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SOME ELEMENTS OF THE THEORY OF QUALITATIVE POSSIBILISTIC NETWORKS

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This paper presents some results concerning the qualitative behaviour of possibilistic networks. The behaviour of singly connected networks is analysed, providing the foundations for qualitative reasoning about changes in possibility values in both predictive and evidential directions. The problems inherent in handling multiply connected networks are also discussed, and a possible solution is proposed. The behaviour of qualitative possibilistic networks is compared to qualitative probabilistic networks, and an example of the kind of reasoning that is permitted by the use of these networks is provided.

Keywords: Qualitative behaviour, directed graphs, possibility theory.

1. Introduction

The study of qualitative probabilistic reasoning in networks has become well established, both in the context of planning[1], explanation[2], and engineering design[3]. This paper extends such work by considering qualitative possibilistic reasoning in networks, that is how to determine the qualitative changes that take place when uncertainty values are propagated through directed graphs similar to those studied by Pearl[4] and Lauritzen and Spiegelhalter[5] using possibility theory[6,7].

When we consider the propagation of probability and possibility values through a network there are two operations that are of interest. Firstly we want to determine the prior values of every node in the network from those prior values that are known and the conditional values that relate the nodes of the network together. Secondly we are interested in establishing the new values of the nodes when certain pieces of evidence are discovered. When considering the qualitative behaviour of such networks we are only interested in the second operation, the way in which the values of the nodes change when evidence is obtained, since all prior values are qualitatively equivalent. When an event is observed, the value of the node relating to that event changes, increasing or decreasing. The change propagates through the network, causing the value of other nodes to change, and we can thus determine the effect of the observed event on the nodes in which we are interested. Thus the qualitative analogy of updating with new evidence is propagating qualitative changes in value.

The structure of the paper is as follows. Section 2 discusses the idea of possibilistic networks, which stems from work by Farreny and Prade[8] and Fonck and Straszecka[9]. Then Section 3 introduces the basic ideas behind qualitative possibilistic networks, and uses them to provide an analysis of the propagation of qualitative changes in value in singly connected networks of binary valued nodes. This section includes a discussion of normalisation and compares the behaviour of qualitative possibilistic networks with that of qualitative probabilistic networks and qualitative belief networks[10]. Section 4 extends the approach to consider multiply connected networks and variables with multiple values, and Section 5 gives an example of the kind of reasoning that can be carried out with qualitative probabilistic networks. Section 6 discusses some of the issues that have been raised, and finally Section 7 concludes.

2. Possibilistic Networks

In possibility theory[8] the information available about the value of a single-valued attribute a for a given item x, is represented by a possibility distribution $\Pi_{a(x)}$. This is a mapping from the attribute domain U to $[0,1]$ which restricts the more or less possible values of $a(x)$. The possibility value $\Pi_{a(x)}(u)$ estimates to what extent it is possible that $a(x) = u$. The distribution $\Pi_{a(x)}$ is assumed to be normalised so that $\sup_{u \in U} \Pi_{a(x)}(u) = 1$, and this is satisfied as soon as at least one value in U is considered to be completely possible. The state of total ignorance about the value of $a(x)$ is represented by $\Pi_{a(x)}(u) = 1$, $\forall u \in U$.

To build a possibilistic network[9] we take a set of variables $\mathcal{V} = \{X, Y, \ldots, Z\}$ such that X takes values from $U = \{A_1, \ldots, A_n\}$, Y takes values from $V = \{B_1, \ldots, B_m\}$, and Z takes values from $W = \{C_1, \ldots, C_p\}$, and construct a network based upon the influences between the variables. The variables are represented by the nodes of the network, and the influences between the variables are represented by the links between the nodes. The strength of the influences is represented by the numerical possibility value assigned to the links. Any node N, representing a given variable X, is only connected to those nodes that represent variables that influence X or are influenced by X. Thus the network encodes all the available information about the dependencies between the variables in $\mathcal{V}$, and the strength of those dependencies. If two nodes in a network are not explicitly connected, then the variables that they represent have no direct influence on one another.

Consider a link from a node representing variable X to that representing variable Y. This link represents the information that "if X is A_i then Y is B_i" where $\forall i, A_i \subseteq U$, $B_i \subseteq V$. Now, the strength of this influence is quantified in terms of possibilities so that[11]:

$$\forall u \in U, \forall v \in V, \Pi_{X,Y}(u,v) = \min\left(\Pi_{Y|X}(v,u), \Pi_X(u)\right) \tag{1}$$

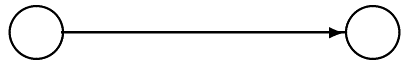

Fig. 1. A simple network

which gives:

$$\forall v \in V, \Pi_Y(v) = \sup_{u \in U} \min \Big(\Pi_{Y|X}(v, u), \Pi_X(u) \Big) \tag{2}$$

Considering only binary valued variables $u \in \{a, \neg a\}$, $v \in \{c, \neg c\}$, as in the network of Fig. 1 we can rewrite these equations as:

$$\Pi(a, c) = \min \Big(\Pi(a \mid c), \Pi(c) \Big) \tag{3}$$

$$\Pi(c) = \sup \Big\{ \min \Big(\Pi(c \mid a), \Pi(a) \Big), \min \Big(\Pi(c \mid \neg a), \Pi(\neg a) \Big) \Big\} \tag{4}$$

The uncertainty attached to the link is represented by the possibility distribution $(\Pi(c \mid a), \Pi(\neg c \mid a)) \in [0, 1]^2$ on the set $\{c, \neg c\}$ in the context of a. In this binary case we have the normalisation condition $\max(\Pi(c \mid a), \Pi(\neg c \mid a)) = 1$. We also have similar information in the context of $\neg a$.

3. Singly Connected Networks

Having established what possibilistic networks are, and how to construct them, we turn to the problem of predicting how values will propagate through them. To do this we start with the simplest possible class of network and then extend our analysis to cover a larger class.

3.1. *Propagating qualitative changes*

When considering how a change in the value of A affects the value of C, in Fig. 1 we find that there are three basic relationships that can hold between them[12]. The possibility of C taking value c is said to *follow* the possibility A taking value a if $\Pi(c)$ increases when $\Pi(a)$ increases, and decreases when $\Pi(a)$ decreases. The possibility of C taking value c is said to *vary inversely with* the possibility of A taking value a if $\Pi(c)$ decreases when $\Pi(a)$ increases, and increases when $\Pi(a)$ decreases. The possibility of C taking value c is said to *be independent of* the possibility of variable A taking value a if $\Pi(c)$ does not change as $\Pi(a)$ increases and decreases.

These relationships may be identified with three possible values of the derivative that relates the values of $\Pi(c)$ and $\Pi(a)$. It is possible for the derivative $\delta\Pi(c)/\delta\Pi(a)$ to be positive, in which case $\Pi(c)$ increases with $\Pi(a)$, negative in which case $\Pi(c)$ varies inversely with $\Pi(a)$, and zero in which case $\Pi(c)$ is independent of $\Pi(a)$. This state of affairs is captured by a statement about the qualitative value of the derivative, which is written as $[\delta\Pi(c)/\delta\Pi(a)]$, so that $[\delta\Pi(c)/\delta\Pi(a)] = [+]$ if $\Pi(c)$ follows $\Pi(a)$. Clearly, if $[\delta\Pi(c)/\delta\Pi(a)] = [-]$ then $\Pi(c)$ varies inversely with $\Pi(a)$ and if $[\delta\Pi(c)/\delta\Pi(a)] = [0]$ then $\Pi(c)$ is independent of $\Pi(a)$. As we shall see, in

Table 1. Qualitative combinator tables

$\oplus$	$[+]$	$[0]$	$[-]$	$[?]$
$[+]$	$[+]$	$[+]$	$[?]$	$[?]$
$[0]$	$[+]$	$[0]$	$[-]$	$[?]$
$[-]$	$[?]$	$[-]$	$[-]$	$[?]$
$[?]$	$[?]$	$[?]$	$[?]$	$[?]$

$\otimes$	$[+]$	$[0]$	$[-]$	$[\uparrow]$	$[\downarrow]$	$[?]$
$[+]$	$[+]$	$[0]$	$[-]$	$[+,0]$	$[0]$	$[?]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[-]$	$[-]$	$[0]$	$[+]$	$[0]$	$[0,-]$	$[?]$
$[?]$	$[?]$	$[0]$	$[?]$	$[+,0]$	$[0,-]$	$[?]$

some situations it may not be possible to tell whether or not a relationship holds, so that, it is only possible to say that $\Pi(c)$ may follow $\Pi(a)$ up, or $\Pi(c)$ may follow $\Pi(a)$ down. These cases are captured by the statements $[\delta\Pi(c)/\delta\Pi(a)] = [\uparrow]$ and $[\delta\Pi(c)/\delta\Pi(a)] = [\downarrow]$, respectively.

The reason for representing changes in terms of qualitative derivatives is that differential calculus may then be used to tell us how to propagate changes in value through networks, since given $\partial x/\partial y$ it is a simple matter to calculate the change Δx from Δy. To determine the change at C in Fig. 1 we have:

$$\Delta\Pi(c) = \Delta\Pi(a) \otimes \left[\frac{\partial\Pi(c)}{\partial\Pi(a)}\right] \oplus \Delta\Pi(\neg a) \otimes \left[\frac{\partial\Pi(c)}{\partial\Pi(\neg a)}\right] \tag{5}$$

$$\Delta\Pi(\neg c) = \Delta\Pi(a) \otimes \left[\frac{\partial\Pi(\neg c)}{\partial\Pi(a)}\right] \oplus \Delta\Pi(\neg a) \otimes \left[\frac{\partial\Pi(\neg c)}{\partial\Pi(\neg a)}\right] \tag{6}$$

where $\oplus$ and $\otimes$ denote qualitative addition and multiplication respectively. These are defined in Table 1. We can express (5) and (6) as a matrix calculation (after Farreny and Prade[8]):

$$\begin{bmatrix} \Delta\Pi(c) \\ \Delta\Pi(\neg c) \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial\Pi(c)}{\partial\Pi(a)}\right] & \left[\frac{\partial\Pi(c)}{\partial\Pi(\neg a)}\right] \\ \left[\frac{\partial\Pi(\neg c)}{\partial\Pi(a)}\right] & \left[\frac{\partial\Pi(\neg c)}{\partial\Pi(\neg a)}\right] \end{bmatrix} \otimes \begin{bmatrix} \Delta\Pi(a) \\ \Delta\Pi(\neg a) \end{bmatrix} \tag{7}$$

3.2. *Simple directed graphs*

Given this background we can start our analysis of possibilistic networks. We will start by considering trees, that is singly connected graphs in which each node is connected to at most one other node. All trees may be constructed from subnetworks of the form of Fig. 1, and so the results of analysing this network will be sufficient to enable us to predict the behaviour of any tree. Writing the graph in Fig. 1 as $A \to C$ we have:

Theorem 3.1. The relation between $\Pi(x)$ and $\Pi(y)$, for all $x \in \{c, \neg c\}$, $y \in \{a, \neg a\}$, for the link $A \to C$ is such that $\Pi(x)$ follows $\Pi(y)$ if $\min\big(\Pi(x \mid y), \Pi(y)\big) > \min\big(\Pi(x \mid \neg y), \Pi(\neg y)\big)$ and $\Pi(y) < \Pi(x \mid y)$. If $\min\big(\Pi(x \mid y), \Pi(y)\big) \leq \min\big(\Pi(x \mid \neg y), \Pi(\neg y)\big)$ and $\Pi(y) < \Pi(x \mid y)$ then $\Pi(x)$ may follow $\Pi(y)$ up if $\Pi(y)$ is increasing, and if $\min\big(\Pi(x \mid y), \Pi(y)\big) > \min\big(\Pi(x \mid \neg y), \Pi(\neg y)\big)$ and $\Pi(y) \geq \Pi(x \mid y)$ then $\Pi(x)$

may follow $\Pi(y)$ down if $\Pi(y)$ is decreasing. Otherwise $\Pi(x)$ is independent of $\Pi(y)$.

Proof. Possibility theory gives $\Pi(c) = \sup\big\{\min\big(\Pi(c \mid a), \Pi(a)\big), \min\big(\pi(c \mid \neg a), \Pi(\neg a)\big)\big\}$. This may not be differentiated, but because possibility theory is essentially qualitative[13], this does not matter. Consider how a small change in $\Pi(a)$ will affect $\Pi(c)$. If $\Pi(a)$ is the value that determines $\Pi(c)$, any change in $\Pi(a)$ will be reflected in $\Pi(c)$. This must happen when $\min\big(\Pi(c \mid a), \Pi(a)\big) > \min\big(\Pi(c \mid \neg a), \Pi(\neg a)\big)$ and $\Pi(a) < \Pi(c \mid a)$. If $\Pi(a)$ is increasing and the second condition does not hold, it may become true at some point, and so the increase may be reflected in $\Pi(c)$. Similar reasoning may be applied when $\Pi(a)$ is decreasing and the first condition is initially false. Thus we can write down the conditions relating $\Pi(c)$ and $\Pi(a)$, while those relating $\Pi(c)$ and $\Pi(\neg a)$ as well as those relating$\Pi(\neg c)$ and $\Pi(a)$ and $\Pi(\neg a)$ may be obtained the same way$\Box$.

Theorem 3.1 allows us to propagate changes from A to C given knowledge of possibilities such as $\Pi(c \mid a)$. Clearly, if we knew all the possibilities of the form $\Pi(a \mid c)$ we could also propagate from C to A. However, we often don't know both sets of values. In a network we usually have predictive values such as $\Pi(c \mid a)$— the values that allow us to tell the possibility of some symptom given the possibility of some disease— since these values are easier to establish than the evidential values such as $\Pi(a|c)$, which tell us the possibility of some disease given the symptom. However, we usually want to reason evidentially from the observation of some symptom to the possibility of a disease, and to do this we must apply the possibilistic version of Bayes' rule[14]. This gives us:

Theorem 3.2. For $A \rightarrow C$ and for all $x \in \{c, \neg c\}$, $y \in \{a, \neg a\}$, if $\Pi(x)$ follows $\Pi(y)$ or $\Pi(x)$ may follow $\Pi(y)$ up, then $\Pi(y)$ may follow $\Pi(x)$ up, and if $\Pi(x)$ may follow $\Pi(y)$ down, or if $\Pi(x)$ is independent of $\Pi(y)$ then $\Pi(y)$ may follow $\Pi(x)$ down.

Proof. To discover how $\Pi(a)$ varies with $\Pi(c)$ we must, by Theorem 3.1, establish whether $\min(\Pi(a \mid c), \Pi(c)) > \min(\Pi(a \mid \neg c), (\neg c))$ and $\Pi(c) < \Pi(a \mid c)$.

Now, if $\Pi(c)$ follows $\Pi(a)$ then $\Pi(c) = \Pi(a)$ and since $\Pi(c)$ follows $\Pi(a)$, $\Pi(a) < \Pi(c \mid a)$. The possibilistic version of Bayes' rule tells us that $\min((\Pi(a \mid c), \Pi(c)) > \min((\Pi(c \mid a), \Pi(a))$ so $\Pi(c) < \Pi(a \mid c)$ and the second condition for $\Pi(a)$ following $\Pi(c)$ holds. Furthermore, $\Pi(c) < \Pi(c \mid a) \leq 1$, so that $\Pi(c) < 1$. Thus normalisation ensures that $\Pi(\neg c) = 1$. Applying possibilistic Bayes' rule again gives $\min((\Pi(a \mid \neg c), \Pi(\neg c)) > \min((\Pi(\neg c \mid a), \Pi(\neg a))$ which means that $\Pi(a \mid \neg c) = min((\Pi(\neg c \mid a), \Pi(a)) \leq \Pi(a)$, and so $\Pi(a \mid \neg c) \leq \Pi(c)$. Thus the first condition for $\Pi(a)$ following $\Pi(c)$ does not necessarily hold, and so we can only determine that $\Pi(a)$ may follow $\Pi(c)$ up.

If $\Pi(c)$ may follow $\Pi(a)$ up then we know that $\Pi(a) < \Pi(c \mid a)$ and $\Pi(a) \leq \min(\Pi(c \mid \neg a), \Pi(\neg a))$. From the first of these $\Pi(a) < 1$, so that $\Pi(\neg a) = 1$, so from the second $\Pi(a) \leq \Pi(c \mid \neg a)$. The possibilistic version of Bayes' rule tells us that $\min(\Pi(c \mid a), \Pi(a)) = \min(\Pi(a \mid c), \Pi(c))$. Since $\Pi(a) < \Pi(c \mid a)$, we can say that $\Pi(a) = \min(\Pi(a \mid c), \Pi(c))$. The possibilistic version of Bayes' rule also gives $\min(\Pi(c \mid \neg a), \Pi(\neg a)) = \min(\Pi(\neg a \mid c), \Pi(c))$. Since $\Pi(\neg a) = 1$ we know that $\Pi(a) \leq \Pi(c \mid \neg a) = \min(\Pi(\neg a \mid c), \Pi(c))$. Thus $\min(\Pi(a|c), \Pi(c)) \geq \min(\Pi(\neg a \mid c), \Pi(c))$. There are four possible ways in which this inequality may be true; (i) $\Pi(c) = \Pi(a|c) = \Pi(\neg a \mid c)$, (ii) $\Pi(c) < \Pi(a \mid c) \leq (\neg a \mid c)$, (iii) $\Pi(c) > \Pi(a|c) \geq \Pi(\neg a \mid c)$, and (iv) $\Pi(\neg a \mid c) > \Pi(c) > \Pi(a \mid c)$. In the first case all must be 1, since $\max((\Pi(a \mid c), \Pi(\neg a \mid c)) = 1$, so that $\Pi(c) = \Pi(a \mid c) = 1$. However, this is impossible since $\Pi(a) \neq 1$, and $\Pi(a) = \min(\Pi(a \mid c), \Pi(c))$. In the second case, $\Pi(c) < \Pi(a|c)$ which satisfies the first condition for $\Pi(a)$ following $\Pi(c)$. Since this also forces $\Pi(\neg c) = 1$, possibilistic Bayes' rule gives $\Pi(a|\neg c) = \min(\Pi(\neg c \mid a), \Pi(a))$. Thus $\Pi(a) \geq \Pi(a \mid \neg c)$ and $\Pi(c) \geq \Pi(a)$ since $\Pi(a) = \min(\Pi(a \mid c), \Pi(c))$. Thus we can be sure that $\Pi(c) \geq (a \mid \neg c)$ so that the second condition on $\Pi(a)$ following $\Pi(c)$ will only definitely be satisfied if $\Pi(c)$ increases. In the third case, $\Pi(c) > \Pi(a \mid c)$, so that $\Pi(\neg a \mid c) = 1$ by normalisation, but $\Pi(c) > \Pi(\neg a \mid c)$, which is impossible. In the fourth case, once again $\Pi(c) > \Pi(a|c)$ so that $\Pi(\neg a \mid c) = 1$. This also means that the second condition on $\Pi(a)$ following $\Pi(c)$ is violated. Furthermore we know that $\Pi(c) < \Pi(\neg a \mid c)$ so that $\Pi(\neg c) = 1$. Thus the first condition on $\Pi(a)$ following $\Pi(c)$ becomes $\Pi(a \mid c) > \Pi(a \mid \neg c)$. Now, the possibilistic version of Bayes' rule says that $\min(\Pi(a \mid c), \Pi(c)) = \min(\Pi(c \mid a), \Pi(a))$. Since $\Pi(c) > \Pi(a \mid c)$ and $\Pi(a) < \Pi(c \mid a)$ we have $\Pi(a) = \Pi(a \mid c)$. Possibilistic Bayes' rule also gives $\min(\Pi(a|\neg c), \Pi(\neg c)) = \min(\Pi(\neg c \mid a), \Pi(a))$ which means that $\Pi(a) \geq \Pi(a \mid \neg c)$, thus $\Pi(a \mid c) \geq \Pi(a \mid \neg c)$ and $\Pi(a)$ is independent of $\Pi(c)$. Thus when $\Pi(c)$ may follow $\Pi(a)$ up, $\Pi(a)$ either may follow $\Pi(c)$ up or is independent of it, which is equivalent to saying that $\Pi(a)$ may follow $\Pi(c)$ up.

If $\Pi(c)$ may follow $\Pi(a)$ down, then $\Pi(a) \geq \Pi(c \mid a)$ and $\Pi(c \mid a) > \min(\Pi(c \mid \neg a), \Pi(\neg a))$. From this, and the definition of $\Pi(c)$ in terms of $\Pi(a)$, $\Pi(\neg a)$, $\Pi(c|a)$ and $\Pi(c \mid \neg a)$, it is clear that $\Pi(c) = \Pi(c \mid a)$, and thus from the possibilistic version of Bayes' rule, $\min(\Pi(a \mid c), \Pi(c)) = \min(\Pi(c \mid a), \Pi(a))$, it is obvious that $\Pi(c) \leq \Pi(a \mid c)$. And so the second condition for $\Pi(c)$ following $\Pi(a)$ is violated. Thus (again using possibilistic Bayes' rule), $\Pi(a \mid c) > \min(\Pi(\neg a \mid c), \Pi(c))$. Now, since $\Pi(c|a) \leq \Pi(a)$ and $\max(\Pi(c \mid a), \Pi(\neg c \mid a)) = 1$, we have four different possible relationships between $\Pi(a)$, and $\Pi(\neg c \mid a)$ from which we can determine $\Pi(a \mid \neg c)$, (i) $\Pi(\neg c \mid a) < \Pi(c \mid a) = 1 = \Pi(a)$, (ii) $\Pi(\neg c \mid a) = \Pi(c \mid a) = \Pi(c) = 1$, (iii) $1 = \Pi(\neg c \mid a) > \Pi(a) > \Pi(c \mid a)$ and (iv) $1 = \Pi(\neg c \mid a) > \Pi(a) = \Pi(c \mid a)$. In the first case, $\min(\Pi(\neg c \mid a), \Pi(a)) = \Pi(\neg c \mid a) = \min(\Pi(a \mid \neg c), \Pi(\neg c))$. Thus $\Pi(c \mid a) > \min(\Pi(a \mid \neg c), \Pi(\neg c))$ and so $\Pi(a)$ may follow $\Pi(c)$ down. In the second case, $\Pi(c|a) = \min(\Pi(a \mid \neg c), \Pi(\neg c))$.Thus $\min(\Pi(a \mid c), \Pi(c)) = \min(\Pi(a \mid \neg c), \Pi(\neg c))$ and $\Pi(a)$ is independent of $\Pi(c)$. In the third case $\min(\Pi(\neg c \mid a), \Pi(a)) = \Pi(a)$, thus $\Pi(\neg a) = 1$ and $\Pi(c \mid a) < \min(\Pi(a \mid \neg c), \Pi(\neg c))$ and $\Pi(a)$ is independent of $\Pi(c)$. In the fourth case, again $\min(\Pi(\neg c \mid a), \Pi(a)) = \Pi(a)$, only this time $\Pi(c|a) = \min(\Pi(a \mid \neg c), \Pi(\neg c))$ which does not change the fact that $\Pi(a)$ is independent of $\Pi(c)$. Thus, overall, when $\Pi(c)$ may follow $\Pi(a)$ down, $\Pi(a)$ may follow $\Pi(c)$ down.

Finally, if $\Pi(a)$ is independent of $\Pi(c)$, then $\Pi(a) \geq \Pi(c \mid a)$ and $\Pi(c|a) \leq \min(\Pi(c \mid \neg a), \Pi(\neg a))$. In addition, the definition of $\Pi(c)$ tells us that $\Pi(c) \geq \Pi(c \mid a)$. Now, possibilistic Bayes' rule gives $\min(\Pi(c \mid a), \Pi(a)) = \min(\Pi(a \mid c), \Pi(c))$ which means that $\min(\Pi(a \mid c), \Pi(c)) = \Pi(c \mid a)$ since $\Pi(c \mid a)$ is always at least as small as $\Pi(a)$. Since $\Pi(c) \geq \Pi(c \mid a)$ it follows that $\Pi(c) \geq \Pi(a \mid c)$ and the first condition on $\Pi(a)$ following $\Pi(c)$ is false. To verify the second condition, we need to establish the relative magnitudes of $\Pi(a, c)$ and $\Pi(a, \neg c)$. If $\Pi(a|c) = 1$ then $\Pi(c) = 1$ and $\Pi(a, c) \geq \Pi(a, \neg c)$ and $\Pi(a)$ may follow $\Pi(c)$ down. If $\Pi(a|c) < 1$, then $\Pi(c) \leq 1$, and $\Pi(a, c)$ might be less than $\Pi(a, \neg c)$ so that $\Pi(a)$ might follow $\Pi(c)$ down, or be independent of it. Thus, overall, $\Pi(a)$ may follow $\Pi(c)$ down. Similar arguments for all $x \in \{c, \neg c\}$ and $y \in \{a, \neg a\}$ complete the proof□.

Having established these two theorems we have completely analysed the network in Fig. 1. In this network the change at C depends only on the change at A, and the change at A depends only on the change at C. Now, differential calculus tells us that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$ so the behaviours of networks such as that in Fig. 1 may be composed. Thus we can predict how qualitative changes in possibility are propagated in any network which is composed of networks of the form of that in Fig. 1 and it is easy to see that this means we can propagate values in any network in which every node has at most a single parent.

3.3. *Normalisation*

In possibility theory normalisation requires that we have $\max(\Pi(a), \Pi(\neg a)) = 1$, ensuring that at least one of $\Pi(a)$ and $\Pi(\neg a)$ is 1. If $\Pi(a)$ is initially 1 and if it decreases, then $\Pi(\neg a)$ must increase to 1 unless, of course, it already is 1. Similarly if $\Pi(\neg a)$ is initially 1 then any change in its value must be accompanied by $\Pi(a)$ becoming 1. Otherwise changes in $\Pi(a)$ and $\Pi(\neg a)$ are unrestricted. If $\Pi(a)$ is 1 and does not change, $\Pi(\neg a)$ may increase, decrease or not change, and if $\Pi(\neg a)$ is 1 and does not change, $\Pi(a)$ may increase, decrease or not change. This may be summarised by:

$\Pi(a) = 1$	If	$\Delta\Pi(a) = [0]$	Then	$\Delta\Pi(\neg a) = [?]$
	If	$\Delta\Pi(a) = [-]$	Then	$\Delta\Pi(\neg a) = [+, 0]$
$\Pi(a) \neq 1$	If	$\Delta\Pi(a) = [+]$	Then	$\Delta\Pi(\neg a) = [0, -]$
	If	$\Delta\Pi(a) = [0]$	Then	$\Delta\Pi(\neg a) = [0]$
	If	$\Delta\Pi(a) = [-]$	Then	$\Delta\Pi(\neg a) = [0]$

Furthermore, in the network of Fig. 1, for any $\Pi(a)$, $\left[\frac{\partial \Pi(c)}{\partial \Pi(a)}\right]$ can be $[+]$, $[\uparrow]$, $[\downarrow]$ or $[0]$ so that $\Pi(c)$ may follow $\Pi(a)$ up, down or both up and down, or be independent of it, while:

$\Pi(c) = 1$	If	$\Delta\Pi(c) = [0]$	Then	$\Delta\Pi(\neg c) = [?]$
	If	$\Delta\Pi(c) = [-]$	Then	$\Delta\Pi(\neg c) = [+, 0]$

$\Pi(c) \neq 1$	If	$\Delta\Pi(c) = [+]$	Then	$\Delta\Pi(\neg c) = [0, -]$
	If	$\Delta\Pi(c) = [0]$	Then	$\Delta\Pi(\neg c) = [0]$
	If	$\Delta\Pi(c) = [-]$	Then	$\Delta\Pi(\neg c) = [0]$

These results summarise the behaviour of a possibilistic network in terms of the kinds of qualitative change that may be propagated across it. That is, the behaviours given are all those that are possible— for a given set of conditional values, a particular type of propagation will take place.

It is constructive to compare the results with similar results for probabilistic networks[12] and belief networks based upon evidence theory[10]. Since probability theory has the strong normalisation condition $p(a) + p(\neg a) = 1$, the relationship between $p(a)$ and $p(\neg a)$ is more constrained than that between $\Pi(a)$ and $\Pi(\neg a)$:

$p(a) = 1$	If	$\Delta p(a) = [0]$	Then	$\Delta p(\neg a) = [0]$
	If	$\Delta p(a) = [-]$	Then	$\Delta p(\neg a) = [+]$
$p(a) \neq 1$	If	$\Delta p(a) = [+]$	Then	$\Delta p(\neg a) = [-]$
	If	$\Delta p(a) = [0]$	Then	$\Delta p(\neg a) = [0]$
	If	$\Delta p(a) = [-]$	Then	$\Delta p(\neg a) = [+]$

For any value of $p(a)$, either $\left[\frac{\partial p(c)}{\partial p(a)}\right] = [+]$, or $\left[\frac{\partial p(c)}{\partial p(a)}\right] = [-]$ Thus $p(c)$ either follows $p(a)$ or varies inversely with it, and changes in $p(c)$ are bound to those in $p(\neg c)$ in the same way that those in $p(a)$ are bound to those in $p(\neg a)$:

$p(c) = 1$	If	$\Delta p(c) = [0]$	Then	$\Delta p(\neg c) = [0]$
	If	$\Delta p(c) = [-]$	Then	$\Delta p(\neg c) = [+]$
$p(c) \neq 1$	If	$\Delta p(c) = [+]$	Then	$\Delta p(\neg c) = [-]$
	If	$\Delta p(c) = [0]$	Then	$\Delta p(\neg c) = [0]$
	If	$\Delta p(c) = [-]$	Then	$\Delta p(\neg c) = [+]$

Thus if $p(c)$ follows $p(a)$, $p(\neg c)$ varies inversely with $p(a)$, and if $p(c)$ varies inversely with $p(a)$ then $p(\neg c)$ follows $p(a)$. Normalisation also ensures that if $p(c)$ follows $p(a)$ it will vary inversely with $p(\neg a)$ and if $p(c)$ varies inversely with $p(a)$ it will follow $p(\neg a)$.

Evidence theory, which only has the very weak normalisation condition that $bel(a) + bel(\neg a) \leq 1$, has less constrained behaviour than either probability or possibility theories. Indeed, when using Dempster's rule[15] to combine mass assignments, there are no constraints on the possible relationships between $bel(a)$ and $bel(\neg a)$:

$bel(a) = 1$	If	$\Delta bel(a) = [0]$	Then	$\Delta bel(\neg a) = [?]$
	If	$\Delta bel(a) = [-]$	Then	$\Delta bel(\neg a) = [?]$
$bel(a) \neq 1$	If	$\Delta bel(a) = [+]$	Then	$\Delta bel(\neg a) = [?]$
	If	$\Delta bel(a) = [0]$	Then	$\Delta bel(\neg a) = [?]$
	If	$\Delta bel(a) = [-]$	Then	$\Delta bel(\neg a) = [?]$

There are also no constraints on the relationship that may hold between $bel(a)$

and $bel(c)$ since for any $bel(a)$, $\left[\frac{\partial bel(c)}{\partial bel(a)}\right]$ can be $[+]$, $[0]$, or $[-]$, and there are no constraints on the possible relationship between $bel(c)$ and $bel(\neg c)$:

$$\begin{array}{llllll} bel(c)=1 & \text{If} & \Delta bel(c)=[0] & \text{Then} & \Delta bel(\neg c)=[?] \\ & \text{If} & \Delta bel(c)=[-] & \text{Then} & \Delta bel(\neg c)=[?] \\ bel(c)\neq 1 & \text{If} & \Delta bel(c)=[+] & \text{Then} & \Delta bel(\neg c)=[?] \\ & \text{If} & \Delta bel(c)=[0] & \text{Then} & \Delta bel(\neg c)=[?] \\ & \text{If} & \Delta bel(c)=[-] & \text{Then} & \Delta bel(\neg c)=[?] \end{array}$$

Thus for a given change in $bel(a)$ it is possible to have any change in $bel(c)$ and any change in $bel(\neg a)$ and $bel(\neg c)$. However, using other rules of combination in evidence theory, such as Smets' disjunctive rule[16] alters the behaviour making it more restrictive[10,12].

Similar differences in behaviour between formalisms occur when we consider reversing the link between A and C in Fig. 1. In probability theory the link behaves the same[12] when values are propagated from C to A as when they are propagated from A to C. Thus if $p(c)$ follows $p(a)$ then $p(a)$ follows $p(c)$, and if $p(c)$ varies inversely with $p(a)$ then $p(a)$ varies inversely with $p(c)$. As we have seen above, possibility theory prevents us from making such clear cut predictions so that we are only able to say that $\Pi(a)$ may follow $\Pi(c)$ up if $\Pi(c)$ follows $\Pi(a)$, or may follow $\Pi(a)$ up, and $\Pi(a)$ may follow $\Pi(c)$ down if $\Pi(c)$ is independent of $\Pi(a)$ or may follow $\Pi(a)$ down. When reversing the link, evidence theory is more constrained than either possibility or probability theory[12] since $bel(a)$ always follows $bel(c)$. This seems to be a direct consequence of using the disjunctive rule of combination in the derivation of the generalisation of Bayes' rule to evidence theory[16].

While the decision about which quantitative formalism is of most use in a particular situation should be made on the basis of the semantics of the different formalisms, this comparison may prove useful when choosing which qualitative formalism to use. It makes clear the fact that when using probability theory a change in the value of one proposition is accompanied by an opposite change in the value of its negation. It also points out that in probability theory it is easy to have the simple network of Fig. 1 invert the change it propagates so that an increase in $p(a)$ becomes a decrease in $p(c)$— a behaviour that is not easy to capture in possibility theory. The comparison also reveals that, unlike possibility theory, probability theory cannot block the effect of $p(a)$ on $p(c)$, and this may bear on its effectiveness. Furthermore, it is clear that evidence theory qualitatively subsumes probability and possibility theory, as one might expect since it is a generalisation of them both.

3.4. *More complex networks*

As discussed above, the analysis carried out in Section 3.2 allows us to predict how qualitative changes in certainty value will be propagated in a simple link between two nodes, and thus in any network in which every node has at most a single parent. We now extend these results to enable us to cope with networks in which

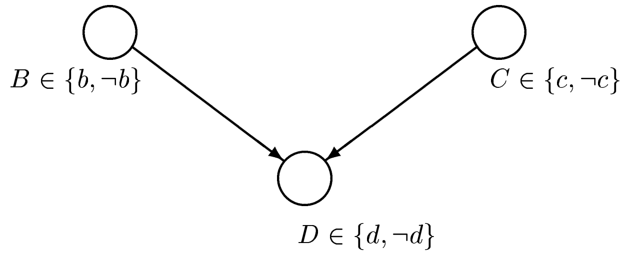

Fig. 2. A more complex network

nodes may have more than one parent. To do this we consider the qualitative effect of two converging links such as those in Fig. 2. Since we are only dealing with singly connected networks, B and C are independent and the overall effect at D is determined by:

$$\begin{bmatrix} \Delta\Pi(d) \\ \Delta\Pi(\neg d) \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial\Pi(d)}{\partial\Pi(b)}\right] & \left[\frac{\partial\Pi(d)}{\partial\Pi(\neg b)}\right] \\ \left[\frac{\partial\Pi(\neg d)}{\partial\Pi(b)}\right] & \left[\frac{\partial\Pi(\neg d)}{\partial\Pi(\neg b)}\right] \end{bmatrix} \otimes \begin{bmatrix} \Delta\Pi(b) \\ \Delta\Pi(\neg b) \end{bmatrix} \oplus \begin{bmatrix} \left[\frac{\partial\Pi(d)}{\partial\Pi(c)}\right] & \left[\frac{\partial\Pi(d)}{\partial\Pi(\neg c)}\right] \\ \left[\frac{\partial\Pi(\neg d)}{\partial\Pi(c)}\right] & \left[\frac{\partial\Pi(\neg d)}{\partial\Pi(\neg c)}\right] \end{bmatrix} \otimes \begin{bmatrix} \Delta\Pi(c) \\ \Delta\Pi(\neg c) \end{bmatrix} \tag{8}$$

When determining how changes are propagated across this kind of network, we have a similar result to that for the simple link, namely:

Theorem 3.3. The relation between $\Pi(x)$, $\Pi(y)$ and $\Pi(z)$, for all $x \in \{b, \neg b\}$, $y \in \{c, \neg c\}$, $z \in \{d, \neg d\}$ for the link $B\&C \rightarrow D$ is such that:
(1) $\Pi(z)$ follows $\Pi(x)$ iff $\Pi(x,y,z) > \sup\big[\Pi(\neg x,y,z), \Pi(x,\neg y,z), \Pi(\neg x,\neg y,z)\big]$ and $\Pi(x) < \min\big(\Pi(z \mid x,y), \Pi(y)\big)$, or $\Pi(x,\neg y,z) > \sup\big[\Pi(x,y,z), \Pi(\neg x,y,z), \Pi(\neg x, \neg y,z)\big]$ and $\Pi(x) < \min\big(\Pi(z \mid x,\neg y), \Pi(\neg y)\big)$.
(2) $\Pi(z)$ may follow $\Pi(x)$ up iff $\Pi(x,y,z) \leq \sup\big[\Pi(\neg x,y,z), \Pi(x,\neg y,z), \Pi(\neg x, \neg y,z)\big]$ and $\Pi(x) < \min\big(\Pi(z \mid x,y), \Pi(y)\big)$, or $\Pi(x,\neg y,z) \leq sup\big[\Pi(x,y,z), \Pi(\neg x,y, z), \Pi(\neg x,\neg y,z)\big]$ and $\Pi(x) < \min\big(\Pi(z \mid x,\neg y), \Pi(\neg y)\big)$.
(3) $\Pi(z)$ may follow $\Pi(x)$ down iff $\Pi(x,y,z) > \sup\big[\Pi(\neg x,y,z), \Pi(x,\neg y,z), \Pi(\neg x, \neg y,z)\big]$ and $\Pi(x) \geq \min\big(\Pi(z \mid x,y), \Pi(y)\big)$, or $\Pi(x,\neg y,z) > \sup\big[\Pi(x,y,z), \Pi(\neg x,y, z), \Pi(\neg x,\neg y,z)\big]$ and $\Pi(x) \geq \min\big(\Pi(z \mid x,\neg y), \Pi(\neg y)\big)$.
(4) Otherwise $\Pi(z)$ is independent of $\Pi(x)$.

Proof. As for Theorem 3.1 we can use the qualitative nature of possibility theory to give the result may be determined directly from careful consideration of $\Pi(d) = \sup_{x \in \{b,\neg b\}, y \in \{c,\neg c\}} \Pi(x,y,z)$ and $\Pi(x,y,z) = \min\big(\Pi(z \mid x,y), \Pi(x), \Pi(y)\big)$, the formulae which determine the possibility of d from that of b, $\neg b$, c and $\neg c$$\square$.

Theorem 3.1 allows us to predict how changes in possibility can be propagated to a node from two parents, and it is clear that similar results can be obtained for

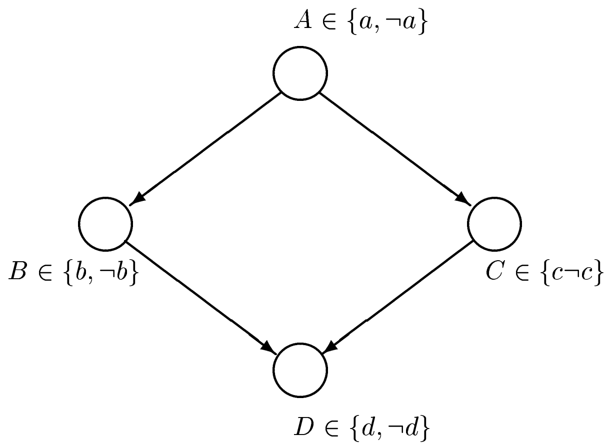

Fig. 3. A loop with four nodes

any number of parents. The only assumption made in the derivation was that the possibility values of the parents are conditionally independent. Now, the differential calculus tells us that changes in value are calculated by $\Delta z = \Delta x \cdot \frac{\partial z}{\partial x} + \Delta y \cdot \frac{\partial z}{\partial y}$, provided that x is not a function of y. Thus we can clearly use the results of this section to propagate qualitative changes in possibility and through any network in which the parents of any node are conditionally independent, that is through any singly connected network.

4. Multiply Connected Networks and More

Although the results presented so far enable us to predict how changes in possibility will be propagated through a large class of networks, we are still not equipped to predict how changes will be propagated in every possible network. This section addresses some of the outstanding problems.

4.1. *From singly to multiply connected networks*

The analysis carried out in Section 3.4 made the explicit assumption that B and C were conditionally independent so that $\Pi(b, c) = \min\big(\Pi(b), \Pi(c)\big)$. This assumption falls apart for multiply connected networks such as those of Fig. 3 where B and C are not conditionally independent when A is not known to be true. To handle this case correctly one should take account of the dependency by writing $\Pi(b, c) = \sup\big(\Pi(a, b, c), \Pi(\neg a, b, c)\big)$. Now, it is clear that it is possible to take any loop and perform a qualitative analysis upon it to establish how qualitative changes in possibility are propagated through it. However, there is no easy way to generalise such an analysis from that of a four node loop like that in Fig. 3 to loops with different numbers of nodes. The qualitative analysis is tied to a particular topology.

However, in qualitative probability and evidence theories[12] it is possible to handle the network of Fig. 3 as if it were two simple causal links combined with a multi-parent link, as in Fig. 4, ignoring the dependency between B and C, and pro-

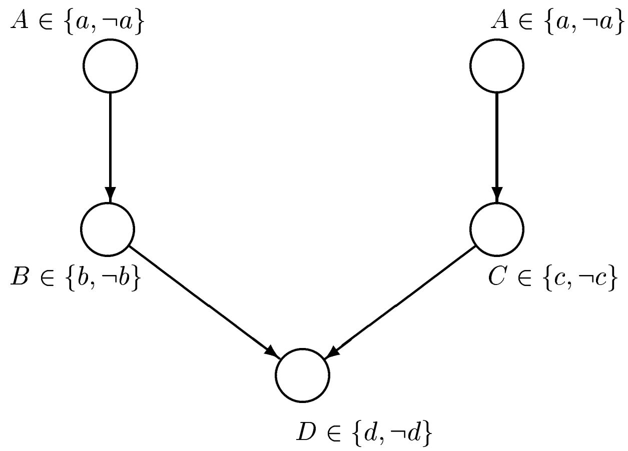

Fig. 4. A naïve view of the loop with four nodes

viding a method of handling loops that can easily be extended to different topologies. Despite the fact that such a naïve approach is incorrect according to the underlying quantitative theory, it does not generate qualitatively incorrect answers. This is due to the fact that ignoring the dependency often only alters the magnitude of the change in value at D rather than the direction of the change (a fact that is ignored by the qualitative analysis), or calculates the change at D to be [?] rather than, say, [+]. Since saying the change at D is [?] is shorthand for the statement "The change at D could be [+], [0] or [−]", this result of the naïve approach is not *incompatible* with the result of the correct approach, and if the naïve approach generates predictions which are never incompatible with the correct approach we say that it is *safe*.

4.2. *Multiply connected networks*

Since the naïve approach is safe in probability and evidence theories, it is worth investigating whether it is safe in possibility theory. To do this we consider propagating a change in possibility from A to D in the network of Fig. 3, which we will refer to as the $\Diamond$-network, comparing the results obtained by the correct and naïve approaches. Disappointingly we have the following result:

Theorem 4.1. It is not safe to use the naïve approach to propagate qualitative values in the $\Diamond$-network using possibility theory.

Proof. For the $\Diamond$-network possibility theory gives us $\Pi(d) = \sup_{b \in \{b, \neg b\}, c \in \{c, \neg c\}} \big\{ \min \big(\Pi(d \mid b, c), \Pi(b, c)\big) \big\}$. Now, in the correct approach when B and C are known to not be independent, $\Pi(b, c) = \sup \big\{\Pi(a, b, c), \Pi(\neg a, b, c)\big\}$ which is equal to $\sup \big\{ \min \big(\Pi(b \mid a), \Pi(c \mid a), \Pi(a)\big), \min \big(\Pi(b \mid \neg a), \Pi(c \mid \neg a), \Pi(\neg a)\big) \big\}$. We can determine the conditions under which $\Pi(d)$ follows $\Pi(a)$ by inspection, and we learn that they are, for any $c \in \{c, \neg c\}$, and $b \in \{b, \neg b\}$; $\Pi(a) < \min \big(\Pi(b \mid a), \Pi(c \mid a)\big)$

(1), $\Pi(a,b,c) > \Pi(\neg a,b,c)$ (2), $\Pi(d \mid b,c) > \sup\{\Pi(a,b,c),\Pi(\neg a,b,c)\}$(3), and $\Pi(a,b,c,d) > \sup_{b\in\{b,\neg b\},c\in\{c,\neg c\}} \Pi(\neg a,b,c,d)$(4). If all conditions hold then $\Pi(d)$ follows $\Pi(a)$, if (1) and (3) hold but (2) and (4) don't then $\Pi(d)$ may follow $\Pi(a)$ up, and if (2) and (4) hold but (1) and (3) don't then $\Pi(d)$ may follow $\Pi(a)$ down. Otherwise $\Pi(d)$ is independent of $\Pi(a)$. If we use the naïve approach we have the possibility at D as before,$\Pi(d) = \sup_{b\in\{b,\neg b\},c\in\{c,\neg c\}} \{\min(\Pi(d \mid b,c),\Pi(b,c))\}$, but $\Pi(b,c) = \min(\Pi(b),\Pi(c))$ and $\Pi(b) = \sup\{\min(\Pi(b \mid a),\Pi(a)),\min(\Pi(b \mid \neg a),\Pi(\neg a))\}$ so that the conditions on $\Pi(d)$ following $\Pi(a)$ are, for any $c \in \{c,\neg c\}$, and $b \in \{b,\neg b\}$; $\Pi(a) < \Pi(b \mid a)$ (1′), $\Pi(a,b) > \Pi(\neg a,b)$ (2′), and $\Pi(b) < \min(\Pi(d \mid b,c),\Pi(c))$ (3′), as well as $\Pi(b,c,d) > \sup_{b\in\{b,\neg b\},c\in\{c,\neg c\}} \Pi(\neg b,c,d)$ (4′). Similar conditions hold for conditionals involving c. If all conditions hold then $\Pi(d)$ follows $\Pi(a)$, if (1′) and (3′) hold but (2′) and (4′) don't then $\Pi(d)$ may follow $\Pi(a)$ up, and if (2′) and (4′) hold but (1′) and (3′) don't then $\Pi(d)$ may follow $\Pi(a)$ down. If we have $\Pi(a) = 0.6$, $\Pi(\neg a) = 1$, $\Pi(b \mid a) = 0.8$, $\Pi(b \mid \neg a) = 0.8$, $\Pi(c \mid a) = 0.5$, $\Pi(c \mid \neg a)$ and $\Pi(d \mid b,c) = 0.9$, then (1′) and (3′) hold while (1), (2) and (2′) don't. Thus the naïve method tells us that $\Pi(d)$ may follow $\Pi(a)$ up, when the exact method tells us $\Pi(d)$ is independent of $\Pi(a)$ and the naïve method is thus unsafe□.

This problem may be related to that reported by Cano *et al.*[17] where propagation of possibility values around a loop was found to be difficult as a result of the idempotence of the function used for combination. The upshot of Theorem 4.1 is that the propagation of qualitative changes around loops in possibility theory must take into account the dependencies between the parents of the node at the base of the loop. The theory, however, does offer another approach to handling loops. It is straightforward to write down the conditions under which $\Pi(d)$ varies with $\Pi(a)$:

Theorem 4.2. In the $\diamondsuit$-network $\Pi(d)$ follows $\Pi(a)$ when, for any $c \in \{c,\neg c\}$ and $b \in \{b,\neg b\}$ (1) $\Pi(a) < \min(\Pi(b \mid a),\Pi(c \mid a))$, (2) $\Pi(a,b,c) > \Pi(\neg a,b,c)$, (3) $\Pi(d \mid b,c) > \sup\{\Pi(a,b,c),\Pi(\neg a,b,c)\}$, and (4) $\Pi(a,b,c,d) > \sup_{b\in\{b,\neg b\},c\in\{c,\neg c\}} \Pi(\neg a,b,c,d)$. If (1) and (3) alone hold then $\Pi(d)$ may follow $\Pi(a)$ up, and if only (2) and (4) hold then $\Pi(d)$ may follow $\Pi(a)$ down. Under all other conditions $\Pi(d)$ is independent of $\Pi(a)$.

Proof. This follows directly from the proof of Theorem 4.1□.

Theorem 4.2 makes it possible to reduce the $\diamondsuit$-network to a simple causal link $A \rightarrow D$ in which the behaviour of this simple link is controlled by the conditions:

$$\Pi(a) < \Pi(d \mid a) \tag{9}$$

$$\Pi(d,a) > \Pi(d \mid \neg a) \tag{10}$$

where (9) is defined to hold if conditions (1) and (3) of Theorem 4.2 hold, and (10) to hold if (2) and (4) of Theorem 4.2 hold. Setting the conditions thus ensures

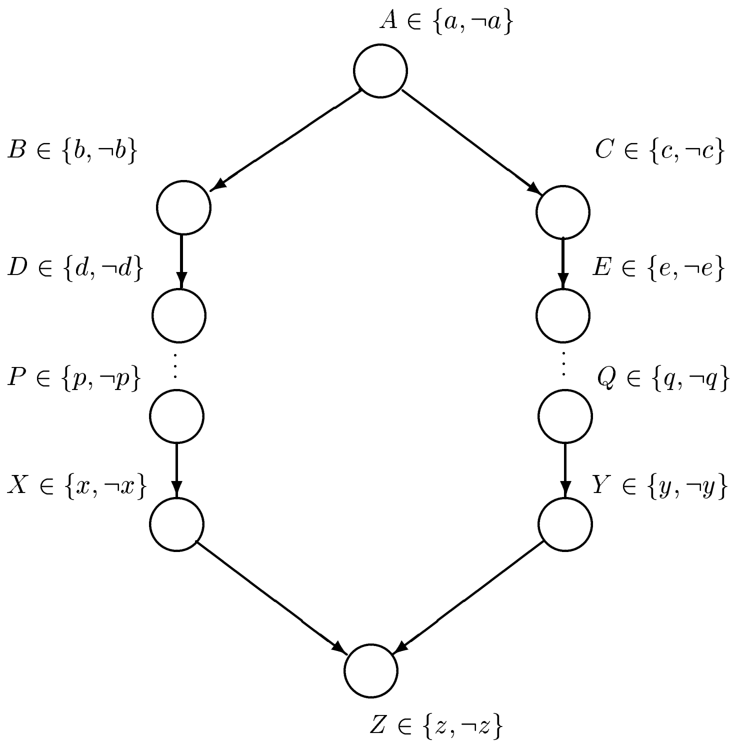

Fig. 5. A loop with an arbitrarily large number of nodes

that the link $A \rightarrow D$ relates A and D in exactly the same way as the $\Diamond$-network does and is thus a similar reduction to that discussed by Wellman[1] for qualitative probabilistic networks.

Reducing the loop to a simple link eliminates the need to consider dependencies between several parents of a node, since these have been dealt with in the reduction. Thus qualitative changes can be propagated through networks obtained by reducing four node loops using methods developed in Section 3 since there are no longer any loops to cause any problems. This kind of reduction does, however, rule out the possibility of determining the changes at B and C, the intermediate nodes along the loop, in the same way that Wellman's reduction does, meaning that the reduction must be targetted at a particular node whose change it is desired to know. Clearly, if the change at an intermediate node is required, this must be determined seperately.

This result can be extended to networks with arbitrarily large numbers of nodes. For the network in Fig. 5, which we will refer to as the $N\Diamond$-network, it is possible to determine that:

Theorem 4.3. In the $N\Diamond$-network $\Pi(z)$ follows $\Pi(a)$ when, for any $c \in \{c, \neg c\}$, $b \in \{b, \neg b\}, \ldots, x \in \{x, \neg x\}$, $y \in \{y, \neg y\}$, all the following conditions hold: $\Pi(a) < \min\big(\Pi(b \mid a), \Pi(d \mid b), \ldots, \Pi(x \mid p), \Pi(c \mid a), \Pi(e \mid c), \ldots, \Pi(y \mid q)$ (1), $\Pi(a, b, c, \ldots, x, y) > \Pi(\neg a, b, c, \ldots, x, y)$ (2), $\Pi(x, y) < \Pi(z \mid x, y)$ (3), and $\Pi(a, b, c, \ldots, x,$

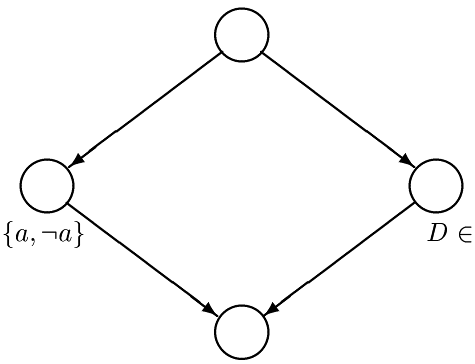

Fig. 6. A new perspective on the four node loop

$y, z) > \sup_{b \in \{b, \neg\}, \ldots, y \in \{y, \neg y\}} \{\Pi(a, b, c, \ldots, x, y, z)\}$ (4). If (1) and (3) hold but (2) and (4) don't, $\Pi(z)$ may follow $\Pi(a)$ up, and if (2) and (4) hold but (1) and (3) don't, then $\Pi(z)$ may follow $\Pi(a)$ down. Under all other conditions $\Pi(z)$ is independent of $\Pi(a)$.

Proof. The theorem follows from the proof of Theorem 4.1 when the equations relating $\Pi(a)$ and $\Pi(d)$ are generalised to the network of Fig. 5□.

Theorem 4.3 permits the reduction of the loop to a single link of the form $A \rightarrow Z$ in the same way as was described in the previous section, and by doing so permits propagations through loops without considering the dependencies between the parents. However, as discussed above, such a reduction will make it impossible to establish changes in value at intermediate nodes such as P and Q.

4.3. *Other loop topologies*

So far only the propagation of values from the top of the loop to the bottom have been considered. In this section a number of different ways in which values may be propagated through loops are dealt with. In particular, considering the $\diamondsuit$-network, it is interesting to try and establish:

(i) the change at B given the change at C;
(ii) the change in D given the change at C;
(iii) the change at D given changes at A and C.

Consider (i) first. In this case the loop in question is that of Fig. 6. where C is the top node and B is the bottom node. In this case, Theorem 4.2 can give us the conditions under which $\Pi(b)$ varies with $\Pi(c)$, and it is possible to propagate changes by reducing the network. Clearly, to apply Theorem 4.2 to this new network

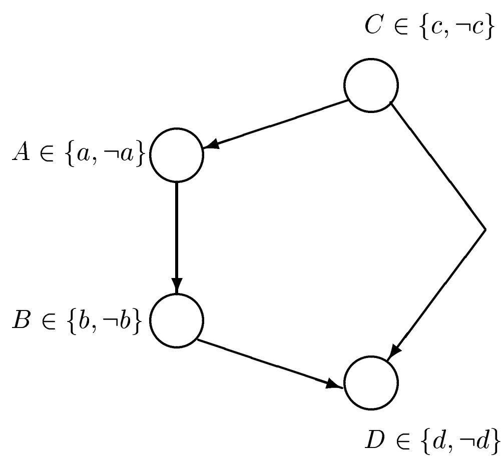

Fig. 7. Another new perspective on the four node loop

will require different possibility values from those required by the analysis of the network of Fig. 3. It may be possible to establish these new values from the old ones by using the possibilistic version of Bayes' theorem, or they may need to be obtained by some form of knowledge aquisition. Case (ii) is similar. Here the network fits into the mould of the $N\Diamond$-network, albeit somewhat lopsidedly (see Fig. 7). It is clear that such a network can be reduced to a link $C \rightarrow D$ given the relevant conditionals. Case (iii) is a little different since it involves the combination of the effects of two different changes. Applying the principle of superposition makes it possible to take the qualitative effect of the change at A alone on D and at C alone on D and sum them to get the total change at D. Clearly, this means reducing the network to $A \rightarrow D$ to compute the change at D due to the change at A and then reducing the network to $C \rightarrow D$ to compute the change at D due to the change at C, before summing the two changes.

Between them cases (i)–(iii), along with the original case, describe all the basic ways in which changes may be propagated around the four node network. Any other propagation of changes are variations on or combinations of these basic patterns. Thus the results given above make it possible for us to predict how any set of changes will be propagated around a four node loop. Now, similar analyses may be carried out for the general loop of Fig. 5, making it possible to predict how any set of changes will be propagated about any loop, and so the results of this section are sufficient to extend the theory of qualitative possibilistic networks to cover any network that is a directed acyclic graph, provided that the variables mentioned in the network are binary valued.

4.4. *Networks with non-binary valued variables*

All the variables considered so far in this paper have been binary valued. The choice of such variables has been entirely pragmatic in that they are easier to work with, lending themselves to simpler proofs and more comprehensible results. However,

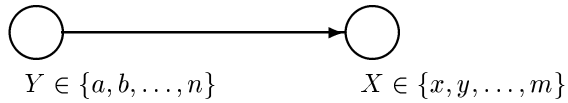

Fig. 8. A network with nodes representing non-binary valued variables

there are many real world situations in which multivalued variables are appropriate, and this section addresses the use of such variables in qualitative possibilistic networks. In particular the propagation of qualitative changes in multivalued variables is considered across the simplest possible network, such as $Y \rightarrow X$ of Fig. 8. Here Y has possible values $\{a, b, \ldots, n\}$ and X has possible values $\{x, y, \ldots, m\}$. Possibility theory tells us that the possibility of X taking value x is

$$\Pi(x) = \sup_{Y \in \{a,b,\ldots,n\}} \min \Big(\Pi(x \mid Y), \Pi(Y) \Big) \tag{11}$$

Using this equation, it is possible to establish how changes in possibility at X may depend upon changes in possibility at Y. For instance, the different ways in which $\Pi(x)$ varies given changes in $\Pi(a)$ are summarised by:

Theorem 4.3. In the network $Y \rightarrow X$, $\Pi(x)$ follows $\Pi(a)$ if $\Pi(a) < \Pi(x \mid a)$ and $\Pi(x, a) > \sup(\Pi(x, b), ..., \Pi(x, n))$, $\Pi(x)$ may follow $\Pi(a)$ up,if $\Pi(a) < \Pi(x \mid a)$ and $\Pi(x, a) < \sup(\Pi(x, b), ..., \Pi(x, n))$, $\Pi(x)$ may follow $\Pi(a)$ down if $\Pi(a) > \Pi(x \mid a)$ and $\Pi(x, a) > \sup(\Pi(x, b), ..., \Pi(x, n))$, otherwise $\Pi(x)$ is independent of $\Pi(a)$.

Proof. In a similar way as for Theorems 3.1 and 3.3, the result follows by inspection from the expression for $\Pi(x)$, namely $\Pi(x) = \sup \Big\{ \min \big(\Pi(x \mid a), \Pi(a) \big), \min \big(\Pi(x \mid b), \Pi(b) \big), ..., \min \big(\Pi(x \mid n), \Pi(n) \big) \Big\}$ $\square$.

Similar results may be obtained for the way in which $\Pi(x)$ varies with $\Pi(b), ..., \Pi(n)$, and for the ways in which $\Pi(y), ..., \Pi(m)$ vary with $\Pi(a), ..., \Pi(n)$. The overall change at X depends upon the changes at Y and all of the relationships between the different possible values of X and Y. The overall change is thus determined by:

$$\begin{bmatrix} \Delta\Pi(x) \\ \Delta\Pi(y) \\ \vdots \\ \Delta\Pi(m) \end{bmatrix} = \begin{bmatrix} \left[\frac{\partial\Pi(x)}{\partial\Pi(a)}\right] & \left[\frac{\partial\Pi(x)}{\partial\Pi(b)}\right] & \cdots \left[\frac{\partial\Pi(x)}{\partial\Pi(n)}\right] \\ \left[\frac{\partial\Pi(y)}{\partial\Pi(a)}\right] & \left[\frac{\partial\Pi(y)}{\partial\Pi(b)}\right] & \cdots \left[\frac{\partial\Pi(y)}{\partial\Pi(n)}\right] \\ \vdots & \vdots & \vdots \\ \left[\frac{\partial\Pi(m)}{\partial\Pi(a)}\right] & \left[\frac{\partial\Pi(m)}{\partial\Pi(b)}\right] & \cdots \left[\frac{\partial\Pi(m)}{\partial\Pi(n)}\right] \end{bmatrix} \otimes \begin{bmatrix} \Delta\Pi(a) \\ \Delta\Pi(b) \\ \vdots \\ \Delta\Pi(n) \end{bmatrix} \tag{12}$$

It is clearly possible to extend the other results in this paper to the case of non-binary valued variables if such results are required, making it possible to analyse the qualitative behaviour of any possibilistic network that is a directed acyclic graph.

5. An Example

In this section we provide an illustration of the kind of reasoning that may be carried out using qualitative possibilistic networks, tackling a version of the dyspnoea problem originally discussed by Lauritzen and Spiegelhalter[5].

5.1. *Preamble*

The previous sections have described how to analyse possibilistic networks in order to establish how qualitative changes will be propagated in a network for which the quantitative values are known. This is the way in which the theory of qualitative possibilistic networks was intended to be used, as part of a scheme for integrating uncertainty handling formalisms[12], and is the way in which the use of qualitative possibilistic networks has previously been discussed[18,19]. However, this is not the only way in which qualitative possibilistic networks may be used. There is an alternative mode of use, and it is this that will be employed in our example.

The other mode of use of qualitative possibilistic networks is that generally proposed for qualitative probabilistic networks— a mode in which the networks are defined in terms of the qualitative, rather than the quantitative, relationships between variables. Thus when considering two binary-valued variables P and Q which are known to influence each other, acquisition centres around establishing whether $\Pi(p)$ follows $\Pi(q)$ rather than what the value of $\Pi(p \mid q)$ is. When establishing this qualitative behaviour, the results of the previous sections identify the assumptions about conditional values that are being made. Having established the qualitative behaviour of the network that represents a given situation, (7) and (8) may be used to propagate changes in possibility, so that the result of various observations may be considered.

It should be noted, however, that whereas in qualitative probabilistic networks a single qualitative value is sufficient to characterise the influence between two variables, in qualitative possibilistic networks a single qualitative value is required for each relevant derivative. Thus the influence between P and Q is characterised by four derivatives, and thus we must seperately acquire the relationships between p and q, between p and $\neg q$, between $\neg p$ and q and between $\neg p$ and $\neg q$.

5.2. *The dyspnoea problem*

The original formualtion of the dyspnoea problem[5] was based upon the following piece of fictitious qualitative medical 'knowledge':

> Dyspnoea (D), that is shortness-of-breath, may be due to tuberculosis (T), lung cancer (L), or bronchitis (B), or none of them, or more than one of them. A recent visit to Asia (A) increases the chances of tuberculosis, while smoking (S) is known to be a risk factor for both lung cancer and bronchitis. The results of a single chest X-ray (X) identifies the presence of either tuberculosis or lung cancer (E) since it does not distinguish between them, as does the presence of absence of dyspnoea.

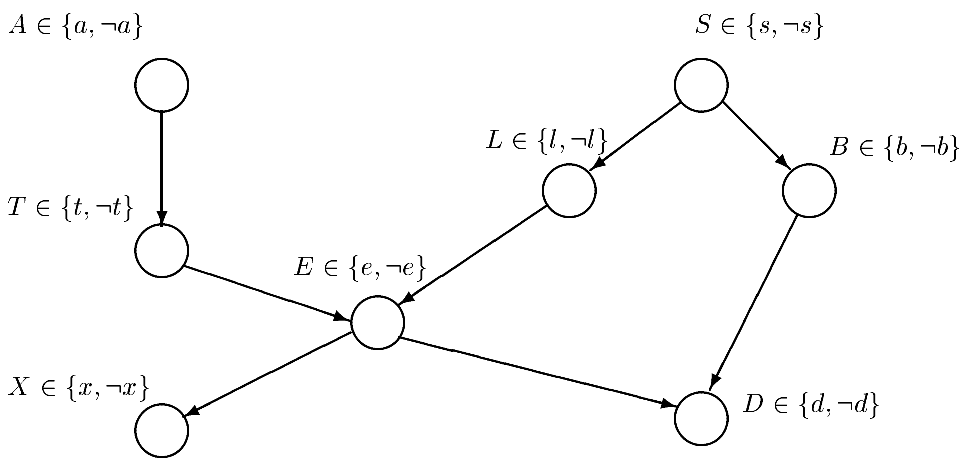

Fig. 9. The network for the dyspnoea example

The bracketed letters are the names of the binary variables representing the conditions, and each variable is represented by a node in the network of Fig. 9. It should be clear that Fig. 9 is the network of influences between the conditions. Now, the situation to which we want to apply this information is, for instance, to establish how the possibility of a patient having tuberculosis changes when it is known that the patient is a smoker, or to establish how the possibility of the patient having bronchitis changes given that a positive x-ray is obtained. That is we want to establish the qualitative change in the possibility of the patient having bronchitis given an increase in the possibility of a positive x-ray, and what the qualitative change in possibility of tuberculosis is given that the possibility of the patient being a smoker increases.

5.3. *Applying qualitative possibilistic networks*

To answer these questions we need information about the qualitative influences between the various values of the variables in the problem, and these can be obtained from the description of the problem given above. In particular we must establish the relationship between $\Pi(s)$ and $\Pi(e)$ in order to reduce the loop in which they both are to a simple link between them, and thus a singly connected network in which it is safe to propagate qualitative possibility values. From the problem description we can say that the possibility of lung cancer or tuberculosis will follow both the possibility of smoking, and the possibility of tuberculosis. Thus, in terms of qualitative derivatives:

$$\left[\frac{\partial \Pi(e)}{\partial \Pi(s)}\right] = [+] \tag{13}$$

$$\left[\frac{\partial \Pi(e)}{\partial \Pi(t)}\right] = [+] \tag{14}$$

It seems reasonable to assume that $\left[\frac{\partial\Pi(\neg e)}{\partial\Pi(\neg s)}\right] = [+]$ and $\left[\frac{\partial\Pi(e)}{\partial\Pi(\neg s)}\right] = \left[\frac{\partial\Pi(\neg e)}{\partial\Pi(s)}\right] = [0]$, and $\left[\frac{\partial\Pi(\neg e)}{\partial\Pi(\neg t)}\right] = [+]$ and $\left[\frac{\partial\Pi(e)}{\partial\Pi(\neg t)}\right] = \left[\frac{\partial\Pi(\neg e)}{\partial\Pi(t)}\right] = [0]$. Thus, given that knowledge of the patient smoking means that $\Delta\Pi(s) = [+]$ and $\Delta\Pi(s) = [-]$ the change in possibility of lung cancer or tuberculosis may be calculated using (7):

$$\begin{bmatrix} \Delta\Pi(e) \\ \Delta\Pi(\neg e) \end{bmatrix} = \begin{bmatrix} [+] & [0] \\ [0] & [+] \end{bmatrix} \otimes \begin{bmatrix} [+] \\ [-] \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} [+] \\ [-] \end{bmatrix} \tag{16}$$

Now, so far reasoning has been predictive, from causes to effects, but to establish how the possibility of tuberculosis will change we must reverse this and reason evidentially from E to T. To do this we apply Theorem 3.2 which tells us that, given the known relationship between $\Pi(t)$ and $\Pi(e)$:

$$\left[\frac{\partial\Pi(t)}{\partial\Pi(e)}\right] = [\uparrow] \tag{17}$$

and we may also establish that $\left[\frac{\partial\Pi(t)}{\partial\Pi(\neg e)}\right] = [\downarrow]$, that $\left[\frac{\partial\Pi(t)}{\partial\Pi(\neg e)}\right] = [\downarrow]$ and $\left[\frac{\partial\Pi(\neg t)}{\partial\Pi(\neg e)}\right] = [\uparrow]$, so that:

$$\begin{bmatrix} \Delta\Pi(t) \\ \Delta\Pi(\neg t) \end{bmatrix} = \begin{bmatrix} [\uparrow] & [\downarrow] \\ [\downarrow] & [\uparrow] \end{bmatrix} \otimes \begin{bmatrix} [+] \\ [-] \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} [?] \\ [0] \end{bmatrix} \tag{19}$$

So that we can say that while it is not possible to predict for sure how the possibility that the patient has tuberculosis will change, we can be sure that the possibility that the patient does not have tuberculosis will not change. Thus knowledge of the patient's history of smoking does not have much relevance to a discussion of whether or not they have tuberculosis.

A similar process could be applied to calculate how a positive X-ray affects the possibility of bronchitis. Consideration of the description of the problem will yield the qualitative influence of E on X, and this may be reversed by Theorem 3.2 to give the change in possibility of E. Then the loop must be reduced by establishing the qualitative influence of B on E, and this in turn may be reversed allowing the change in possibility of B to be established.

6. Discussion

Having introduced a number of features of qualitative possibilistic networks and having demonstrated their application on a small example, this section brings the paper almost to a close with a discussion of what the theory may be used for, some related pieces of work, and some directions in which the theory might be extended.

6.1. *Uses for qualitative possibilistic networks*

As mentioned above, the original motivation for the development of the theory of qualitative possibilistic networks was the need to integrate different uncertainty handling formalisms. It is possible[12,18] to argue that integration may be achieved by only considering qualitative changes in values expressed in different formalisms, and so the study of qualitative possibilistic networks, and the results detailed above, make it possible to integrate possibility theory with other formalisms. The method is flexible, simple to extend, and unlike other schemes for integration does not impose a particular semantics upon the formalisms, and these advantages offset the weak qualitative results that the method provides. These advantages also seem to make the method applicable in the area of distributed artificial intelligence[19,20] when existing systems are coupled together.

As argued above, however, it is also possible to use qualitative possibilistic networks on their own account as a means of representing and reasoning with uncertain information in exactly the same way as qualitative probabilistic networks may be used. In this case exactly the same motivation may be proposed. This is[1] that the use of precise numerical information may be inappropriate since, in certain circumstances, it leads to knowledge bases being applicable only in very narrow areas because of the interaction between values at a fine level of detail. Since they view the world at a higher level of abstraction, qualitative methods are immune to such problems— the small complications such interactions cause simply have no effect at the coarse level of detail with which qualitative methods are concerned.

This means that a system that was tailored to one environment can be moved to another and continue to operate reliably since the same qualitative information applies. Thus, for instance a medical expert system using qualitative possibilistic reasoning could be moved from one clinic to another with no adverse results since the information upon which it makes its diagnoses would be the same in both places. In contrast a system using a numerical formalism would be expected to become unreliable since the numbers on which it was based would have changed, and in order to make it reliable a whole new set of numbers would have to be acquired. In this mode, then, the qualitative formalism provides robust reasoning that still obeys the fundamental tenets of the underlying quantitative theory with all the advantages that it entails.

Finally, qualitative possibilistic reasoning can be used as a means of validating quantitative possibilistic systems. Since the qualitative behaviour of a system is an abstraction of its quantitative behaviour, it is possible to use the qualitative analysis to predict how the system will behave quantitatively. This means that it is possible to carry out a few simple tests to determine if the basic behaviour of a numerical model is that which is desired by its developers, and to make corrections if these are necessary[21].

6.2. *Related work*

This work is closely related to that of Wellman[1] and Henrion and Druzdzel[2,22]. These authors are interested in the propagation of qualitative probability and base their notion of dependency between variables on the idea of forward stochastic dominance. As a result, the scheme that they come up with is rather simpler than mine, which is good from the point of view of clarity. However, their scheme is not built on quite such fundamental notions. These differences are not surprising given that the schemes have different intended uses. Mine was intended as a basis for the integration of different formalisms and it is thus important that it is completely faithful to the underlying theory. Theirs is intended as an efficient and extensible abstraction of probability theory so that simplicity is paramount. However, if qualitative possibilistic networks are to be used in a similar way to that discussed in Section 6, it might well be worth extending the underlying theory to make them as simple and robust as qualitative probabilistic networks.

Another piece of related work is that of Fonck and Straszecka[9] who have studied the propagation of possibility values through acyclic directed graphs and have discovered a means of carrying out such a propagation in a similar way to that in which Pearl[4] propagates probability values in such structures. There are two differences between their approach and mine. Firstly Fonck and Straszecka handle quantitative changes in possibility value while I deal with qualitative changes. If the changes generated by their scheme in a particular scenario are considered in a qualitative way, so that, for instance, a change from $\Pi(a) = 0.1$ to $\Pi(a) = 0.3$ is seen only as an increase $\Pi(a) = [+]$, then both of our approaches will give the same result. In any network, in every situation that Fonck and Straszecka's scheme of propagation generates a quantitative change in possibility at some node as a result of a change in possibility at another node, my scheme will generate an equivalent qualitative change. The second difference is that Fonck and Straszecka are interested in efficient propagation, a subject which is not considered here. However, it seems that the propagation of qualitative values is inherently more efficient than that of numerical values. Druzdzel and Henrion[23] have recently shown that the propagation of qualitative probability values is quadratic in the number of nodes in the graph, making the process considerably more tractable than the propagation of quantitative values which is known to be NP-hard[24].

6.3. *Future work*

There are three obvious ways in which the work reported in this paper may be extended. The first is to generalise the qualitative quantity space from $\{[+], [-], [0], [?], [\uparrow], [\downarrow]\}$ to allow it to represent more precise information when such information is available. There are several ways in which this might be done. It could be addressed by the use of more complex qualitative values[12], semiqualitative values[25] or a combination of numerical and qualitative information[26]— all methods which hybridise qualitative and quantitative representations in an attempt to broaden both.

Alternatively the problem could be handled by some form of order of magnitude reasoning[27] in which the relative magnitude of quantities are explictly manipulated in order to enable reasoning about which changes are significant, or which are more important than others.

This kind of development would make the representation more complex. It would also be possible to make the representation simpler. That is by borrowing ideas from the theory of qualitative probabilistic networks, it might be possible to provide a simpler means of representing the changes that take place at a node, and the way in which changes in value might be propagated along links between nodes. This kind of simplification might also help to ease the search for an efficient algorithm for propagating values in qualitative possibilistic networks, which is the third area in which it seems that the theory could be extended. It should be noted, however, that the lack of a specialised efficient algorithm has not prevented the implementation of the current theory. This implementation[12] has been carried out using the Pulcinella[28] system which itself is based upon the efficient local computation method of Shenoy and Shafer[29].

7. Summary

This paper has introduced the idea of qualitative possibilistic networks, reporting results that complement recent work on qualitative probabilistic networks. Qualitative possibilistic networks were introduced as an abstraction of possibilistic networks, and the conditions for their various types of qualitative behaviour established. The results given are sufficient to establish the qualitative behaviour of any possibilistic network which takes the form of a directed acyclic graph, and permit the way in which changes in possibility will be propagated in both predictive and evidential directions to be predicted. The results may also be viewed in a different light, as an investigation of the possible behaviours that may be encoded by a qualitative possibilistic network. Looked at in this way the results form the basis of a new qualitative means of representing and reasoning with uncertain information that obeys the basic axioms of possibility theory. This view of the results was illustrated by means of a medical example, and other applications of qualitative possibilistic networks were mentioned. The paper closed with a discussion of related work and some of the directions in which the theory might be expanded.

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