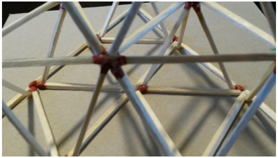
On the Parikh-de-Bruijn grid

Péter Burcsi Zsuzsanna Lipták W. F. Smyth

ELTE Budapest (Hungary), U of Verona (Italy),
McMaster U (Canada) & Murdoch U (Australia)



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Abelian stringology

Def. Given a string $s = s_1 \cdots s_n$ over a finite ordered alphabet Σ of size σ, the Parikh-vector $\mathbf{pv}(s)$ is the vector $(p_1, \ldots, p_\sigma)$ whose i'th entry is the multiplicity of character a_i.

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- Jumbled Pattern Matching
- abelian borders
- abelian periods
- abelian squares, repetitions, runs
- abelian pattern avoidance
- abelian reconstruction
- abelian problems on run-length encoded strings
- ...

Abelian stringology

In this talk, we introduce a new tool for attacking abelian problems.

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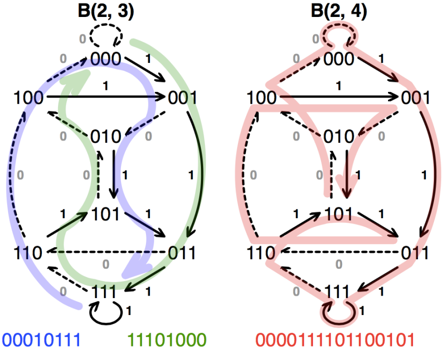
But first: in what way are abelian problems different from their classical counterparts?

N.B.: Recall Σ is finite and ordered, and $\sigma = |\Sigma|$.

Example 1: Parikh-de-Bruijn strings

- **Recall:** A de Bruijn sequence of order k over alphabet Σ is a string over Σ which contains every $u \in \Sigma^k$ exactly once as a substring.
- de Bruijn sequences exist for every Σ and k
- correspond to Hamiltonian paths in the de Bruijn graph of order k
- can be constructed efficiently via Euler-paths in the de Bruijn graph of order $k-1$



Source: Wikipedia

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Def.

- the order of a Parikh vector (Pv) is the sum of its entries (= length of a string with this Pv)
- a Parikh-de-Bruijn string of order k (a (k,σ)-PdB-string) is a string s over an alphabet of size σ s.t.

$$\forall\ p \text{ Parikh vector of order } k\ \exists!(i,j) \text{ s.t. } \mathbf{pv}(s_i \cdots s_j) = p$$

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- but no $(4,3)$-PdB-string exists
- and no $(2,4)$-PdB-string exists

Example 2: Covering strings

Next best thing: covering strings.

Def.

- We call a string s (k,σ)-covering if

$$\forall\ p \text{ Parikh vector of order } k\ \exists(i,j) \text{ s.t. } \mathbf{pv}(s_i \cdots s_j) = p$$

  (There is at least one substring in s which has Pv p.)

- The excess of s is: $|s| - \underbrace{\binom{\sigma+k-1}{k} + k - 1}_{\text{length of a PdB-string}}$.

Ex.

- `aaaabbbbccccaacabcb` is a shortest $(4,3)$-covering string, with excess 1.
- `aabbcadbccdd` is a shortest $(2,4)$-covering string, with excess 1.

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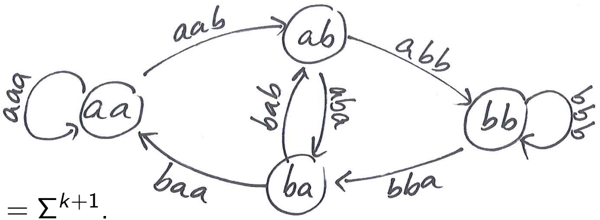
For PdB-strings, this is not always true, e.g.

aaaaabbbbbcaaaadbbbcccccddddaaaccdbcbaccaccddbddbadacddbbbb

is a $(5, 4)$-PdB-string but is not $(4, 4)$-covering: no substring with Pv $(1, 1, 1, 1)$.

The Parikh-de-Bruijn grid

Recall: de Bruijn graphs $B_k = (V, E)$, where $V = \Sigma^k$ and $(\mathtt{x}u, u\mathtt{y}) \in E$ for all $\mathtt{x}, \mathtt{y} \in \Sigma$ and $u \in \Sigma^{k-1}$



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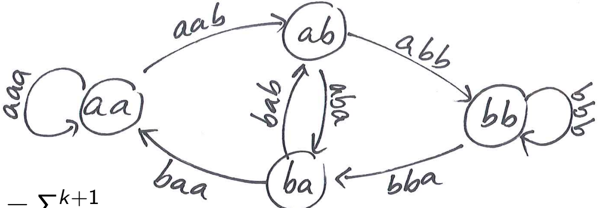
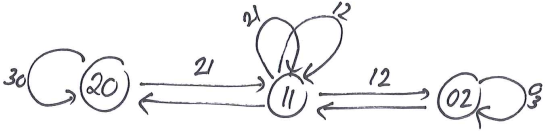
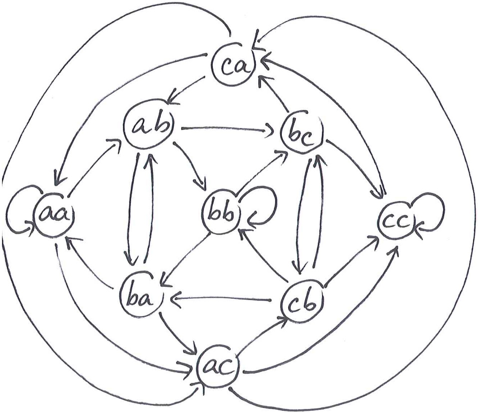


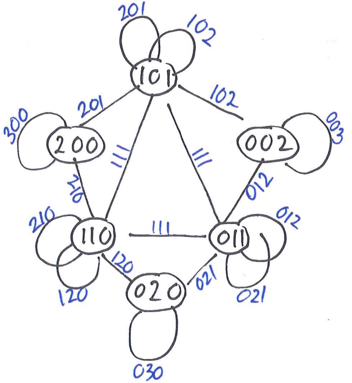
Note that $E = \Sigma^{k+1}$.

A straightforward generalization to Pv's does not work, because edges do not uniquely correspond to $(k+1)$-order Pv's:



Let’s look at another example: Here, $\sigma = 3, k = 2$.





Again, in the abelian version, we have that several edges have the same label (i.e. here: the same 3-order Pv).

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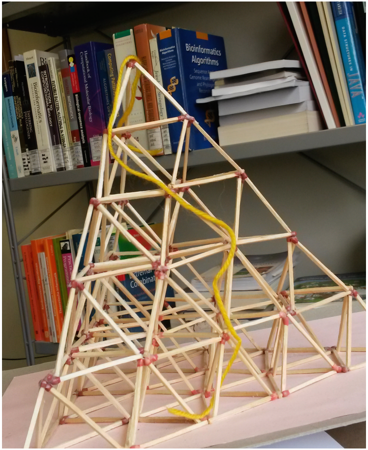
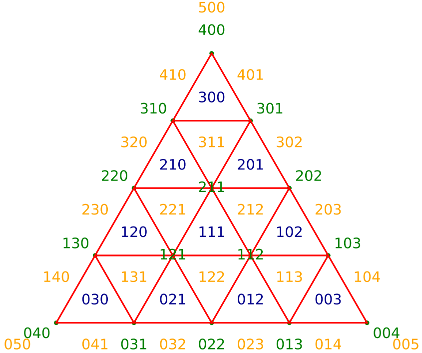
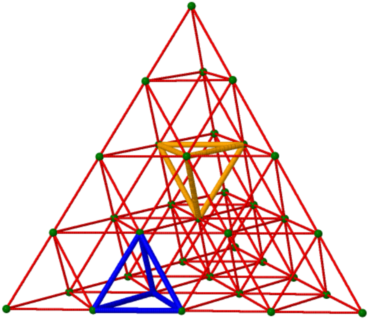


The Parikh-de-Bruijn grid

The $(4,3)$-PdB-grid



The $(4,4)$-PdB-grid

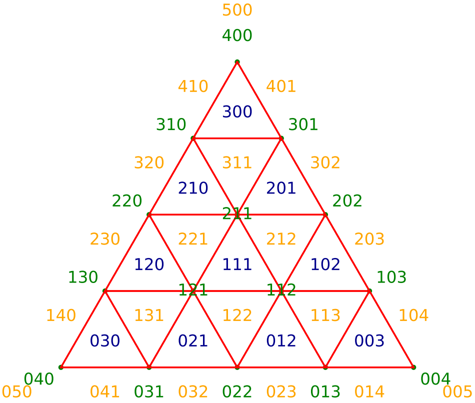


green: k-order Pv's (vertices), yellow: $(k+1)$-order Pv's (downward triangles/tetrahedra), blue: $(k-1)$-order Pv's (upward triangles/tetrahedra).

The Parikh-de-Bruijn grid

PdB-grid:

- $V = k$-order Pv's
- $pq \in E$ iff exist $\mathtt{x}, \mathtt{y} \in \Sigma$ s.t. $p = q - \mathtt{x} + \mathtt{y}$
- undirected edges (or: bidirectional edges)
- $(k-1)$- and $(k+1)$-order Pv's correspond to sub-simplices (triangles for $\sigma = 3$, tetrahedra for $\sigma = 4$ etc.)
- every string corresponds to a walk in the PdB-grid, but not every walk corresponds to a string



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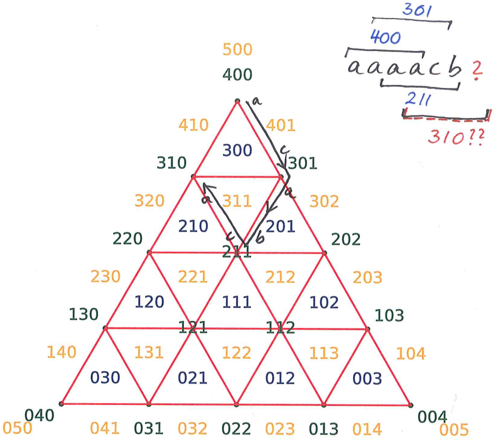

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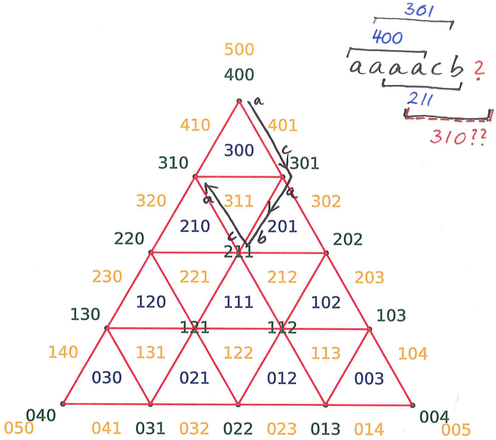
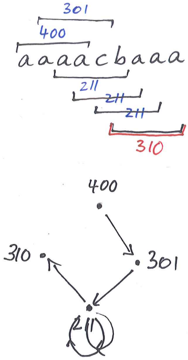


But with loops it's possible!



The Parikh-de-Bruijn grid

Lemma

A set of k-order Parikh vectors is realizable if and only if the induced subgraph in the k-PdB-grid is connected.

realizable = exists string with exactly these k-order sub-Pv's.

Proof sketch

Use loops until undesired character $\mathtt{x}$ exits, replace by new character $\mathtt{y}$.

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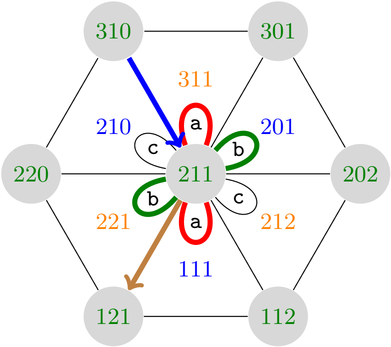
Actually, better name: loops $\rightarrow$ bows (see next slide); one for each character.

The Parikh-de-Bruijn grid

$k = 4$, $\sigma = 3$



$(k+1)$	a	3	3	2	2				
	b	1	1	2	2				
	c	1	1	1	1				
		a	a	b	a	c	a	b	b
k	a	3	2	2	2	1			
	b	1	1	1	1	2			
	c	0	1	1	1	1			
$(k-1)$	a		2	1	2	1			
	b		1	1	0	1			
	c		0	1	1	1			

Walk corresponding to aabacabb. $(k+1)$- and $(k-1)$-order Pv's: triangles incident to the edges traversed by the walk. The $(k+1)$ and $(k-1)$-order Pv's for loops (same k-order Pv twice) lie in opposite direction, hence the name bow.

Back to Parikh-de-Bruijn and covering strings

Theorem 1

No $(k, 3)$-PdB strings exist for $k \geq 4$.

Theorem 2

A $(2, \sigma)$-PdB string exists if and only if σ is odd.

Theorem 3

For every $\sigma \geq 3$ and $k \geq 4$, there exist (k, σ)-covering strings which are not $(k - 1, \sigma)$-covering.

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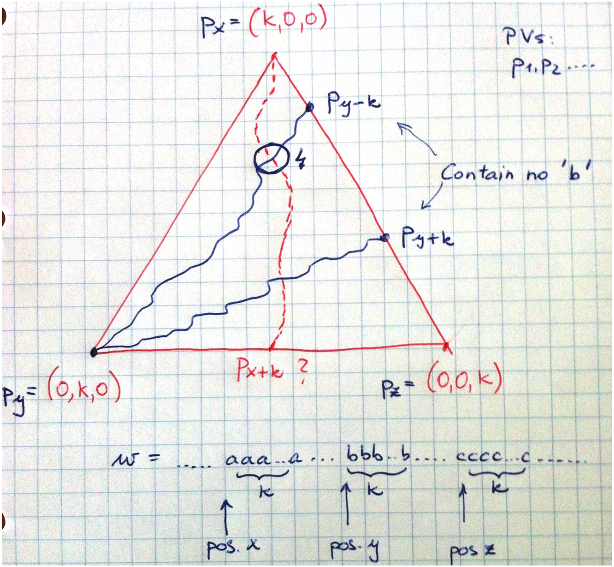


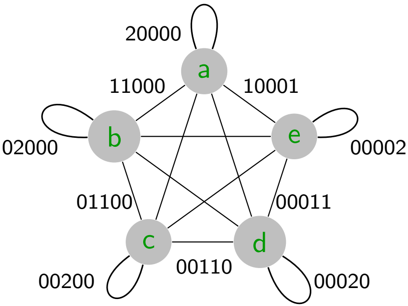
Parikh-de-Bruijn and covering strings

Theorem

A $(2, \sigma)$-PdB string exists if and only if σ is odd.

Proof

Pv's of order 2 have either the form $(0...0, 2, 0..0)$ or $(0...0, 1, 0...0, 1, 0..0)$. So s has to have exactly one substring of the form $\mathtt{aa}$ for all $\mathtt{a} \in \Sigma$, and either $\mathtt{ab}$ or $\mathtt{ba}$ for all $\mathtt{a}, \mathtt{b} \in \Sigma$. Consider the undirected complete graph $G = (V, E)$ with loops where $V = \Sigma$ (N.B.: not the PdB-grid!): an Euler path exists iff σ is odd.



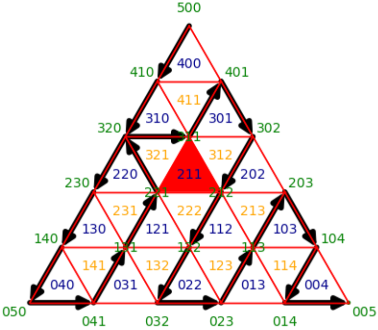
Parikh-de-Bruijn and covering strings

Theorem 3

For every $\sigma \geq 3$ and $k \geq 4$, there exist (k, σ)-covering strings which are not $(k-1, \sigma)$-covering.

Proof

$w =$ aaaaabbbbbcabbaaacacbbcbccacaccccbccccc



General construction:

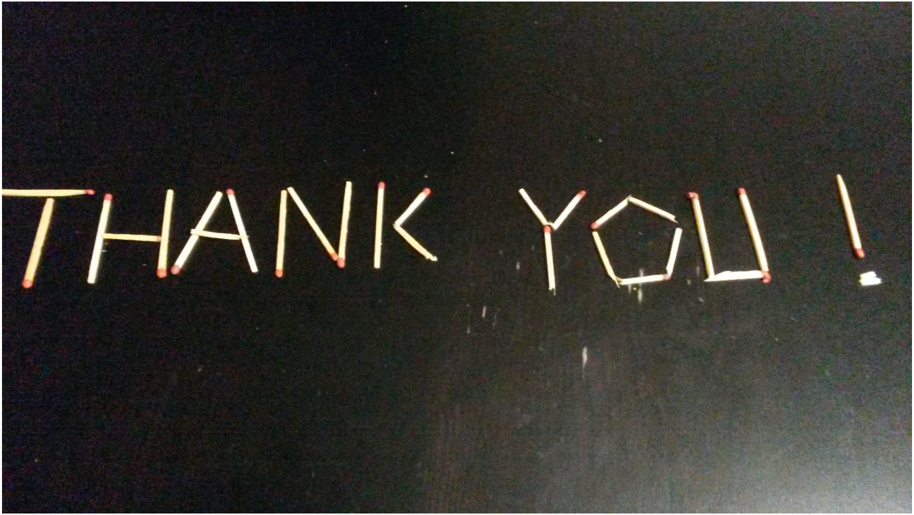
- remove $(k-1)$-order Pv $p = (k-3, 1, 1, 0, \ldots, 0)$ with incident edges and vertices
- the rest is connected, hence a string exists (Lemma)
- add vertices of p without traversing edges incident to p
- can be done by detours from corners of PdB-grid

Experimental results

k	σ	*string*	*length (excess)*
2	3	aabbcca	7 (0)
3	3	abbbcccaaabc	12 (0)
4	3	aaaabbbbccccaacabcb	19 (1)
5	3	aaaaabbbacccccbbbbaacaaccb	27 (2)
6	3	aaaabccccccaaaaabbbbbcccbbcabbaca	35 (2)
7	3	aabbbccbbcccabacaaabcbbbbbbaaaaaacccccccba	44 (2)
2	4	aabbcadbccdd	12 (1)
3	4	aaabbbcaadbdbccadddccc	22 (0)
4	4	aabbbbcaacadbddbccacddddaaaabdbbccccdd	38 (0)
5	4	aaaaabbbbbcaaaadbbbcccccddddaaaccdbcbaccaccddbddbadacddbbbb	60 (0)
2	5	aabbcadbeccddeea	16 (0)
3	5	aaabbbcaadbbeaccbdddcccebededadceeeaa	37 (0)
4	5	aaaabbbbcaaadbbbeaaccbbddaaeaebcccadbeeeadddcccceeeedddd...	73 (0)

Conclusion and open problems

- new tool for modeling and solving abelian problems
- find good characterization for walks which correspond to strings
- several open problems on PdB- and covering strings (see paper on Arxiv)
- apply PdB-grid to other abelian problems


THANK YOU !