $\psi_{m}\left(x_{0}, \ldots, x_{f(m)-1}\right)$ holds in $\mathfrak{C}_{k}$. By the definition of splitting and (14), we obtain that

$$
f(k) \leq f(m) \leq f(k)+2
$$

Since $f(k+1)>f(k)+2$ whenever $k \geq 2$, this shows that there are no elements $x_{0}, \ldots, x_{f(m)-1}$ satisfying $\psi_{m}$ in $\mathfrak{A}$, unless $k=m$. Thus $\varphi_{m}$ is valid in $\mathfrak{C}_{k}$, for all
$k \neq m$. In case $k=m$, choose $x_{0}, \ldots, x_{f(m)-1}$ to be the permutational singletons of $\mathcal{C} s_{n}(m)$ such that

$$
x_{0}=\{\langle 0,1, \ldots, n-1\rangle\} .
$$

Then $\psi_{m}\left(x_{0}, \ldots, x_{f(m)-1}\right)$ holds in $\mathfrak{C}_{m}$, but by (Sp4) $x_{0}$ is not an atom of $\mathfrak{C}_{m}$, thus $\varphi_{m}$ is not valid in $\mathfrak{C}_{m}$.

By ( Sp 1 ), $\mathrm{K}_{H}$ is a class of simple $n$-dimensional cylindric algebras. Thus there is an equation $e_{m}$ with the same free variables as $\varphi_{m}$ such that $e_{m} \leftrightarrow \varphi_{m}$ is valid in $\mathrm{K}_{H}$. Let

$$
\mathbf{V}_{H}=\mathbf{H S P} \mathrm{K}_{H}
$$

Since all the full set algebras are in $\mathrm{K}_{H}$, we have

$$
\begin{equation*}
\mathrm{V}_{H} \cap \mathrm{RCA}_{n}=\mathrm{RCA}_{n} . \tag{15}
\end{equation*}
$$

On the other hand, Claim 5.1 implies that, for all $m \in \omega-n$,

$$
e_{m} \text { is valid in } \mathrm{V}_{H} \quad \text { iff } \quad \mathfrak{C}_{m} \notin \mathrm{~K}_{H} \quad \text { iff } \quad m \in H
$$

thus
the degree of unsolvability of $\mathrm{Eq}\left(\mathrm{V}_{H}\right)$ is $\geq$ the degree of unsolvability of $H$.
Claim 5.1 also implies that if $H_{1} \neq H_{2}$ then $\mathrm{V}_{H_{1}} \neq \mathrm{V}_{H_{2}}$. Thus (15) and (16) prove Theorem 3.

## References

[1] H. Andréka, Complexity of equations valid in algebras of relations, part I: strong non-finitizability, Annals of Pure and Applied Logic, 89 (1997), 149-209.
[2] H. Andréka, S. Givant, and I. Németi, Decision problems for equational theories of relation algebras, Memoirs Amer. Math. Soc., 126, no. 604, Amer. Math. Soc., 1997.
[3] J. Barwise (ed.), Handbook of Mathematical Logic, North Holland, 1977.
[4] R. H. Bruck and H. J. Ryser, The nonexistence of certain finite projective planes, Canad. J. Math., 1 (1949), 88-93.
[5] H. S. M. Coxeter, Projective geometry, University of Toronto Press, 2nd edition, 1974.
[6] L. Henkin, J. D. Monk, and A. Tarski, Cylindric Algebras, Part II, North Holland, 1985.
[7] R. C. Lyndon, Relation algebras and projective geometries, Michigan Math. J., 8 (1961), 21-28.
[8] R. Maddux, Topics in relation algebras, Ph.D. dissertation, Univ. of California, Berkeley, 1978.


[9] J. D. Monk, Studies in cylindric algebra, Doctoral dissertation, Univ. of California, Berkeley, 1961.
[10] J. D. Monk, Connections between combinatorial theory and algebraic logic, Studies in Math., vol. 9, Amer. Math. Soc. (1974), pp. 58-91.
[11] A. Simon, Non-representable algebras of relations, Ph.D. dissertation, Hungarian Academy of Sciences, Budapest (1997).

Agnes Kurucz
Department of Computer Science, King's College London, Strand, London WC2R 2LS, UK e-mail: kuag@dcs.kcl.ac.uk
URL: http://www.dcs.kcl.ac.uk/staff/kuag


[^0]:    Presented by I. Sain.
    Received July 27, 1999; accepted in final form October 2, 2001.
    2000 Mathematics Subject Classification: 03G15.
    Key words and phrases: Cylindric algebras, relation algebras, decision problem, algebraic logic.
    Research supported by UK EPSRC grant GR/L85978 and by Hungarian National Foundation for Scientific Research grants T16448, F17452, T23234 and T30314. Thanks to István Németi for the problems and discussions. I am grateful to the anonymous referee for his thorough reading of the manuscript, and for his many suggestions on improving the presentation.

