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Arrow Logic and Infinite Counting

Abstract. We consider arrow logics (i.e., propositional multi-modal logics having three — a dyadic, a monadic, and a constant — modal operators) augmented with various kinds of infinite counting modalities, such as 'much more', 'of good quantity', 'many times'. It is shown that the addition of these modal operators to weakly associative arrow logic results in finitely axiomatizable and decidable logics, which fail to have the finite base property.

Keywords: arrow logic, weakly associative relation algebras, graded modalities.

1. Introduction

In this paper we add several new connectives — modeling counting features of natural language — to locally square arrow logic (or, in algebraic setting, to weakly associative relation algebras) and show that the resulting logics are finitely axiomatizable and decidable. These logics are of interest also from a purely technical point of view. First, they are examples of decidable modal logics of relations without the finite base property. Second, our finite axiomatization theorems are consequences of (in fact, equivalent to) the result of Andréka et al. [1] saying that weakly associative relation algebras have the finite base property (see below for definitions).

Arrow logic. Arrow logic is a broadly applicable modal formalism. Possible applications range from dynamic semantics of natural language to transition systems in computer science, see Venema [20] for an overview of motivations and application areas. Arrow logic is of interest within logic itself, mainly because of its close connections with relation algebras, one of the most well-known paradigms of algebraic logic.

The *language of arrow logic* consists of the usual connectives of propositional logic together with modal connectives $\circ$ (dyadic), $\smile$ (monadic), and *Id* (constant). Formulas of this language — *arrow formulas* — are built up from a fixed countable set of propositional variables with the help of the above connectives in the usual way. (We use infix notation for $\circ$, and suffix notation for $\smile$).

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An *arrow frame* is a quadruple $\mathscr{F} = \langle W, \boldsymbol{C}, \boldsymbol{R}, \boldsymbol{E}\rangle$ such that W is a non-empty set (of *worlds*), $\boldsymbol{C} \subseteq W \times W \times W$, $\boldsymbol{R} \subseteq W \times W$, and $\boldsymbol{E} \subseteq W$ are (*accessibility*) relations on W. An *arrow model* is a pair $\mathfrak{M} = \langle \mathscr{F}, v\rangle$, where $\mathscr{F} = \langle W, \boldsymbol{C}, \boldsymbol{R}, \boldsymbol{E}\rangle$ is an arrow frame, and v is a function mapping propositional variables to subsets of W. *Truth* of arrow formulas in a world of an arrow model is defined inductively, for propositional connectives the usual way, and for the modal operators as follows. For any world $w \in W$,

$$\begin{aligned}
\mathfrak{M}, w \Vdash \varphi \circ \psi &\overset{\text{def}}{\iff} \exists u, v \text{ with } \boldsymbol{C}(w, u, v),\ \mathfrak{M}, u \Vdash \varphi \text{ and } \mathfrak{M}, v \Vdash \psi\,,\\
\mathfrak{M}, w \Vdash \varphi^{\smile} &\overset{\text{def}}{\iff} \exists u \text{ with } \boldsymbol{R}(w, u) \text{ and } \mathfrak{M}, u \Vdash \varphi\,,\\
\mathfrak{M}, w \Vdash Id &\overset{\text{def}}{\iff} \boldsymbol{E}(w) \text{ holds.}
\end{aligned}$$

Validity of arrow formulas in arrow models and frames is defined as usual.

By selecting various classes of arrow frames one can define various arrow logics. Since arrow logics are intended to talk about some kinds of arrows, those frames where worlds are pairs of objects and the accessibility relations represent composition, converse and identity are of particular interest. An arrow frame $\mathscr{F} = \langle W, \boldsymbol{C}, \boldsymbol{R}, \boldsymbol{E}\rangle$ is called a *pair frame* if $W \subseteq U \times U$, for some set U, and

$$\begin{aligned}
\boldsymbol{C} &= \{\langle\langle a, c\rangle, \langle a, b\rangle, \langle b, c\rangle\rangle \in W \times W \times W \ :\ a, b, c \in U\}\,,\\
\boldsymbol{R} &= \{\langle\langle a, b\rangle, \langle b, a\rangle\rangle \in W \times W \ :\ a, b \in U\}\,,\\
\boldsymbol{E} &= \{\langle a, a\rangle \in W \ :\ a \in U\}\,.
\end{aligned}$$

A pair frame $\mathscr{F} = \langle W, \boldsymbol{C}, \boldsymbol{R}, \boldsymbol{E}\rangle$ is called a *square frame* if $W = U \times U$, for some set U. $\mathscr{F}$ is called a *locally square frame* if W is a reflexive and symmetric relation on U. (The reason for this name is that if W is reflexive and symmetric then $\langle a, b\rangle \in W$ implies $\{a, b\} \times \{a, b\} \subseteq W$ as well.) *Arrow logics* $\mathcal{AL}_{\mathrm{SQ}}$ and $\mathcal{AL}_{\mathrm{LSQ}}$ are defined to be the sets of all arrow formulas which are valid in all square frames and in all locally square frames, respectively. As well-known results of Tarski and Monk show, the logic $\mathcal{AL}_{\mathrm{SQ}}$ is non-finitely axiomatizable and undecidable, and its expressive power is the same as that of first-order logic with binary predicates and with three variables (see e.g. Tarski–Givant [19], Henkin et al. [5]). On the other hand, the logic $\mathcal{AL}_{\mathrm{LSQ}}$ is finitely axiomatizable (Maddux [9]) and decidable (Németi [16]), but much of the expressive power of $\mathcal{AL}_{\mathrm{SQ}}$ is lost. There is an extensive literature on increasing the expressive power of $\mathcal{AL}_{\mathrm{LSQ}}$ by adding new connectives, cf. e.g. Andréka et al. [1], Kurucz [8], Marx [10], Mikulás [15], Stebletsova [18].

Infinite counting. We consider three kinds of additional connectives, all corresponding to some infinite counting concepts of natural language. Our metatheory is Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

The first new connective M is a dyadic one with the intended meaning '*much more*' (or '*infinitely more*'):

$$\mathfrak{M}, w \Vdash \mathsf{M}(\varphi, \psi) \overset{\text{def}}{\iff} |\{u \in W : \mathfrak{M}, u \Vdash \varphi\}| - |\{u \in W : \mathfrak{M}, u \Vdash \psi\}| \geq \aleph_0 \,.$$

(Throughout, $|X|$ denotes the cardinality of set X. Cardinal subtraction is defined the usual way by taking

$$\kappa - \lambda = \begin{cases} 0 & \text{if } \kappa \leq \lambda, \\ k - \ell & \text{if } \kappa = k,\ \lambda = \ell \text{ for some } k, \ell \in \omega,\ k > \ell, \\ \kappa, & \text{otherwise.)} \end{cases}$$

Second, take some infinite cardinal κ and define the following monadic connective $\langle \kappa \rangle$, with the intended meaning '*many times*' (or '*at least κ-times*'):

$$\mathfrak{M}, w \Vdash \langle \kappa \rangle \varphi \quad \overset{\text{def}}{\iff} \quad |\{u \in W : \mathfrak{M}, u \Vdash \varphi\}| \geq \kappa \,.$$

Finite variants of these connectives, the so-called counting or graded modalities are extensively studied in the literature, see e.g. Gargov et al. [3], van der Hoek [6], van der Hoek–de Rijke [7], Mikulás [15], Ohlbach [17].

Third, fix some class H of infinite cardinals (in the language of ZFC). We consider the following monadic connective Q_H, with the intended meaning '*of good quantity*':

$$\mathfrak{M}, w \Vdash \mathsf{Q}_H \varphi \quad \overset{\text{def}}{\iff} \quad |\{u \in W : \mathfrak{M}, u \Vdash \varphi\}| \in H \,.$$

Results and plan of paper. We show that the addition of some of the above counting features to the arrow logic $\mathcal{AL}_{\mathrm{LSQ}}$ results in finitely axiomatizable and decidable logics. All the results are proved in an algebraic setting.

Section 2 is the core of the paper. We define the algebraic counterparts of $\mathcal{AL}_{\mathrm{LSQ}}$ and its various extensions with infinite counting. We state and prove the finite axiomatizability theorems.

In section 3 we show that all the classes of algebras defined in the previous section have the effective finite algebra property, thus their equational theories are decidable. We also point out why they do not have the finite base property.

Logic and algebra. There is a well-known correspondence between arrow logics augmented with further modal operators and their algebraic counterparts (relation algebras with additional operations), see e.g. Marx–Venema [14] on this connection in general. For the purposes of this paper, it is enough to mention the following. Given a multi-modal language and a class

$\mathcal{C}$ of its frames, one can define a first-order language (which has an n-ary function symbol, for each n-adic connective of the multi-modal language in question) and a class $\mathsf{Alg}(\mathcal{C})$ of algebras of this first-order language such that the following properties hold.

(I) The multi-modal logic determined by $\mathcal{C}$ is decidable iff the equational theory of $\mathsf{Alg}(\mathcal{C})$ is decidable.

(II) The multi-modal logic determined by $\mathcal{C}$ is finitely axiomatizable iff $\mathsf{Alg}(\mathcal{C})$ generates a finitely axiomatizable variety (= equational class).

2. Axiomatization

2.1. Definitions

In this section we define the algebraic counterparts of the arrow logic $\mathcal{AL}_{\mathrm{LSQ}}$ and its extensions with infinite counting.

Notation. For any set W, let $\mathcal{P}(W)$ denote the powerset of W. $X \subseteq_\omega W$ means that X is a finite subset of W, and $\mathcal{P}_\omega(W) \stackrel{\mathrm{def}}{=} \{X : X \subseteq_\omega W\}$. The usual Boolean operations on sets are denoted by $\cup$, $\cap$, and $-^W$.

Algebras are denoted by gothic letters with the corresponding roman letters denoting their universes. We assume as known some basic concepts of universal algebra, such as homomorphisms and subdirectly irreducible algebras (see e.g. Burris–Sankappanavar [2]). Given some class K of algebras, $\mathrm{Sir}\ \mathsf{K}$ denotes the class of all subdirectly irreducible members of K; and $\mathrm{I}\mathsf{K}$, $\mathrm{S}\mathsf{K}$, $\mathrm{P}\mathsf{K}$, and $\mathrm{Up}\mathsf{K}$ denote, respectively, the classes of all isomorphic copies, isomorphic copies of subalgebras, isomorphic copies of direct products, and isomorphic copies of ultraproducts of members of K.

Given a set Ax of formulas of some first-order language, $\mathrm{Mod}\, Ax$ denotes the class of all models of Ax.

The algebraic counterpart of $\mathcal{AL}_{\mathrm{LSQ}}$ is the class WA of *weakly associative relation algebras* which is defined as follows.

$$\begin{aligned}\mathsf{WA} \stackrel{\mathrm{def}}{=} \ & \mathrm{S}\{\langle \mathcal{P}(W), \cup, \cap, -^W, W, \emptyset, \circ^W, {}^{\smile}, Id_U\rangle : \\ & U \text{ is a set, } W \text{ is a reflexive, symmetric binary relation on } U\},\end{aligned}$$

where for all $X, Y \subseteq W$,

$$\begin{aligned} X \circ^W Y &= \{\langle a,c\rangle \in W : \langle a,b\rangle \in X,\ \langle b,c\rangle \in Y, \text{ for some } b \in U\}, \\ X^{\smile} &= \{\langle b,a\rangle : \langle a,b\rangle \in X\}, \\ Id_U &= \{\langle a,a\rangle : a \in U\}. \end{aligned}$$

Sets W and U in this definition are called the *unit* and the *base* of the algebras in question, respectively.

THEOREM M. (Maddux [9]) WA *is a finitely axiomatizable variety.*

Here we will consider the following — non-equational — axiomatization of WA, cf. Maddux [9]. (The symbols of the language of WA are $\vee$, $\wedge$, $-$, 1, 0, $\circ$, $\breve{}$, Id. Below we will also use the predicate symbol $\leq$, which is defined as usual.) The name 'weakly associative' is justified by axiom (wa) below.

Boolean axioms

$(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$

$x^{\breve{}\breve{}} = x$

$(x \vee y)^{\breve{}} = x^{\breve{}} \vee y^{\breve{}}$

$x \circ Id = Id \circ x = x$

the *cycle law*: $x \wedge (y \circ z) = 0 \iff y \wedge (x \circ z^{\breve{}}) = 0 \iff z \wedge (y^{\breve{}} \circ x) = 0$

(wa) $\quad (Id \wedge x) \circ (1 \circ 1) = ((Id \wedge x) \circ 1) \circ 1$

We will also use the following property of WA.

THEOREM AHN. (Andréka, Hodkinson and Németi [1]) WA *has the* finite base property*: any finite* $\mathfrak{A} \in \mathsf{WA}$ *is isomorphic to some* $\mathfrak{B} \in \mathsf{WA}$ *with a finite base.*

Next, we turn to the extensions of WA with various infinite counting. First, we extend the language of WA with a binary function symbol M. WAM is the following class of algebras of this extended language.

$$\begin{aligned}\mathsf{WAM} \stackrel{\text{def}}{=} \ & \mathrm{S}\{\langle \mathcal{P}(W), \cup, \cap, -^W, W, \emptyset, \circ^W, \breve{}, Id_U, \mathsf{M}^W \rangle \ : \\ & U \text{ is a set, } W \text{ is a reflexive, symmetric binary relation on } U\},\end{aligned}$$

where for all $X, Y \subseteq W$,

$$\mathsf{M}^W(X,Y) = \begin{cases} W & \text{if } |X| - |Y| \geq \aleph_0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Next, take some finite set S of infinite cardinals, and extend the language of WA with one unary function symbol $\langle \kappa \rangle$, for each $\kappa \in S$. WAC_S is the following class of algebras of this extended language.

$$\begin{aligned}\mathsf{WAC}_S \stackrel{\text{def}}{=} \ & \mathrm{S}\{\langle \mathcal{P}(W), \cup, \cap, -^W, W, \emptyset, \circ^W, \breve{}, Id_U, \langle \kappa \rangle^W \rangle_{\kappa \in S} \ : \\ & U \text{ is a set, } W \text{ is a reflexive, symmetric binary relation on } U\},\end{aligned}$$

where for every $X \subseteq W$, $\kappa \in S$,

$$\langle \kappa \rangle^W(X) = \begin{cases} W & \text{if } |X| \geq \kappa, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that $\langle\aleph_0\rangle^W$ is expressible with the help of M^W as follows: $\langle\aleph_0\rangle^W(X) = \mathsf{M}^W(X,\emptyset)$. Concerning finite variants of $\langle\kappa\rangle^W$, Mikulás [15] proved that if $S = n$ for some $n \leq \omega$ then $\mathrm{S}\,\mathrm{P}\,\mathsf{WAC}_S$ is finitely axiomatizable as an existential variety — that is, formulas of form $\forall x \exists y(\tau = \sigma)$ are among the axioms. He also proved that in this case $\mathrm{S}\,\mathrm{P}\,\mathsf{WAC}_S$ is a variety, but a finite equational axiomatization is still unknown.

Next, fix some class H of infinite cardinals, and extend the language of WA with a unary function symbol Q_H. We define the class WAQ_H of algebras of the extended language as follows.

$$\begin{aligned}\mathsf{WAQ}_H \stackrel{\text{def}}{=} \ & \mathrm{S}\{\langle \mathcal{P}(W), \cup, \cap, -^W, W, \emptyset, \circ^W, \breve{}, Id_U, \mathsf{Q}_H^W\rangle \ : \\ & U \text{ is a set, } W \text{ is a reflexive, symmetric binary relation on } U\},\end{aligned}$$

where for every $X \subseteq W$,

$$\mathsf{Q}_H^W(X) = \begin{cases} W & \text{if } |X| \in H, \\ \emptyset & \text{otherwise.}\end{cases}$$

This new kind of operation can be considered as a generalization of $\langle\kappa\rangle^W$, since $\langle\kappa\rangle^W = \mathsf{Q}_H^W$ for $H = \{\lambda : \lambda \geq \kappa\}$. But now, for any class H of cardinals, we extend the language of WA only with one new unary function symbol Q_H.

Finally, we also consider M and Q_H together. For any class H of infinite cardinals, let WAMQ_H be the following class of algebras.

$$\begin{aligned}\mathsf{WAMQ}_H \stackrel{\text{def}}{=} \ & \mathrm{S}\{\langle \mathcal{P}(W), \cup, \cap, -^W, W, \emptyset, \circ^W, \breve{}, Id_U, \mathsf{M}^W, \mathsf{Q}_H^W\rangle \ : \\ & U \text{ is a set, } W \text{ is a reflexive, symmetric binary relation on } U\}.\end{aligned}$$

2.2. Results

THEOREM 1. *The class* WAM *of algebras generates a finitely axiomatizable variety. Namely,* $\mathrm{S}\,\mathrm{P}\,\mathrm{Up}\,\mathsf{WAM} = \mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAM})$*, where* $\mathrm{Ax}(\mathsf{WAM})$ *consists of the following equations:*

(a1) *equations axiomatizing* WA

(a2) $\mathsf{M}(x,x) = 0$

(a3) $\mathsf{M}(x \vee y, z) = \mathsf{M}(x,z) \vee \mathsf{M}(y,z)$ *and* $\mathsf{M}(z, x \vee y) = \mathsf{M}(z,x) \wedge \mathsf{M}(z,y)$

(a4) $(\mathsf{M}(x,y))\breve{} = \mathsf{M}(x,y)$

(a5) $(1 \circ \mathsf{M}(x,y)) \vee (\mathsf{M}(x,y) \circ 1) \leq \mathsf{M}(x,y)$

(a6) $(1 \circ -\mathsf{M}(x,y)) \vee (-\mathsf{M}(x,y) \circ 1) \leq -\mathsf{M}(x,y)$

(a7) $\mathsf{M}(x,y) \wedge \mathsf{M}(z,w) = \mathsf{M}(x \wedge \mathsf{M}(z,w), y \wedge \mathsf{M}(z,w)) \wedge \mathsf{M}(z,w)$

(a8) $\mathsf{M}(x, \mathsf{max}(\mathsf{Do}\, x, \mathsf{Rg}\, x)) \vee \mathsf{M}(\mathsf{max}(\mathsf{Do}\, x, \mathsf{Rg}\, x), x) = 0$

(where $\mathsf{max}(x, y) \stackrel{\text{def}}{=} (x \wedge -\mathsf{M}(y, x)) \vee (y \wedge \mathsf{M}(y, x))$,

$\mathsf{Do}\, x \stackrel{\text{def}}{=} (x \circ 1) \wedge Id$ *and* $\mathsf{Rg}\, x \stackrel{\text{def}}{=} (1 \circ x) \wedge Id$*)*

(a9) $\mathsf{M}(x, z) \leq \mathsf{M}(x, y) \vee \mathsf{M}(y, z)$

(a10) $\mathsf{M}(x, y) \wedge \mathsf{M}(y, z) \leq \mathsf{M}(x, z)$

THEOREM 2. *For any finite set S of infinite cardinals,* WAC_S *generates a finitely axiomatizable variety. Namely,* $\mathrm{S\,P\,Up}\,\mathsf{WAC}_S = \mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAC}_S)$, *where* $\mathrm{Ax}(\mathsf{WAC}_S)$ *consists of the following equations:*

(b1) *equations axiomatizing* WA

$(\mathrm{b2})_\kappa$ $\langle\kappa\rangle 0 = 0$ *($\kappa \in S$)*

$(\mathrm{b3})_\kappa$ $\langle\kappa\rangle(x \vee y) = \langle\kappa\rangle x \vee \langle\kappa\rangle y$ *($\kappa \in S$)*

$(\mathrm{b4})_\kappa$ $(\langle\kappa\rangle x)^{\smile} = \langle\kappa\rangle x$ *($\kappa \in S$)*

$(\mathrm{b5})_\kappa$ $(1 \circ \langle\kappa\rangle x) \vee (\langle\kappa\rangle x \circ 1) \leq \langle\kappa\rangle x$ *($\kappa \in S$)*

$(\mathrm{b6})_\kappa$ $(1 \circ -\langle\kappa\rangle x) \vee (-\langle\kappa\rangle x \circ 1) \leq -\langle\kappa\rangle x$ *($\kappa \in S$)*

$(\mathrm{b7})_{\kappa\lambda}$ $\langle\kappa\rangle\langle\lambda\rangle x \leq \langle\lambda\rangle x$ *($\kappa, \lambda \in S$)*

$(\mathrm{b8})_{\kappa\lambda}$ $\langle\kappa\rangle -\langle\lambda\rangle x) \leq -\langle\lambda\rangle x$ *($\kappa, \lambda \in S$)*

$(\mathrm{b9})_\kappa$ $\langle\kappa\rangle x = \langle\kappa\rangle\, \mathsf{Do}\, x \vee \langle\kappa\rangle\, \mathsf{Rg}\, x$ *($\kappa \in S$)*

$(\mathrm{b10})_{\kappa\lambda}$ $\langle\kappa\rangle x \leq \langle\lambda\rangle x$ *($\lambda < \kappa \in S$)*

THEOREM 3. *For any class H of infinite cardinals,* WAMQ_H *generates a finitely axiomatizable variety.*

REMARK 1. It is not so straightforward what this theorem means. Take some statement concerning cardinals which is independent of ZFC, e.g. the continuum hypothesis, and define a class H by taking $H = \{\kappa : \aleph_0 < \kappa < 2^{\aleph_0}\}$. Then H behaves differently in different models of set theory. Therefore statements like "$\mathsf{WAMQ}_H \vDash \forall x(\mathsf{Q}_H(x) = 0)$" are independent of ZFC as well. Nevertheless, it is meaningful to ask whether the statement "WAMQ_H generates a finitely axiomatizable variety" is a theorem of ZFC or not. And if so then it can happen that the finitely many equations axiomatizing this variety vary from model to model. Also, if the statement "the equational theory of WAMQ_H is decidable" is a theorem of ZFC then it can happen that, though we have some decision algorithm in every model of ZFC, it is not the same everywhere.

We want to find properties of WAMQ_H which are enough for an axiomatization. To this purpose, we have to exclude the 'impossible cardinal constellations' in the following sense. For instance, let H be the class of cardinals strictly larger than $\aleph_0$, and let $\mathfrak{A} \in \operatorname{Sir}\mathsf{WAMQ}_H$. Since with the help of M we are able to express that the cardinality of one infinite set is larger than that of some other one, there can be no sets $X, Y \in A$ such that (i) $|X| < |Y|$ and $\mathsf{Q}_H(X) = 1$, $\mathsf{Q}_H(Y) = 0$; or (ii) $|X| < |Y|$ and $\mathsf{Q}_H(X) = 0$, $\mathsf{Q}_H(Y) = 0$. To exclude such algebras axiomatically, we need axioms like

$$\mathsf{M}(y,x) \wedge \mathsf{Q}_H(x) \wedge -\mathsf{Q}_H(y) = 0 \qquad \text{(in case (i))}$$
$$\mathsf{M}(y,x) \wedge -\mathsf{Q}_H(x) \wedge -\mathsf{Q}_H(y) = 0 \qquad \text{(in case (ii))}.$$

It may seem that there are infinitely many possibilities to exclude, but below we show that this is not the case. To this end, we discuss some properties of 0-1 sequences. A 0-1 *sequence* is a function $s\colon n \to \{0,1\}$, for some $n \leq \omega$. A 0-1 sequence s is *finite* if its domain $\operatorname{dom} s$ is finite. For any 0-1 sequences s and t, t is a *subsequence* of s ($t \sqsubseteq s$) if, as usual, $\operatorname{dom} t \leq \operatorname{dom} s$ and

$$(\exists i_0 < \cdots < i_{\operatorname{dom} t-1})(\forall j < \operatorname{dom} t)\ t_j = s_{i_j}\,.$$

We say that a 0-1 sequence s *occurs in* H if there is some strictly increasing mapping λ from $\operatorname{dom} s$ to infinite cardinals such that

$$(\forall i < \operatorname{dom} s) \quad \lambda_i \in H \iff s_i = 1\,.$$

The range of such a λ is called an *occurence of* s *in* H. Now let

$$Bad_H = \{s : s \text{ is a finite 0-1 sequence which does not occur in } H\}\,.$$

It is easy to see that $Bad_H = \emptyset$ iff the ω-long 0-1 sequence 010101... occurs in H. We show that Bad_H is always 'finitely generated' in the following sense.

Claim 3.1. *For any class H of infinite cardinals, there is some finite $Gen_H \subseteq Bad_H$ such that*

$$(\forall s \in Bad_H)(\exists t \in Gen_H)\ t \sqsubseteq s\,.$$

Proof. If $Bad_H = \emptyset$ then $Gen_H = Bad_H = \emptyset$ is a good choice. So assume $Bad_H \neq \emptyset$. We consider *intervals* of cardinals in the usual sense. A 1-*interval* (of H) is an interval consisting only of cardinals which belong to H. Similarly, a 0-*interval* has cardinals which are not in H. An *i-block* ($i = 0$, 1) *of* H is a maximal i-interval. A *block of* H is either a 0-block or a 1-block.

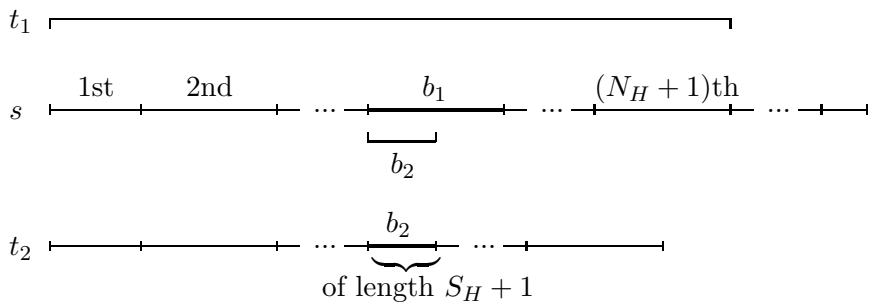

Fig. 1: Reducing the length of a bad 0-1 sequence.

A *block of a* 0-1 *sequence* is defined analogously. If $Bad_H \neq \emptyset$ then there are only finitely many blocks of H. Let N_H be the number of blocks of H. Some of these finitely many blocks may be finite, some may be infinite. For $i = 0, 1$, let S^i be the sum of the lengths of the finite i-blocks of H, and let $S_H = \max(S^0, S^1)$. We claim that

$$Gen_H = \{t \in Bad_H : \text{dom } t \leq (N_H + 1)(S_H + 1)\}$$

will do. Indeed, let $s \in Bad_H$. We will give some $t \in Gen_H$ with $t \sqsubseteq s$. Let

$$t_1 = \begin{cases} s & \text{if } s \text{ has} \leq N_H + 1 \text{ blocks,} \\ \text{the first } N_H + 1 \text{ blocks of } s & \text{otherwise.} \end{cases}$$

Then clearly $t_1 \sqsubseteq s$ and $t_1 \in Bad_H$. If the length of each block of t_1 is $\leq S_H + 1$ then $t_1 \in Gen_H$ holds. If not then choose some block b_1 of t_1 of length $> S_H + 1$. Let b_2 consist of the first $S_H + 1$ digits of b_1, and let t_2 be the same as t_1 but considering block b_2 in place of b_1 (cf. Figure 1). Then $t_2 \sqsubseteq t_1 \sqsubseteq s$ holds, and we claim that $t_2 \in Bad_H$. Assume indirectly that $t_2 \notin Bad_H$ that is, t_2 occurs in H. Choose some occurrence of t_2. Let $G \subseteq H$ be the union of those blocks of H which intersect the occurrence of block b_2 of t_2. Since the length of b_2 is $S_H + 1$, G must be infinite. This implies that t_1 also occurs in H, a contradiction. Now, reducing the size of the long blocks of t_2 step by step this way, we obtain some $t \sqsubseteq s$ with $t \in Gen_H$. $\square$

Now we are in a position to define a finite axiomatization for WAMQ_H. ('To define' does not mean 'to give explicitly', see Remark 1 above.) Given a class H of infinite cardinals, let Gen_H be a finite set of finite 0-1 sequences, generating Bad_H in the above sense, and for any $s \in Gen_H$, $i <$ dom s, let

$$s_i\mathrm{Q}_H = \begin{cases} \mathrm{Q}_H & \text{if } s_i = 1, \\ -\mathrm{Q}_H & \text{if } s_i = 0. \end{cases}$$

Let $\mathrm{Ax}(\mathsf{WAMQ}_H)$ be $\mathrm{Ax}(\mathsf{WAM})$ plus the following equations:

(no)$_s$ $\quad [\bigwedge_{i<\mathrm{dom}\ s-1} \mathsf{M}(x_{i+1}, x_i) \wedge \bigwedge_{i<\mathrm{dom}\ s} s_i \mathsf{Q}_H(x_i)] = 0 \qquad (s \in Gen_H)$

(a11) $\quad \mathsf{Q}_H(0) = 0$

(a12) $\quad \mathsf{Q}_H(x \vee y) = \mathsf{Q}_H(\mathsf{max}(x, y))$

(a13) $\quad \mathsf{Q}_H(x) \wedge \mathsf{M}(y, z) = \mathsf{Q}_H(x \wedge \mathsf{M}(y, z)) \wedge \mathsf{M}(y, z)$

(a14) $\quad \mathsf{Q}_H(x) \wedge \mathsf{M}(y, z) = \mathsf{Q}_H(x) \wedge \mathsf{M}(y \wedge \mathsf{Q}_H(x), z \wedge \mathsf{Q}_H(x))$

(a15) $\quad \mathsf{Q}_H(x)^{\smile} = \mathsf{Q}_H(x)$

(a16) $\quad (1 \circ \mathsf{Q}_H(x)) \vee (\mathsf{Q}_H(x) \circ 1) \leq \mathsf{Q}_H(x)$

(a17) $\quad (1 \circ -\mathsf{Q}_H(x)) \vee (-\mathsf{Q}_H(x) \circ 1) \leq -\mathsf{Q}_H(x)$

(a18) $\quad \mathsf{Q}_H(\mathsf{Q}_H(x)) \leq \mathsf{Q}_H(x)$

(a19) $\quad \mathsf{Q}_H(-\mathsf{Q}_H(x)) \leq -\mathsf{Q}_H(x)$

(a20) $\quad \mathsf{Q}_H(x) = \mathsf{Q}_H(\mathsf{max}(\mathsf{Do}\, x, \mathsf{Rg}\, x))$

(a21) $\quad \mathsf{Q}_H(x) \oplus \mathsf{Q}_H(y) \leq \mathsf{M}(x, y) \vee \mathsf{M}(y, x) \qquad (\oplus$ is symmetric difference)

2.3. Proofs

In this section we prove Theorems 1, 2 and 3 above. All proofs consist of the following five steps. Let $\mathsf{K} \in \{\mathsf{WAM}, \mathsf{WAC}_S, \mathsf{WAMQ}_H\}$.

(1) It is easily checked that the axioms are sound — that is, $\mathrm{Ax}(\mathsf{K})$ holds in K, thus also in $\mathrm{S\,P\,Up}\,\mathsf{K}$.

(2) We prove that each finite subdirectly irreducible model of $\mathrm{Ax}(\mathsf{K})$ is isomorphic to some algebra in K (Lemmas 1, 2 and 3).

(3) We prove that, for every $\mathfrak{A} \in \mathrm{Sir\ Mod\,Ax}(\mathsf{K})$ and $X \subseteq_\omega A$, there is some finite $\mathfrak{A}_X \in \mathrm{Sir\ Mod\,Ax}(\mathsf{K})$ such that

- $X \subseteq A_X \subseteq A$;
- the $\circ$-free reduct of $\mathfrak{A}_X$ is a subalgebra of the $\circ$-free reduct of $\mathfrak{A}$;
- for all $a, b \in X$, if $a \circ^{\mathfrak{A}} b \in X$ then $a \circ^{\mathfrak{A}} b = a \circ^{\mathfrak{A}_X} b$

(Lemmas 4, 5 and 6). Such a sequence $\langle \mathfrak{A}_X : X \subseteq_\omega A \rangle$ of algebras is called a *finite approximation of* $\mathfrak{A}$. If there is some recursive function $g : \omega \to \omega$ such that $|A_X| \leq g(|X|)$, for any $X \subseteq_\omega A$, then $\langle \mathfrak{A}_X : X \subseteq_\omega A \rangle$ is called an *effective approximation of* $\mathfrak{A}$.

(4) It is not hard to see that if $\langle \mathfrak{A}_X : X \subseteq_\omega A \rangle$ is a finite approximation of some algebra $\mathfrak{A}$ then $\mathfrak{A}$ is embeddable into some ultraproduct of the $\mathfrak{A}_X$'s. Indeed, consider the following well-known ultrafilter over $\mathcal{P}_\omega(A)$. For every $a \in A$, let $E_a = \{X \in \mathcal{P}_\omega(A) : a \in X\}$, and let $E = \{E_a : a \in A\}$. Then $E \subseteq \mathcal{P}(\mathcal{P}_\omega(A))$ and E has the finite intersection property, thus there is some

ultrafilter D including E. For any $y \in A$, $X \subseteq_\omega A$, we let

$$\lceil y \rceil^X = \bigvee \{a \in A_X : a \text{ is an atom of the finite algebra } \mathfrak{A}_X, \text{ and } a \wedge y \neq 0\}\,.$$

Thus $\lceil y \rceil^X \in A_X$ always holds, and $y \in A_X$ implies $\lceil y \rceil^X = y$. Now, for each $y \in A$, let

$$f(y) = \text{the } D\text{-class of the sequence } \langle \lceil y \rceil^X : X \subseteq_\omega A \rangle\,.$$

It is easy to check that f is a homomorphism from $\mathfrak{A}$ into the ultraproduct of the $\mathfrak{A}_X$'s (modulo D), e.g. f preserves $\circ$ by the following.

$$\begin{aligned}&\{X \subseteq_\omega A : \lceil y \circ^{\mathfrak{A}} z \rceil^X = \lceil y \rceil^X \circ^{\mathfrak{A}_X} \lceil z \rceil^X\} \supseteq \{X \subseteq_\omega A : y, z, y \circ^{\mathfrak{A}} z \in X\}\\ &= E_y \cap E_z \cap E_{y \circ z} \;\in\; D\,.\end{aligned}$$

Finally, for any $X \subseteq_\omega A$, $y \leq \lceil y \rceil^X$ always holds, since $1^{\mathfrak{A}} = 1^{\mathfrak{A}_X} = \bigvee\{a : a$ is an atom of $\mathfrak{A}_X\}$. Thus if $y \in A$, $y \neq 0$ then $f(y) \neq 0$, proving that f is one-one.

(5) Now, using (2)–(4), the proofs can be completed as follows.

$$\begin{aligned}\mathrm{Mod\,Ax}(\mathsf{K}) &= \mathrm{S\,P\,Sir\,Mod\,Ax}(\mathsf{K})\\ &\overset{(3),(4)}{\subseteq} \mathrm{S\,P\,S\,Up}\{\mathfrak{A} \in \mathrm{Mod\,Ax}(\mathsf{K}) : \mathfrak{A} \text{ is finite}\}\\ &= \mathrm{S\,P\,S\,Up\,S\,P}\{\mathfrak{A} \in \mathrm{Sir\,Mod\,Ax}(\mathsf{K}) : \mathfrak{A} \text{ is finite}\}\\ &\overset{(2)}{\subseteq} \mathrm{S\,P\,S\,Up\,S\,P\,I}\,\mathsf{K} = \mathrm{S\,P\,Up}\,\mathsf{K}\,.\end{aligned}$$

It remains to prove (2) and (3) for the particular classes of algebras.

Lemma 1. *Every finite, subdirectly irreducible* $\mathfrak{A} \in \mathrm{Mod\,Ax}(\mathsf{WAM})$ *is isomorphic to some* $\mathfrak{C} \in \mathsf{WAM}$.

Proof. In this proof — as well as in the proofs of Lemmas 2 and 3 below — we will use the following technique which may be called '*blowing up* finite WA's'. Let $\mathfrak{B} \in \mathsf{WA}$ be finite with base U and unit W. We will 'enlarge' some points of U in such a way that the algebra $\mathfrak{B}$ 'does not recognize' it, thus we obtain some algebra $\mathfrak{C}$ isomorphic to $\mathfrak{B}$, but having a larger base. To this end, let E be any set with

$$E \subseteq \{e \in B : e \text{ is an atom of } \mathfrak{B}, \text{ and } e \leq Id\}\,.$$

For any $e \in E$, take some non-empty set P_e such that $P_e \cap U = \emptyset$, and $P_e \cap P_f = \emptyset$ whenever $e \neq f$. Further, for any $e \in E$, fix some $q_e \in U$ such

Fig. 2: Blowing up atom a.

that $\langle q_e, q_e\rangle \in e$ holds. (Since atoms are disjoint, $q_e \neq q_f$ if $e \neq f$ follows.) Let $Q = \{q_e : e \in E\}$, and let $U' = (U - Q) \cup \bigcup\{P_e : e \in E\}$. Now for any atom a of $\mathfrak{B}$, we define the set $a' \subseteq U' \times U'$ as follows.

$$\begin{aligned} a' \;=\; & (a - (Q \times U \;\cup\; U \times Q)) \;\cup \\ & \cup\, \{\langle p, u\rangle : p \in P_e,\ u \in U - Q,\ \langle q_e, u\rangle \in a,\ e \in E\} \;\cup \\ & \cup\, \{\langle u, p\rangle : p \in P_e,\ u \in U - Q,\ \langle u, q_e\rangle \in a,\ e \in E\} \;\cup \\ & \cup\, \{\langle p, p\rangle : p \in P_e,\ \langle q_e, q_e\rangle \in a,\ e \in E\} \;\cup \\ & \cup\, \{\langle p, r\rangle \;:\; p \in P_e,\ r \in P_f,\ \langle q_e, q_f\rangle \in a,\ e \neq f \in E\} \end{aligned}$$

(cf. Figure 2). Let $W' = \bigcup\{a' : a \text{ is an atom of } \mathfrak{B}\}$. It is easily shown that

$$\text{for any atoms } a \neq b \text{ of } \mathfrak{B}, \quad a' \cap b' = \emptyset\,, \tag{1}$$

and that W' is a reflexive and symmetric relation on U'. For any $x \in B$, define

$$\begin{aligned} x' &= \bigcup\{a' : a \text{ is an atom, and } a \leq x\}\,, \\ C &= \{x' : x \in B\}\,, \\ \mathfrak{C} &= \langle C, \cup, \cap, -^{W'}, W', \emptyset, \circ^{W'}, \breve{\ }, Id_{U'}\rangle\,. \end{aligned}$$

It is a routine computation to show that the function $x \mapsto x'$ is an isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$.

Next, we give a characterization of the subdirectly irreducibles.

CLAIM 1.1. *If* $\mathfrak{A} \in \mathrm{ModAx}(\mathsf{WAM})$ *is subdirectly irreducible then for all* $a, b \in A$*, either* $\mathsf{M}(a,b) = 0$ *or* $\mathsf{M}(a,b) = 1$*.*

Proof. Given some $a, b \in A$, one can define the following, so-called 'relativization functions' on A. For each $x \in A$, let

$$\mathsf{rl}_{\mathsf{M}(a,b)}(x) \stackrel{\text{def}}{=} x \wedge \mathsf{M}(a,b) \quad \text{and} \quad \mathsf{rl}_{-\mathsf{M}(a,b)}(x) \stackrel{\text{def}}{=} x \wedge -\mathsf{M}(a,b)\,.$$

Relativizations are always Boolean homomorphisms, and they respect $\breve{}$ by (a1) and (a4); $\circ$ by (a1), (a5) and (a6); M by (a7). Thus, if $\mathsf{M}(a,b) \neq 1$ and $\mathsf{M}(a,b) \neq 0$ held then $\mathsf{rl}_{\mathsf{M}(a,b)}$ and $\mathsf{rl}_{-\mathsf{M}(a,b)}$ would give a non-trivial subdirect decomposition of $\mathfrak{A}$. □

Now assume $\mathfrak{A} \in \mathrm{ModAx}(\mathsf{WAM})$ is finite and subdirectly irreducible. We will show that $\mathfrak{A}$ is isomorphic to some $\mathfrak{C} \in \mathsf{WAM}$. To this end, define a binary relation $\sim$ on A as follows. For any $a, b \in A$,

$$a \sim b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathsf{M}(a,b) \vee \mathsf{M}(b,a) = 0\,.$$

CLAIM 1.2. (i) $\sim$ *is an equivalence relation on* A*.*

(ii) $(\forall a, b \in A)\ a \sim b \implies (\forall c \in A)\ \mathsf{M}(a,c) = \mathsf{M}(b,c)$ *and* $\mathsf{M}(c,a) = \mathsf{M}(c,b)$*.*

Proof. For (i): $\sim$ is reflexive by (a2), symmetric by the commutativity of $\vee$, and transitive by (a9). For (ii):

$$\mathsf{M}(a,c) \stackrel{\text{(a9)}}{\subseteq} \mathsf{M}(a,b) \vee \mathsf{M}(b,c) \stackrel{a \sim b}{=} 0 \vee \mathsf{M}(b,c) = \mathsf{M}(b,c). \qquad \square$$

Next we define a binary relation $<$ on the $\sim$-classes of A. For any $a \in A$, let $[a]$ denote the $\sim$-class of a. For any $a, b \in A$,

$$[b] < [a] \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathsf{M}(a,b) = 1$$

($<$ is well-defined by Claim 1.2(ii)).

CLAIM 1.3. $<$ *is an irreflexive linear order.*

Proof. $<$ is irreflexive by (a2), transitive by (a10), and for any $a, b \in A$, either $[a] < [b]$, or $[b] < [a]$, or $a \sim b$ hold, by Claim 1.1. □

The M-free reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is a finite WA which, by Theorem AHN, is isomorphic to some $\mathfrak{B}^- \in \mathsf{WA}$ having a finite base.[2] Let $\mathfrak{B} = \langle \mathfrak{B}^-, \mathsf{M} \rangle$ (where M denotes the operation on B induced by the above isomorphism between the reducts), and let U and W denote the base and the unit of $\mathfrak{B}$, respectively. We define a special blow-up $\mathfrak{C}^-$ of $\mathfrak{B}^-$ as follows. Let $At_{\mathfrak{B}}$ denote the set of atoms of $\mathfrak{B}$. Let

$$E = \{e \in At_{\mathfrak{B}} \ : \ e \leq Id, \ \mathsf{M}(e, 0) = 1\}.$$

Let $E_0, \ldots, E_{n-1}$ be the enumeration of the equivalence classes of $\sim \cap (E \times E)$ such that $E_0 < E_1 < \ldots < E_{n-1}$. Take arbitrary infinite cardinals $\lambda_0, \ldots, \lambda_{n-1}$ such that $\lambda_0 < \lambda_1 < \ldots < \lambda_{n-1}$. For each $i < n$, $e \in E_i$, choose the 'blow-up' set P_e such that $|P_e| = \lambda_i$, and let $\mathfrak{C}^-$ be the blow-up of $\mathfrak{B}^-$, defined as above. Let $\mathfrak{C} = \langle \mathfrak{C}^-, \mathsf{M}^{W'} \rangle$. We want to show that the function $x \mapsto x'$ is an isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$ — that is, for any $x, y \in B$,

$$\mathsf{M}(x, y) = 1 \iff |x'| - |y'| \geq \aleph_0$$

holds. To this end, for any cardinals κ, λ, let

$$\mathrm{MAX}_{\aleph_0}(\kappa, \lambda) = \begin{cases} \lambda & \text{if } \lambda - \kappa \geq \aleph_0 \\ \kappa & \text{otherwise} \end{cases}$$

Recall that $\mathsf{max}(x, y) = (x \wedge -\mathsf{M}(y, x)) \vee (y \wedge \mathsf{M}(y, x))$.

Claim 1.4. (i) $(\forall e, f \in At_{\mathfrak{B}},\ e, f \leq Id)\ \mathsf{M}(e, f) = 1 \iff |e'| - |f'| \geq \aleph_0$.
(ii) $(\forall e, f \in At_{\mathfrak{B}},\ e, f \leq Id)\ |\mathsf{max}(e, f)'| = \mathrm{MAX}_{\aleph_0}(|e'|, |f'|)$.
(iii) $(\forall a \in At_{\mathfrak{B}})\ |(\mathsf{Do}; a)'| = |\operatorname{dom} a'|$ *and* $|(\mathsf{Rg}\, a)'| = |\operatorname{rng} a'|$.
(iv) $(\forall a, b \in At_{\mathfrak{B}})\ \mathsf{M}(a, b) = 1 \iff |a'| - |b'| \geq \aleph_0$.

Proof. For (i): Assume first that $\mathsf{M}(e, f) = 1$. Then $e \in E_i$, $f \in E_j$ for some $E_i > E_j$. By definition, $e' = e - \{\langle q_e, q_e \rangle\} \cup \{\langle p, p \rangle : p \in P_e\}$, $f' = f - \{\langle q_f, q_f \rangle\} \cup \{\langle p, p \rangle : p \in P_f\}$. Therefore, since e and f are finite sets, $|e'| = |P_e| = \lambda_i > \lambda_j = |P_f| = |f'|$. Now assume $\mathsf{M}(e, f) = 0$. If $\mathsf{M}(f, e) = 1$ then, by the previous argument, $|e'| - |f'| \ngeq \aleph_0$. If $\mathsf{M}(f, e) = 0$ then $e \sim f$ holds. Thus either $e, f \in E_i$ for some $i < n$, or, by Claim 1.2(ii), $e \notin E$, $f \notin E$. In the first case, $|e'| = |P_e| = \lambda_i = |P_f| = |f'|$. In the latter case, $e' = e$, $f' = f$ and, since both e and f are finite sets, $|e'| - |f'| \ngeq \aleph_0$.

[2] In fact, Theorem AHN is equivalent with Lemma 1 by the following. Take some finite $\mathfrak{A} \in \mathsf{WA}$, we may assume that $\mathfrak{A}$ is subdirectly irreducible. Define M on $\mathfrak{A}$ by $\mathsf{M}(a, b) = 0$, for any $a, b \in A$. This new algebra will satisfy $\mathrm{Ax}(\mathsf{WAM})$ thus, by Lemma 1, it is isomorphic to some $\mathfrak{B}^+ \in \mathsf{WAM}$. Its M-free reduct $\mathfrak{B} \in \mathsf{WA}$ must have a finite base, since $\mathsf{M}(b, 0) = 0$, for any $b \in B$.

Item (ii) follows from (i) and from the definitions of $\mathfrak{max}$ and $\mathrm{MAX}_{\aleph_0}$. Item (iii) can be proved by an inspection of the definition of the blow-up function $'$.

For (iv): By (a8), $a \sim \mathfrak{max}(\mathsf{Do}\, a, \mathsf{Rg}\, a)$, $b \sim \mathfrak{max}(\mathsf{Do}\, b, \mathsf{Rg}\, b)$. Thus, by Claim 1.2(ii), $\mathsf{M}(a,b) = \mathsf{M}(\mathfrak{max}(\mathsf{Do}\, a, \mathsf{Rg}\, a), \mathfrak{max}(\mathsf{Do}\, b, \mathsf{Rg}\, b))$. Therefore

$$\begin{aligned}
\mathsf{M}(a,b) = 1 &\overset{\text{(i)}}{\Leftrightarrow} \aleph_0 \le |\mathfrak{max}(\mathsf{Do}\, a, \mathsf{Rg}\, a)'| - |\mathfrak{max}(\mathsf{Do}\, b, \mathsf{Rg}\, b)'| \qquad (2)\\
&\overset{\text{(ii)}}{=} \mathrm{MAX}_{\aleph_0}(|(\mathsf{Do}\, a)'|, |(\mathsf{Rg}\, a)'|) - \mathrm{MAX}_{\aleph_0}(|(\mathsf{Do}\, b)'|, |(\mathsf{Rg}\, b)'|)\\
&\overset{\text{(iii)}}{=} \mathrm{MAX}_{\aleph_0}(|\operatorname{dom} a'|, |\operatorname{rng} a'|) - \mathrm{MAX}_{\aleph_0}(|\operatorname{dom} b'|, |\operatorname{rng} b'|)\\
&\Leftrightarrow |a'| - |b'| \ge \aleph_0 .
\end{aligned}$$

□

Now, for any $x, y \in B$,

$$\mathsf{M}(x,y) \overset{\text{(a3)}}{=} \bigvee_{a \in At_B, a \le x} \ \bigwedge_{b \in At_B, b \le y} \mathsf{M}(a,b) . \qquad (3)$$

Therefore, by Claims 1.1 and 1.4(iv),

$$\begin{aligned}
\mathsf{M}(x,y) = 1 \quad &\Longleftrightarrow \quad (\exists a \in At_{\mathfrak{B}},\ a \le x)(\forall b \in At_{\mathfrak{B}}, b \le y)\ |a'| - |b'| \ge \aleph_0\\
&\Longleftrightarrow \quad (\text{since } B \text{ is finite and by (1)})\ |x'| - |y'| \ge \aleph_0 . \qquad (4)
\end{aligned}$$

Since $\mathfrak{C} \in \mathsf{WAM}$ holds by definition, this completes the proof of Lemma 1. ■

Lemma 2. *Let S be some finite set of infinite cardinals. Then every finite, subdirectly irreducible $\mathfrak{A} \in \mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAC}_S)$ is isomorphic to some $\mathfrak{C} \in \mathsf{WAC}_S$.*

Proof. We start with a characterization of the subdirectly irreducibles.

Claim 2.1. *If $\mathfrak{A} \in \mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAC}_S)$ is subdirectly irreducible then for all $a \in A$, $\kappa \in S$, either $\langle\kappa\rangle a = 0$ or $\langle\kappa\rangle a = 1$.*

Proof. Given $a \in A$, $\kappa \in S$, define the relativization functions $\mathsf{rl}_{\langle\kappa\rangle a}$ and $\mathsf{rl}_{-\langle\kappa\rangle a}$ as follows. For each $x \in A$, let

$$\mathsf{rl}_{\langle\kappa\rangle a}(x) = x \wedge \langle\kappa\rangle a \quad \text{and} \quad \mathsf{rl}_{-\langle\kappa\rangle a}(x) = x \wedge -\langle\kappa\rangle a .$$

They are clearly Boolean homomorphisms, and they respect $\breve{}$ by (b1), $(\mathrm{b4})_\kappa$; $\circ$ by (b1), $(\mathrm{b5})_\kappa$, $(\mathrm{b6})_\kappa$; $\langle\lambda\rangle$ (for $\lambda \in S$) by (b1), $(\mathrm{b3})_\kappa$, $(\mathrm{b7})_{\lambda\kappa}$, $(\mathrm{b8})_{\lambda\kappa}$. Thus, if $\langle\kappa\rangle a \ne 1$ and $\langle\kappa\rangle a \ne 0$ held then $\mathsf{rl}_{\langle\kappa\rangle a}$ and $\mathsf{rl}_{-\langle\kappa\rangle a}$ would give a non-trivial subdirect decomposition of $\mathfrak{A}$. □

Now assume $\mathfrak{A} \in \mathrm{ModAx}(\mathsf{WAC}_S)$ is finite and subdirectly irreducible. As in the proof of Lemma 1, the appropriate reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is a finite algebra in WA which is, by Theorem AHN, always isomorphic to some $\mathfrak{B}^- \in \mathsf{WA}$ having a finite base. Let $\mathfrak{B} = \langle \mathfrak{B}^-, \langle\kappa\rangle\rangle_{\kappa\in S}$ (where, for each $\kappa \in S$, $\langle\kappa\rangle$ denotes the operation on B induced by the above isomorphism between the reducts), and let U and W denote the base and the unit of $\mathfrak{B}$, respectively. We define a blow-up $\mathfrak{C}^-$ of $\mathfrak{B}^-$ as follows. Let

$$E = \{e \in At_{\mathfrak{B}} \ : \ e \leq Id, \text{ and } \langle\kappa\rangle e = 1, \text{ for some } \kappa \in S\}.$$

Assume that $S = \{\lambda_0, \ldots, \lambda_{n-1}\}$ such that $\lambda_0 < \ldots < \lambda_{n-1}$, and for each $i < n$, let

$$E_i = \{e \in E \ : \ \langle\lambda_i\rangle e = 1, \text{ and } \langle\kappa\rangle e = 0, \text{ for all } \kappa > \lambda_i, \ \kappa \in S\}.$$

Then some of the E_i's can be empty but, by axioms $(\mathrm{b10})_{\kappa\lambda}$ $(\lambda < \kappa \in S)$, the non-empty E_i's form a partition of E. For any $i < n$, $e \in E_i$, choose the 'blow-up' set P_e (see the proof of Lemma 1 for notation) such that $|P_e| = \lambda_i$, and let $\mathfrak{C} = \langle \mathfrak{C}^-, \langle\kappa\rangle^{W'}\rangle_{\kappa\in S}$. We show that the function $x \mapsto x'$ is an isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$ — that is, it preserves $\langle\kappa\rangle$, for each $\kappa \in S$.

Claim 2.2. *For any $a \in At_{\mathfrak{B}}$, $\kappa \in S$,*

$$\langle\kappa\rangle a = 1 \iff |a'| \geq \kappa.$$

Proof. For $\Rightarrow$: It is easy to see that if $a \in At_{\mathfrak{B}}$ then $\mathsf{Do}\, a \in At_{\mathfrak{B}}$ and $\mathsf{Rg}\, a \in At_{\mathfrak{B}}$ also hold. If $\langle\kappa\rangle a = 1$ then, by $(\mathrm{b9})_\kappa$, either $\langle\kappa\rangle \mathsf{Do}\, a = 1$ or $\langle\kappa\rangle \mathsf{Rg}\, a = 1$. Say, $\langle\kappa\rangle \mathsf{Do}\, a = 1$. Then $\mathsf{Do}\, a \in E_i$ for some $i < n$ with $\lambda_i \geq \kappa$. Thus $|P_{\mathsf{Do}\, a}| = \lambda_i \geq \kappa$. Further, $\langle q_{\mathsf{Do}\, a}, q_{\mathsf{Do}\, a}\rangle \in \mathsf{Do}\, a$, thus $\langle q_{\mathsf{Do}\, a}, u\rangle \in a$, for some $u \in U$.

(a) $u \in U - Q$. Then $P_{\mathsf{Do}\, a} \times \{u\} \subseteq a'$.

(b) $u = q_e \in Q$, $e \neq \mathsf{Do}\, a$. Then $P_{\mathsf{Do}\, a} \times P_e \subseteq a'$.

(c) $u = q_{\mathsf{Do}\, a}$. Then $\{\langle p, p\rangle : p \in P_{\mathsf{Do}\, a}\} \subseteq a'$.

In all the cases (a)–(c), $|a'| \geq |P_{\mathsf{Do}\, a}| = \lambda_i \geq \kappa$ holds.

For $\Leftarrow$: If $|a'| \geq \kappa$ then, since a is a finite set and κ is infinite, there is some $u \in U'$, some $P_e \subseteq P$ with $|P_e| \geq \kappa$ such that either $P_e \times \{u\} \subseteq a'$ or $\{u\} \times P_e \subseteq a'$. Therefore either $\langle q_e, u\rangle \in a$ or $\langle u, q_e\rangle \in a$ must hold. Thus $q_e \in \mathrm{dom}\, a \cup \mathrm{rng}\, a$, i.e. $\langle q_e, q_e\rangle \in \mathsf{Do}\, a \cup \mathsf{Rg}\, a$, say, $\langle q_e, q_e\rangle \in \mathsf{Do}\, a$. So $\mathsf{Do}\, a = e \in E$. Let $\lambda_i = |P_{\mathsf{Do}\, a}| = |P_e| \geq \kappa$. By the definition of λ_i,

$$1 = \langle\lambda_i\rangle \mathsf{Do}\, a \leq \langle\lambda_i\rangle \mathsf{Do}\, a \vee \langle\lambda_i\rangle \mathsf{Rg}\, a \overset{(\mathrm{b9})_{\lambda_i}}{=} \langle\lambda_i\rangle a \overset{(\mathrm{b10})_{\lambda_i\kappa}}{\leq} \langle\kappa\rangle a. \qquad \square$$

Now, for any $x \in B$, $\kappa \in S$,

$$\langle\kappa\rangle x \overset{(\mathrm{b3})_\kappa}{=} \bigvee_{a \in At_B, a \leq x} \langle\kappa\rangle a\,.$$

Thus, by Claims 2.1 and 2.2,

$$\begin{aligned} \langle\kappa\rangle x = 1 \quad &\Longleftrightarrow \quad (\exists a \in At_{\mathfrak{B}},\ a \leq x)\ |a'| \geq \kappa \\ &\Longleftrightarrow \quad (\text{since } B \text{ is finite and by (1)})\ |x'| \geq \kappa\,. \end{aligned}$$

Since $\mathfrak{C} \in \mathsf{WAC}_S$ holds by definition, this completes the proof of Lemma 2. ■

Lemma 3. *Let H be some class of infinite cardinals. Then every finite, subdirectly irreducible $\mathfrak{A} \in \mathrm{Mod\,Ax}(\mathsf{WAMQ}_H)$ is isomorphic to some $\mathfrak{C} \in \mathsf{WAMQ}_H$.*

Proof. First, we give a characterization of the subdirectly irreducibles.

Claim 3.1. *If $\mathfrak{A} \in \mathrm{Mod\,Ax}(\mathsf{WAMQ}_H)$ is subdirectly irreducible then for all $a, b \in A$,*
(i) *either $\mathsf{M}(a,b) = 0$ or $\mathsf{M}(a,b) = 1$; and*
(ii) *either $\mathsf{Q}_H(a) = 0$ or $\mathsf{Q}_H(a) = 1$.*

Proof. As it was shown in the proof of Claim 1.1, both $\mathsf{rl}_{\mathsf{M}(a,b)}$ and $\mathsf{rl}_{-\mathsf{M}(a,b)}$ are homomorphisms w.r.t. the operations of the language of WAM. They also respect Q_H, by (a13).

Now we show that for any $b \in B$, $\mathsf{rl}_{\mathsf{Q}_H(b)}$ and $\mathsf{rl}_{-\mathsf{Q}_H(b)}$ are homomorphisms. Indeed, they are clearly Boolean homomorphisms, and they respect $\breve{}$ by (a1), (a15); $\circ$ by (a1), (a16), (a17); M by (a1), (a14); Q_H by (a1), (a11), (a18), and (a19). Therefore, if either $\mathsf{M}(a,b) \neq 1$ and $\mathsf{M}(a,b) \neq 0$, or $\mathsf{Q}_H(a) \neq 1$ and $\mathsf{Q}_H(a) \neq 0$ held then there would exist some non-trivial subdirect decomposition of $\mathfrak{A}$. □

Assume $\mathfrak{A} \in \mathrm{ModAx}(\mathsf{WAMQ}_H)$ is finite and subdirectly irreducible. Again, we proceed as in the proof of Lemma 1. The appropriate reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is a finite algebra in WA which, by Theorem AHN, is isomorphic to some $\mathfrak{B}^- \in \mathsf{WA}$ having a finite base. Let $\mathfrak{B} = \langle \mathfrak{B}^-, \mathsf{M}, \mathsf{Q}_H \rangle$ (where M and Q_H denote the operations on B induced by the above isomorphism between the reducts), and let U and W denote the base and the unit of $\mathfrak{B}$, respectively. We define the following blow-up $\mathfrak{C}^-$ of $\mathfrak{B}^-$. Let the sets $At_{\mathfrak{B}}$, E, E_i $(i < n)$, and the binary relations $\sim$ and $>$ be defined as in the proof

of Lemma 1. Now for any $i < n$, take some $e_i \in E_i$ and let t be the 0-1 sequence with $\operatorname{dom} t = n$ defined by $t_i = \mathsf{Q}_H(e_i)$, $i < n$. (This definition of t does not depend on the choice of the representing elements e_i since, by axiom (a21), $x \sim y$ implies $\mathsf{Q}_H(x) = \mathsf{Q}_H(y)$.) Then, for each $s \sqsubseteq t$, there are some $x_0, \ldots, x_{\operatorname{dom} s-1} \in B$ such that $[x_0] < \cdots < [x_{\operatorname{dom} s-1}]$ and $\mathsf{Q}_H(x_j) = s_j$ hold, for all $j < \operatorname{dom} s$. Thus $(\mathrm{no})_s \notin \mathrm{Ax}(\mathsf{WAMQ}_H)$, which implies $s \notin \mathit{Gen}_H$. Therefore, by the definition of Gen_H, $t \notin \mathit{Bad}_H$ follows, i.e. t does occur in H. Choose some occurrence — i.e., choose some infinite cardinals $\lambda_0 < \ldots < \lambda_{n-1}$ such that for any $i < n$, $\lambda_i \in H$ iff $t_i = 1$. Then for each $i < n$, $e \in E_i$, choose the 'blow-up' set P_e such that $|P_e| = \lambda_i$. Let $\mathfrak{C} = \langle \mathfrak{C}^-, \mathsf{M}^{W'}, Q_H^{W'} \rangle$. We prove that the function $x \mapsto x'$ is an isomorphism between $\mathfrak{B}$ and $\mathfrak{C}$. It has already been shown that it is an isomorphism between the Q_H-free reducts. Now we show that

$$(\forall x \in B) \quad \mathsf{Q}_H(x) = 1 \iff |x'| \in H\,. \tag{5}$$

Indeed, for atoms below the identity (5) holds by the definition of the blow-up function $'$. Now let a be an arbitrary atom of B. Then

$$1 = \mathsf{Q}_H(a) \overset{\text{(a20)}}{=} \mathsf{Q}_H(\mathfrak{max}(\mathsf{Do}\, a, \mathsf{Rg}\, a)) \overset{\text{Claim 1.4(ii)–(iii)}}{\iff} |a'| \in H\,.$$

Next, assume that (5) holds for $y, z \in B$. We want to prove it for $y \vee z$. To this end, it is easily seen that, by Claim 1.4(i)–(ii), (2), (3) and (4),

$$(\forall y, z \in B) \quad |\mathfrak{max}(y, z)'| = \mathrm{MAX}_{\aleph_0}(|y'|, |z'|) \tag{6}$$

holds. Now

$$\begin{aligned}
1 = \mathsf{Q}_H(y \vee z) &\overset{\text{(a12)}}{=} \mathsf{Q}_H(\mathfrak{max}(y, z)) \\
&\overset{(6)}{\iff} \begin{cases} \text{either } \mathsf{Q}_H(y) = 1,\ |z'| - |y'| \not\geq \aleph_0 \overset{\text{i.h.}}{\iff} |y'| \in H,\ |z'| - |y'| \not\geq \aleph_0 \\ \text{or } \mathsf{Q}_H(z) = 1,\ |z'| - |y'| \geq \aleph_0 \overset{\text{i.h.}}{\iff} |z'| \in H,\ |z'| - |y'| \geq \aleph_0 \end{cases} \\
&\iff |(y \vee z)'| = |y' \cup z'| \in H\,.
\end{aligned}$$

Since $\mathfrak{C} \in \mathsf{WAMQ}_H$ by definition, this completes the proof of Lemma 3. ■

It remains to show that each subdirectly irreducible model of the axioms has some finite (in fact, effective) approximation (cf. step (3) in the beginning of this section).

Lemma 4. *Any subdirectly irreducible model of* $\mathrm{Ax}(\mathsf{WAM})$ *has some effective approximation.*

Proof. Given $\mathfrak{A} = \langle A, \vee, \wedge, -, 1, 0, \circ, \breve{}, Id, \mathsf{M}\rangle \in \mathrm{Sir\,Mod\,Ax}(\mathsf{WAM})$ and some $X \subseteq_\omega A$, we will construct a finite algebra $\mathfrak{A}_X$ with the following properties.

(i) $X \subseteq A_X \subseteq A$ and $|A_X| \leq 2^{2^{8|X|+2}}$;

(ii) the $\circ$-free reduct of $\mathfrak{A}_X$ is a subalgebra of the $\circ$-free reduct of $\mathfrak{A}$;

(iii) for all $y, z \in X$, if $y \circ z \in X$ then $y \circ z = y \circ^{\mathfrak{A}_X} z$;

(iv) $\mathfrak{A}_X \in \mathrm{Mod\,Ax}(\mathsf{WAM})$.

To this end, we let

$$\begin{aligned} Y &= X \cup \{x^{\smile} : x \in X\} \cup \{Id\} \\ Z &= Y \cup \{-y : y \in Y\} \\ A_X &= \text{Boolean closure of } Z \cup \{\mathsf{Do}\, z : z \in Z\}\,. \end{aligned}$$

We note that similar constructions were used in Henkin [4, 2.5.4] and in Németi [16, Lemma 1.1]. The set A_X is closed under the Booleans and Id, by definition. It is also closed under $\breve{}$, since $\breve{}$ is a Boolean homomorphism and $(\mathsf{Do}\, y)^{\smile} = \mathsf{Do}\, y$ always holds. Finally, it is closed under M, by Claim 1.1. Thus the following definition is sensible. Let

$$\mathfrak{A}_X^- = \langle A_X, \vee, \wedge, -, 1, 0, \circ^{\mathfrak{A}_X}, \breve{}, Id, \rangle \quad \text{and} \quad \mathfrak{A}_X = \langle \mathfrak{A}_X^-, \mathsf{M}\rangle\,,$$

where $\vee, \wedge, -, 1, 0, \breve{}, Id$ and M are the operations of $\mathfrak{A}$ restricted to A_X, and for any $y, z \in A_X$, $y \circ^{\mathfrak{A}_X} z$ is defined as follows:

$$y \circ^{\mathfrak{A}_X} z \stackrel{\text{def}}{=} \bigvee\{a \in A_X : a \text{ is an atom of } \mathfrak{A}_X, \text{ and } a \wedge (y \circ z) \neq 0\}\,.$$

Then properties (i)–(iii) above hold by definition. We show that $\mathfrak{A}_X$ satisfies property (iv) — that is, $\mathfrak{A}_X \in \mathrm{Mod\,Ax}(\mathsf{WAM})$.

Claim 4.1. $\mathfrak{A}_X^- \in \mathsf{WA}$.

Proof. We show that $\mathfrak{A}_X^-$ satisfies the axiomatization of WA, given in section 2.1. $\mathfrak{A}_X^-$ obviously satisfies those axioms which do not involve $\circ$. It is an easy consequence of the definiton of $\circ^{\mathfrak{A}_X}$ that axioms $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$ and $x \circ Id = Id \circ x = x$ hold in $\mathfrak{A}_X^-$. Concerning the cycle law, it holds for atoms of $\mathfrak{A}_X^-$ by the definition of $\circ^{\mathfrak{A}_X}$ and by the cycle law in $\mathfrak{A}$. For arbitrary elements of $\mathfrak{A}_X^-$, one can use the additivity of $\circ^{\mathfrak{A}_X}$. Finally, we show that axiom (wa) holds in $\mathfrak{A}_X^-$. One direction is obvious by the other axioms:

$$\begin{aligned} &(Id \wedge y) \circ^{\mathfrak{A}_X} (1 \circ^{\mathfrak{A}_X} 1) \leq (Id \wedge y) \circ^{\mathfrak{A}_X} 1 = \\ &\quad = ((Id \wedge y) \circ^{\mathfrak{A}_X} Id) \circ^{\mathfrak{A}_X} 1 \leq ((Id \wedge y) \circ^{\mathfrak{A}_X} 1) \circ^{\mathfrak{A}_X} 1\,. \end{aligned}$$

For the other direction, let us collect some basic properties of Do and Rg, which will be used later as well.

Claim 4.1.1. (i) *A_X is closed under taking* Do *and* Rg *— that is, for any* $y \in A_X$, $\mathsf{Do}\, y \in A_X$ *and* $\mathsf{Rg}\, y \in A_X$.

(ii) Do *and* Rg *are the same in* $\mathfrak{A}$ *and in* $\mathfrak{A}_X^-$ *— that is, for any* $y \in A_X$, $\mathsf{Do}\, y = (y \circ^{\mathfrak{A}_X} 1) \wedge Id$ *and* $\mathsf{Rg}\, y = (1 \circ^{\mathfrak{A}_X} y) \wedge Id$.

(iii) *For each atom* a *of* $\mathfrak{A}_X^-$, $\mathsf{Do}\, a$ *is also an atom.*

(iv) *For all atoms* a, b *of* $\mathfrak{A}_X^-$, *if* $a \wedge (b \circ 1) \neq 0$ *then* $\mathsf{Do}\, a = \mathsf{Do}\, b$.

Proof of Claim 4.1.1. For (i): It is easy to check that Do has the following properties.

$$\begin{aligned} \mathsf{Do}(x \vee y) &= \mathsf{Do}\, x \vee \mathsf{Do}\, y \\ \mathsf{Do}(x \wedge y) &= \mathsf{Do}\, x \wedge \mathsf{Do}\, y \\ \mathsf{Do}(x \wedge -\mathsf{Do}\, y) &= (x \wedge Id \wedge -\mathsf{Do}\, y) \vee \mathsf{Do}(x \wedge -Id) \end{aligned}$$

Now for any $a \in A_X$, $a = \bigvee\{Z_i : i \in I_a\}$, where each Z_i $(i \in I_a)$ is a conjunction of some elements of the set $Y \cup \{\mathsf{Do}\, y : y \in Y\} \cup \{-\mathsf{Do}\, y : y \in Y\}$. Therefore, by the above equations, $\mathsf{Do}\, a \in A_X$ and $\mathsf{Rg}\, a = \mathsf{Do}\, a^{\smile} \in A_X$ hold.

For (ii): First, $\mathsf{Do}\, y \leq (y \circ^{\mathfrak{A}_X} 1) \wedge Id$, by the definition of $\circ^{\mathfrak{A}_X}$. On the other hand, take some atom a of $\mathfrak{A}_X^-$ with $a \leq (y \circ^{\mathfrak{A}_X} 1) \wedge Id$. Then

$$0 \neq (y \circ 1) \wedge a = (y \circ 1) \wedge Id \wedge a = \mathsf{Do}\, y \wedge a\,,$$

which implies, by (i), that $a \leq \mathsf{Do}\, y$.

Items (iii) and (iv) can be proved by an easy WA-computation. (We may think of elements of A_X as sets of pairs, since $A_X \subseteq A$, and $\mathfrak{A} \in \operatorname{Mod} \mathrm{Ax}(\mathsf{WAM})$, thus we may assume that the M-free reduct of $\mathfrak{A}$ is in fact in WA.) □

To complete the proof of Claim 4.1, take an atom $a \leq ((Id \wedge y) \circ^{\mathfrak{A}_X} 1) \circ^{\mathfrak{A}_X} 1$. Then $a \wedge (((Id \wedge y) \circ^{\mathfrak{A}_X} 1) \circ 1) \neq 0$, thus there is some atom b such that

$$b \wedge ((Id \wedge y) \circ 1) \neq 0 \quad \text{and} \quad a \wedge (b \circ 1) \neq 0 \tag{7}$$

hold. Therefore, by Claim 4.1.1.(iv), $\mathsf{Do}\, a = \mathsf{Do}\, b$. This and (7) together imply that $a \wedge ((Id \wedge y) \circ 1) \neq 0$, by a simple WA-computation. That is,

$$a \leq (Id \wedge y) \circ^{\mathfrak{A}_X} 1 = (Id \wedge y) \circ^{\mathfrak{A}_X} (Id \circ^{\mathfrak{A}_X} 1) \leq (Id \wedge y) \circ^{\mathfrak{A}_X} (1 \circ^{\mathfrak{A}_X} 1)\,,$$

which proves that $\mathfrak{A}_X^-$ satisfies (wa). □

Finally, we show that $\mathfrak{A}_X$ satisfies the axioms concerning M. Axioms not involving $\circ$ hold in $\mathfrak{A}_X$, since the $\circ$-free reduct of $\mathfrak{A}_X$ is a subalgebra of

that of $\mathfrak{A}$. By Claim 1.1, axioms (a5) and (a6) hold, since $y \circ 0 = 0 \circ y = 0$ holds in $\mathfrak{A}_X$. Finally, $\mathfrak{A}_X$ satisfies (a8) by Claim 4.1.1(ii). This completes the proof of Lemma 4. ■

LEMMA 5. *For any finite set S of infinite cardinals, every subdirectly irreducible model of* $\mathrm{Ax}(\mathsf{WAC}_S)$ *has some effective approximation.*

PROOF. Given some finite set S of infinite cardinals, some subdirectly irreducible algebra $\mathfrak{A} = \langle A, \vee, \wedge, -, 1, 0, \circ, \breve{}, Id, \langle\kappa\rangle\rangle_{\kappa\in S}$ in $\mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAC}_S)$, and some $X \subseteq_\omega A$, let the set A_X and the algebra $\mathfrak{A}_X^- \in \mathsf{WA}$ be defined as in the proof of Lemma 4, and let

$$\mathfrak{A}_X = \langle \mathfrak{A}_X^-, \langle\kappa\rangle\rangle_{\kappa\in S}\,,$$

where, for each $\kappa \in S$, $\langle\kappa\rangle$ is that of $\mathfrak{A}$, restricted to A_X. (A_X is closed under $\langle\kappa\rangle$, by Claim 2.1.) It remains to show that $\mathfrak{A}_X$ satisfies axioms $(\mathrm{b2})_\kappa$–$(\mathrm{b10})_{\kappa\lambda}$, for all $\kappa, \lambda \in S$. Axioms not involving $\circ$ hold in $\mathfrak{A}_X$, since the $\circ$-free reduct of $\mathfrak{A}_X$ is a subalgebra of that of $\mathfrak{A}$. Axioms $(\mathrm{b5})_\kappa$ and $(\mathrm{b6})_\kappa$ hold, by Claim 2.1. $\mathfrak{A}_X$ satisfies $(\mathrm{b10})_{\kappa\lambda}$, by Claim 4.1.1(ii). ■

LEMMA 6. *For any class H of infinite cardinals, every subdirectly irreducible model of* $\mathrm{Ax}(\mathsf{WAMQ}_H)$ *has some effective approximation.*

PROOF. Given some class H of infinite cardinals, some subdirectly irreducible algebra $\mathfrak{A} = \langle A, \vee, \wedge, -, 1, 0, \circ, \breve{}, Id, \mathsf{M}, \mathsf{Q}_H\rangle$ in $\mathrm{Mod}\,\mathrm{Ax}(\mathsf{WAMQ}_H)$, and some $X \subseteq_\omega A$, let the set A_X and the algebra $\mathfrak{A}_X^- \in \mathsf{WA}$ be defined as in the proof of Lemma 4, and let

$$\mathfrak{A}_X = \langle \mathfrak{A}_X^-, \mathsf{M}, \mathsf{Q}_H\rangle\,,$$

where M and Q_H are those of $\mathfrak{A}$, restricted to A_X. (A_X is closed under M and Q_H, by Claim 3.1.) Again, we have to prove that $\mathfrak{A}_X \in \mathrm{Mod}\mathrm{Ax}(\mathsf{WAMQ}_H)$. $\mathfrak{A}_X$ satisfies $\mathrm{Ax}(\mathsf{WAM})$, by the proof of Lemma 4. All the other axioms which do not include $\circ$ hold in $\mathfrak{A}_X$, since the $\circ$-free reduct of $\mathfrak{A}_X$ is a subalgebra of that of $\mathfrak{A}$. Axioms (a16), (a17) hold, by Claim 3.1. Finally, $\mathfrak{A}_X$ satisfies (a12) and (a20), by Claims 3.1 and 4.1.1(ii). ■

3. Lack of finite base property and decidability

It is easy to see that WAM and WAMQ_H do not have the finite base property. Indeed, if $\mathfrak{A}$ is a finite algebra in WAM with some infinite unit then $\mathsf{M}(1,0) = 1$ holds in $\mathfrak{A}$, which cannot hold in an algebra with a finite base. Similarly, if $\aleph_0 \in S$ then WAC_S does not have the finite base property.

As mentioned in section 1, Németi [16] showed that the equational theory of WA is decidable. Actually he proved that already the universal theory of WA is decidable by showing that WA has the following *effective finite algebra property*: there is some recursive function f from the set of universal formulas of the language of WA into ω such that, for each universal formula φ, if φ fails in WA then φ fails in some finite $\mathfrak{A} \in \mathsf{WA}$ with $|A| \leq f(\varphi)$. The theorems below state that our classes also have this property.

THEOREM 4. WAM *has the effective finite algebra property. Therefore the universal (and thus the equational) theory of* WAM *is decidable.*

PROOF. Assume $\mathsf{WAM} \not\models \varphi$, for some universal formula φ. Then, by Theorem 1, there is some $\mathfrak{A} \in \mathrm{Sir}\ \mathrm{ModAx}(\mathsf{WAM})$, and some n-tuple $\bar{a}$ in A with $\mathfrak{A} \not\models \varphi[\bar{a}]$. Let

$$X = \{\tau^{\mathfrak{A}}[\bar{a}] : \tau \text{ is a subterm of } \varphi\}\,.$$

Then, by Lemma 4, there is some recursive function g, and some finite $\mathfrak{A}_X \in \mathrm{ModAx}(\mathsf{WAM}) \overset{\text{Thm. 1}}{=} \mathsf{WAM}$ such that $\mathfrak{A}_X \not\models \varphi[\bar{a}]$ and $|A_X| \leq g(|X|) \leq g(\text{number of subterms of } \varphi)$. This way, the refutable universal formulas are recursively enumerable. The universal formulas which are valid in WAM are also recursively enumerable by Theorem 1, thus the universal theory of WAM is decidable.

In fact, one can describe a decision algorithm as follows. Let g be as above. Given some universal formula φ, first check whether φ is valid in the (finitely many, finite) algebras of the language of WAM, having size $\leq g(\text{number of subterms of } \varphi)$. If φ is valid in these 'small' algebras then $\mathsf{WAM} \models \varphi$ follows by the effective finite algebra property. If we find some small algebra $\mathfrak{A}$ with $\mathfrak{A} \not\models \varphi$ then, by Theorem 1, we can decide whether $\mathfrak{A} \in \mathrm{SPUp}\ \mathsf{WAM}$ holds. If so then we can conclude $\mathsf{WAM} \not\models \varphi$, otherwise we can continue checking whether φ is valid in the small algebras. ∎

Similar arguments prove the following theorems.

THEOREM 5. *For any finite set S of infinite cardinals,* WAC_S *has the effective finite algebra property. Therefore the universal (and thus the equational) theory of* WAC_S *is decidable.*

Concerning finite variants, in case of $S = n$ for some $n \leq \omega$ Mikulás [15] proved that the equational theory of WAC_S is decidable, Marx [11] proved that it is in EXPTIME, and Andréka et al. [1] showed that WAC_S has the finite base property.

THEOREM 6. *For any class H of infinite cardinals,* WAMQ_H *has the effective finite algebra property. Therefore the universal (and thus the equational) theory of* WAMQ_H *is decidable.*

THEOREM 7. *For any class H of infinite cardinals, the universal (and thus the equational) theory of* WAQ_H *is decidable.*

This last theorem is a consequence of Theorem 6 and the fact that

$$\mathsf{WAQ}_H = \{\mathfrak{A} : \mathfrak{A} \text{ is the } \mathsf{M}\text{-free reduct of some } \mathsf{WAMQ}_H\}\,.$$

REMARK 2. (on complexity) The problem whether an equation is valid in any of the classes discussed here is EXPTIME-hard, since already that of WA is such (cf. Marx [11]). However, the proofs above give only a double exponential bound on the size of the finite algebras constructed. It would be interesting to know whether a mosaic-type proof can deliver a better upper bound. Note that neither the difference operator nor nominals are expressible in the above algebras.

QUESTION. (on infinite coordinate-wise counting)
For any cardinal κ and set W, consider the following unary operations $\langle\kappa\rangle_0^W$ and $\langle\kappa\rangle_1^W$ on $\mathcal{P}(W)$. For any $X \subseteq W$, let

$$\begin{aligned}\langle\kappa\rangle_0^W(X) &\stackrel{\text{def}}{=} \{\langle u,v\rangle \in W : |\{z : \langle z,v\rangle \in X\}| \geq \kappa\}, \text{ and}\\ \langle\kappa\rangle_1^W(X) &\stackrel{\text{def}}{=} \{\langle u,v\rangle \in W : |\{z : \langle u,z\rangle \in X\}| \geq \kappa\}\,.\end{aligned}$$

Marx and Mikulás [12] prove, with the help of interpreting a tiling problem, that '$\mathsf{WA} + \{\langle\kappa\rangle_0, \langle\kappa\rangle_1 : \kappa \leq n\}$' has an undecidable equational theory if $n \in \omega$. But what happens for infinite κ's? E.g., is the equational theory of '$\mathsf{WA}+\langle\aleph_0\rangle_0, \langle\aleph_0\rangle_1$' decidable? Or, does the class '$\mathsf{WA}+\langle\aleph_0\rangle_0, \langle\aleph_0\rangle_1$' generate a finitely axiomatizable variety?

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