An Analysis of Call Admission Problems on Grids

LSD & LAW

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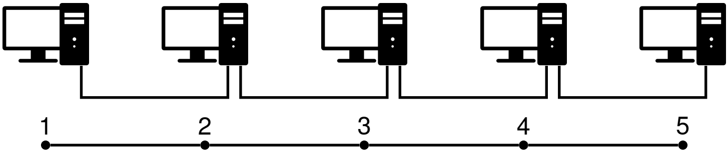
Outline

Online Problems

Definition (Online Maximization Problem Π)

- Sequence of requests
- Satisfy one request before the next one arrives
- Maximize the gain

The Disjoint Path Allocation Problem (DPA)



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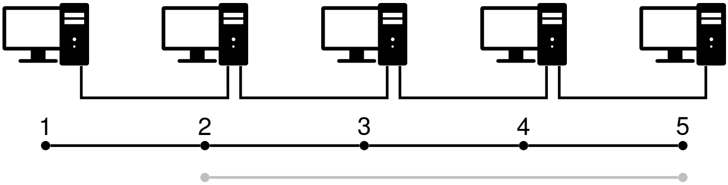

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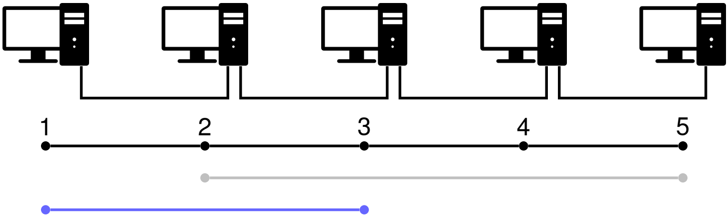

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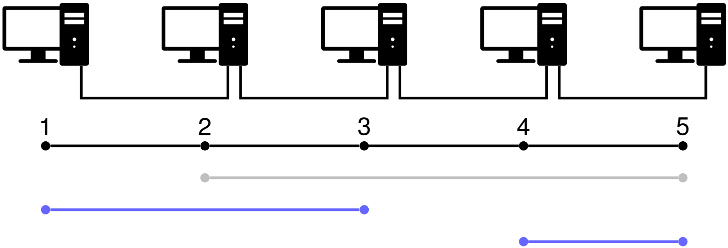

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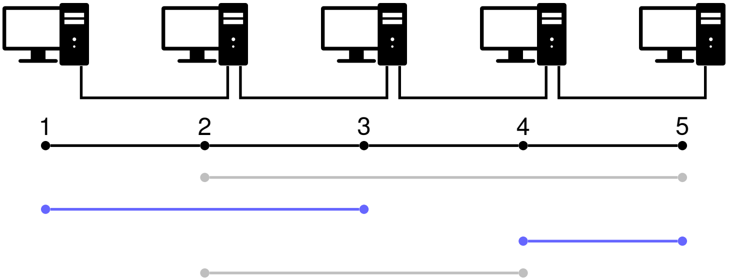


Online Setting with Advice

How much information are we missing

- . . . to be optimal?
- . . . to achieve some competitive ratio?

$\Longrightarrow$ New measure for complexity of online problems

Online Setting with Advice

Definition (Online Algorithm ALG with Advice for Π)

- Adversary chooses online input instance
- Oracle with unlimited power knows instance and chooses infinite advice string
- ALG can read an arbitrary long, but finite prefix
- $q(\cdot)$ is the advice complexity of ALG $\iff$ ALG reads at most first $q(\cdot)$ bits of advice from start
- Advice complexity $s(n)$ of Π: maximum over a all inputs of length n, for best pair of oracle and algorithm

Online Setting with Advice

Definition (Competitive ratio with advice)

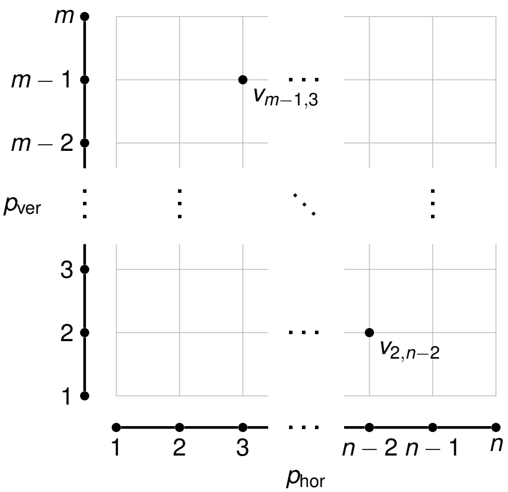
- Π is online maximization problem
- ALG is online algorithm with advice for Π
- OPT(I) is an optimal (offline) solution for instance I of Π

ALG is c-competitive for Π if there exists a constant $\alpha \geq 0$ such that

$$\text{gain}\left(\text{OPT}(I)\right) \leq c \cdot \text{gain}\left(\text{ALG}(I)\right) + \alpha$$

for all instances I of Π.

Extended to Grids



- Height: $m-1$
- Length: $n-1$

The Call Admission Problem on Grids (CAPG)

Definition (CAPG)

- Online maximization problem Π_{CAPG}
- Request is a pair of servers asking for a connection
- Every connection is fixed (no termination or modification)
- Only one connection per wire
- Goal: maximize the number of granted connections

Outline

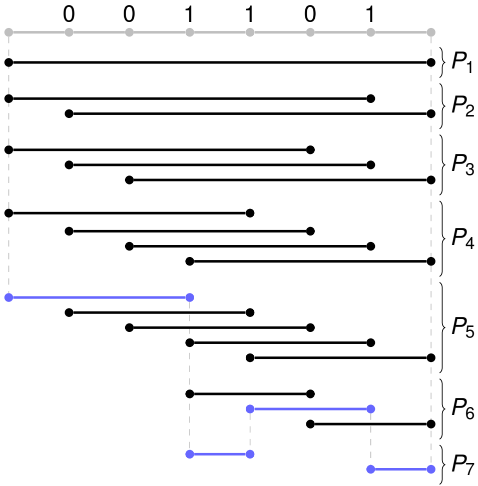
1. Motivation and Definitions
2. Results
   - Lower Bounds
   - Upper Bounds
3. Conclusion

$|E| - 1$ Advice Bits for DPA

Theorem ([BBF+14])

To solve DPA optimally, $|E| - 1$ advice bits are necessary and sufficient.

$|E| - 1$ Advice Bits for DPA



- Optimal solution OPT(I) indicated by bit string: 1 $\iff$ end one request, start another one
- P_i contains all requests of length $|E| - i + 1$ which do not contradict requests in OPT(I) of earlier phase

$|E| - 1$ Advice Bits for DPA

- Optimal solution is unique
- $2^{|E|-1}$ different instances
- Optimal solutions S_1, S_2 have to differ before instances I_1, I_2 are distinct on their asked prefixes of requests
  $\implies \geq \log_2(2^{|E|-1}) = |E| - 1$ advice bits required for optimality

Almost $|E|$ Advice Bits for CAPG

Can we just ask these instances on each column and row for CAPG?

- Consider long request in solution indicated by bit string
- If not satisfied we have much space for detours

$\Longrightarrow$ bit string solution is not optimal anymore

Mitigation:

- Ask only sufficiently small requests, i.e., only last four phases

  $\Longrightarrow$ bit string solution is optimal again
- Still no unique optimal solution in general

Almost $|E|$ Advice Bits for CAPG

Lemma

There are t_{n+2} bit strings of length $n \in \mathbb{N}$ which contain at most three consecutive 0s, where t_n denotes the nth tetranacci number [a]*.*

[a] $t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4}$, $t_0 = t_1 = t_2 = 0$, $t_3 = 1$

Almost $|E|$ Advice Bits for CAPG

Proof sketch:

- Optimal solutions differ only in requests satisfied with paths of length 4 and have specific forms
- Every optimal solution has to grant a detour before rejecting a request intended by bit string

  $\implies$ optimal solutions are distinct before the prefixes of respective instances are different
- $t_n^m \cdot t_m^n$ instances with different optimal solutions

  $\implies \geq m \cdot \log_2(t_n) + n \cdot \log_2(t_m)$ advice bits required for optimality

Almost $|E|$ Advice Bits for CAPG

Theorem

Every optimal online algorithm with advice for CAPG on an $(m \times n)$-grid G has to read at least $m \cdot \log_2(t_n) + n \cdot \log_2(t_m)$ advice bits.

Almost $|E|$ Advice Bits for CAPG

Corollary

Every optimal online algorithm with advice for CAPG on an $(m \times n)$-grid G has to read at least $0.94677 \cdot |E(G)| - m - n$ advice bits.

Bit Guessing for CAPG

Theorem

Every online algorithm with advice for CAPG which achieves a competitive ratio of $c \leq \frac{12}{11}$ on a grid G has to read at least

$$\left(1+\left(6-\frac{6}{c}\right)\log_2\left(6-\frac{6}{c}\right)+\left(\frac{6}{c}-5\right)\log_2\left(\frac{6}{c}-5\right)\right)\frac{|E(G)|}{2}$$

bits of advice.

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Trivial Bound

Theorem

There is an optimal online algorithm with advice for CAPG which reads at most $2|E| \cdot \lceil \log_2(|V|) \rceil \leq 2|E| \cdot \log_2(|E| + m + n)$ *bits of advice for every* $(m \times n)$*-grid* $G = (V, E)$.

- Oracle chooses some optimal solution
- $\leq |E|$ requests
- Encode both endpoints of every granted request $\implies 2 \cdot \lceil \log_2(|V|) \rceil$ bits per satisfied request
- Overall: $\leq 2|E| \cdot \lceil \log_2(|V|) \rceil$

How can we improve?

- Knowing which edges are used in an optimal solution sometimes not helpful (e.g., in case all edges are used, but some requests are contradicting)
- Need to transmit the “membership” to a request
- “Neighbouring” paths need to be distinguishable

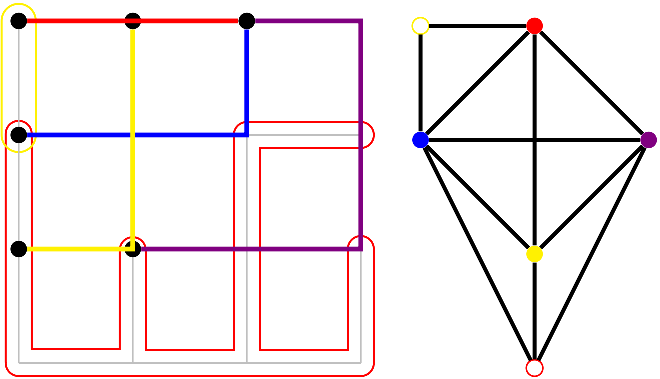
$\Longrightarrow$ Coloring problem in some auxiliary graph

The Auxiliary Graph $\widehat{G}$

Definition ($\widehat{G}(S) = (\widehat{V}, \widehat{E})$)

- Path p in S satisfying a request $\implies v_p \in \widehat{V}$
- $\{v_g, v_h\} \in \widehat{E} \iff g$ and h share some vertex in G
- Edges of G unused by S are split up into connected components, s.t. component q corresponds to $v_q \in \widehat{V}$ and the chromatic number $\chi(\widehat{G}(S))$ is minimized

The Auxiliary Graph $\widehat{G}$



$\widehat{G}$ Bound

Theorem

Let $\mathcal{I}$ denote all possible instances of CAPG on a grid $G = (V, E)$, and let $\mathcal{S}_{\text{opt}}(I)$ be the set of optimal solutions for an instance $I \in \mathcal{I}$. Then, there is an optimal online algorithm with advice for CAPG using at most

$$\max_{I \in \mathcal{I}} \min_{S \in \mathcal{S}_{\text{opt}}(I)} \lceil |E| \cdot \log_2(\chi(\widehat{G})) \rceil + 2\lceil \log_2(\chi(\widehat{G}(S))) \rceil$$

advice bits.

$\widehat{G}$ Bound

- Oracle:
  - Can compute all optimal solutions
  - Selects optimal solution S, s.t. $\lceil |E| \cdot \log_2(\chi(\widehat{G}))\rceil + 2\lceil \log_2(\chi(\widehat{G}(S)))\rceil$ is minimal
  - Uses $2\lceil \log_2(\chi(\widehat{G}(S)))\rceil$ bits to transmit $\chi(\widehat{G}(S))$ in a self-delimiting encoding
  - Colors corresponding connected components of G according to $\chi(\widehat{G}) \implies \chi(\widehat{G})^{|E|}$ possibilities
  - $\lceil |E| \cdot \log_2(\chi(\widehat{G}))\rceil$ bits for transmitting the coloring

$\widehat{G}$ Bound

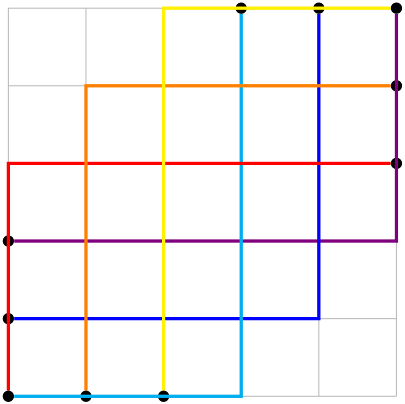
- Algorithm (receiver):
    - Recomputes the length of the encoding
    - Reads off the coloring
    - Decides accordingly

$\widehat{G}$ Bound

Corollary

There is an optimal online algorihm with advice for CAPG that reads at most $\lceil |E| \cdot \log_2(\frac{1}{3}(2|E| + 7)) \rceil$ bits of advice.

$\widehat{G}$ Bound – Limitations



- All paths are "neighboring" $\implies$ $\widehat{G}$ contains n clique
- Since $\chi(\widehat{G}) \geq \omega(\widehat{G})$ this upper bound can not be stronger than $\lceil |E| \cdot \log_2(n) \rceil = \lceil |E| \cdot \log_2(\sqrt{|V|}) \rceil \in \mathcal{O}(|E| \cdot \log(|E|))$

Can we still improve?

It suffices to be able to

- Distinguish the end vertices of different satisfied requests
- Follow the path that is used to satisfy the request
  - Only three possible direction at every inner vertex

Our Best Upper Bound $3|E|$

Theorem

There is an online algorithm with advice for CAPG that computes an optimal solution using at most $3|E|$ advice bits.

Corollary

Let $\mathcal{I}$ denote all possible instances of CAPG on a grid $G = (V, E)$, and let $\mathcal{S}_{\text{opt}}(I)$ be the set of optimal solutions for an instance $I \in \mathcal{I}$. Then, there is an optimal online algorithm with advice for CAPG that uses at most

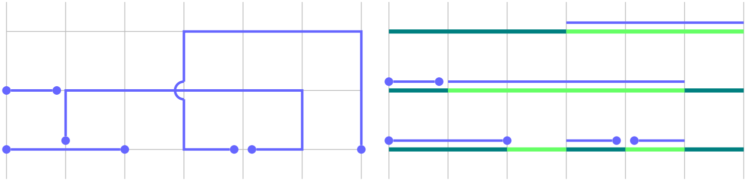
$$\lceil \log_2(5) \cdot k + \log_2(3) \cdot |V| \rceil + \lceil 2 \log_2(k) \rceil$$

advice bits, where k is the number of requests in I.

Proof Sketch

- Choose some optimal solution
- Treat each row and each column separately
- Cut off requests
- Color edges as before using the auxiliary graph

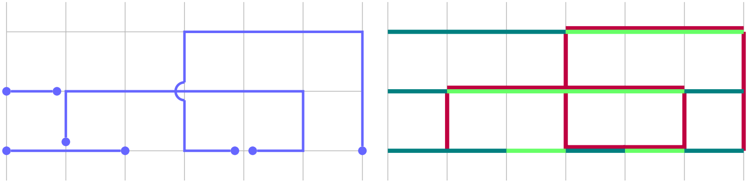
$\Longrightarrow$ Can distinguish aligned paths in solution



Proof Sketch

- Color edges of unaligned paths in optimal solution additionally red

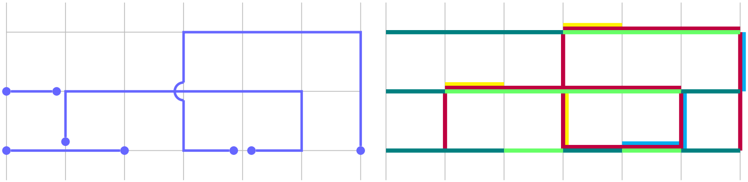
  $\Longrightarrow$ Can distinguish aligned and non-aligned paths in solution



Proof Sketch

- Yellow: next edge in clockwise direction belongs to same path, pivot around lower, left vertex of edge
- Cyan: next edge in clockwise direction belongs to same path, pivot around lower, left vertex of edge

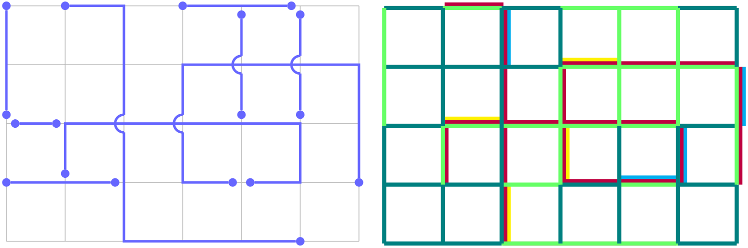
$\Longrightarrow$ Can follow non-aligned paths in solution



Proof Sketch

$\Longrightarrow$ Eight color combinations

$\Longrightarrow$ $log_2(8) = 3$ bits per edge



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Conclusion

- Lower and upper bound already close
- Upper bounds applicable for other graphs

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