Make Pals Your Pals

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Outline

Palindromes

- Palindrome is a string that reads the same in both directions
  - like rotator
- There is also a generalized version (palindromes with involution) inspired by the Watson–Crick palindromes in DNA/RNA strands

Topics

- Find/count palindromes in a string
- Compare palindromes in two or more strings
- Factorize a string into palindromes
- Strings with maximum number of palindromes
- Expected distribution of palindromes in strings

Notation and Definitions

- Array notation for strings (words): $S = S[1..n]$
  - $n = |S|$, $\sigma = alph(S)$
- Substring $S[i..j]$, prefix $S[1..i]$, suffix $S[j..n]$
- Reversal: $\overleftarrow{S} = S[n]S[n-1] \cdots S[1]$
- Palindrome: $S = \overleftarrow{S}$
- Involution: letter-to-letter map θ such that $\theta^2 = id$
- Involution palindrome: $S = \theta(\overleftarrow{S})$
- Gapped palindrome: $ST\overleftarrow{S}$, where $T[1] \neq T[|T|]$
- Subpalindrome: substring $S[i..j]$ which is a palindrome
  - has center $(j+i)/2$ and radius $\lceil (j-i)/2 \rceil$
  - the set of centers is $\{1, \frac{3}{2}, 2, \ldots, n - \frac{1}{2}, n\}$

Agreements

Alphabets:

- general ordered
  - only comparisons; sorting in $n \log n$ time
- integer of polynomial size
  - many tricks including sorting in linear time

Computation:

- Word-RAM model
- input string usually arrives online, symbol by symbol
  - an algorithm solves a problem for a string S online if it gives the answer for every prefix $S[1..i]$ before reading $S[i+1]$
- Streaming model (sublinear space available)

Disclaimer:

- All results are formulated for palindromes, many translate to involution pals, none translates to gapped pals

Outline

Array of Radiuses

Rad[*c*]: maximum radius of a subpalindrome centered at *c*

Theorem

The array *Rad* can be computed "almost" online in linear time (Manacher's algorithm, Manacher, 1975). The algorithm can be made real-time using lazy computation (Galil, 1976).

Example: (the red part is computed online)

S	a		a		a		a		b		c		b		c		b		a
Rad	0	1	1	2	1	1	0	0	0	0	1	0	3	0	1	0	0	0	0

As a data structure, the array *Rad*

- Compactly represents all subpalindromes of a string
- Answers the queries "is $S[i..j]$ a palindrome?" in O(1) time
  - Compare $Rad[(j+i)/2]$ to $\lceil (j-i)/2 \rceil$

Array of Radiuses (2)

Manacher's algorithm allows one to compute online

- the longest prefix palindrome, the longest suffix palindrome and the longest subpalindrome of a string
  - in particular, to check whether the string is a palindrome
- the total number of (occurrences of) palindromes in a string

... but gives no information about the number of distinct palindromes which occur in the string

Palindrome or Not?

Checking whether a string is a palindrome

- In the RAM model:
  - online in linear/real time by Manacher's algorithm/ Galil's modification
- On a multi-tape Turing machine:
  - online in linear time (Slisenko 1973, simplified by Galil 1975)
- In the streaming model:
  - in real time, w.h.p. (by Karp–Rabin hashes)

Factorizations into Palindromes

Checking whether a string can be factorized into

- Even-length palindromes
  - online in real time (Knuth, Morris, Pratt 1977 + later improvements)
- Palindromes of length > 1
  - online in linear time (Galil, Seiferas 1978)
- Two palindromes
  - online in linear time (Galil, Seiferas 1978)
- Three/four palindromes
  - linear time (Galil, Seiferas 1978)
- k palindromes, for any constant k

  ··· conjectured to be linear time

Outline

Distribution of Palindromes

$E(n, \sigma)$ is the expected number of palindromes in a σ-ary string of length n

Theorem (RS 2016)

For any fixed $\sigma > 1$, $E(n, \sigma) = \Theta(\sqrt{n})$.
The function $E(n, \sigma)/\sqrt{n}$ oscillates between the values of size $\Theta(1)$ and $\Theta(\sqrt{\sigma})$.

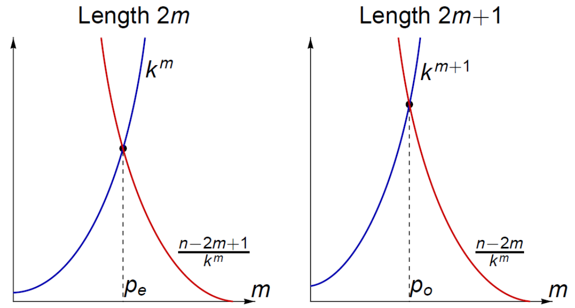
$L(n, \sigma)$ is the expected length of a subpalindrome of a σ-ary string of length n

Proposition (easily follows from RS 2016)

For any fixed $\sigma > 1$, $L(n, \sigma) = (2 + o(1)) \log_\sigma n$.

Distribution of Palindromes: picture

Upper bounds for the expected number of distinct palindromes of length $s \in \{2m, 2m+1\}$ are the total number of palindromes and the expected number of subpalindromes of length s.



Matching lower bounds from the estimations of the number of strings without a given substring (Guibas, Odlyzko 1981)

Rich String, Poor String

The minimum number of distinct palindromes in a σ-ary long (even infinite) string is constant for every $\sigma > 1$ (

- poor strings, not very interesting

The maximum number of distinct palindromes in a length-n string is n

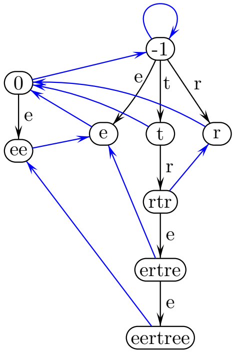
- rich strings, have many nice properties:
  - A substring or reversal of a rich string is rich
  - Any rich string can be extended to a longer rich strings
  - Include sturmian strings and their generalizations
  - Several combinatorial characterizations

The number $R_\sigma(n)$ of rich strings grows with length unusually:

- $R_\sigma(n) > R_2(n) \geq C\sqrt{n}$ for $C \approx 37.6$ (Guo, Shallit, S 2016)
- $R_\sigma(n) = O(2^{\frac{n \log \log n}{\log n}})$ (Rukavička 2017)

Eertree

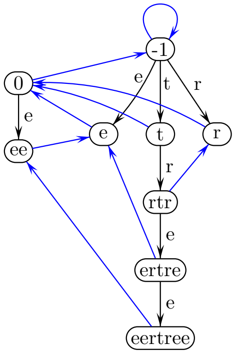
Eertree: linear-size tree-like data structure capturing all information about subpalindromes of a string
Introduced by Mikhail Rubinchik (RS, IWOCA 2015)



- Vertices: palindromes + $\{0, -1\}$
- Edges: $W \to aWa$, labeled by a
  - two trees with roots 0 and -1
- Suffix links: longest suffix palindrome
  - reversed tree with root -1
- Lengths are stored (not strings!)
- Optional: “fast track” suffix links
- ★ Space is often sublinear
  - $O(\sqrt{\sigma n})$ for random σ-ary strings

Eertree Construction Time

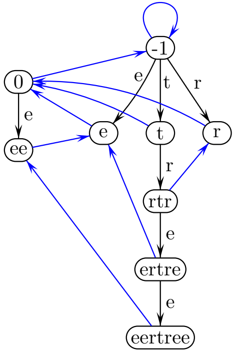
For a string $S[1..n]$, *eertree*(S) can be built in the time



- $O(n \log \sigma)$ online for a general alphabet
  - $\log n$ per step or
  - $\log \sigma$ per step + some extra space
- $O(n)$ offline for an integer alphabet
- $O(n\alpha(n))$ online for an integer alphabet
  - $\alpha(n)$: insertion time for a hash-based dictionary
  - randomized: $\alpha(n) = O(1)$
  - deterministic: $\alpha(n) =? O((\log \log \sigma)^2)$

Free with Eertree

Some problems can be solved just by building an eertree (possibly with some additional fields stored in vertices)



- Find/count distinct subpalindromes
- Compare palindromes in two strings (build "joint" eertree)
  - number of common palindromes
  - longest common palindrome
  - shortest distinctive palindrome
  - palindromes having the same / different numbers of occurrences
  - ...

Factorizations with Eertree

Two main factorization problems:

- k-factorization: given k, can $S[1..n]$ be factorized into exactly k palindromes?
- Palindromic length: what is the minimum k such that $S[1..n]$ admits a k-factorization?
  - the first problem does not reduce to the second, because a string with a k-factorization can have no $(k+1)$-factorization
- Both problems solved with eertree in $O(n \log n)$ time
  - same time bound for palindromic length was first obtained by Fici et al and by I et al (2014)
  - method: dynamic programming, testing every suffix palindrome of the current string as the last palindrome in the factorization; $O(n^2)$ in a naive way, reduced to $O(n \log n)$ using series of palindromes

Counting Hard with Eertree

Consider the following query problem:

- Palindromes in substrings: a string S arrives online, and each symbol is followed by zero or more queries $count(i, j)$ which should be answered by the number of distinct palindromes in $S[i..j]$

Theorem (RS, SPIRe 2017)

Palindromes in substrings can be solved in time $O(n \log n)$ plus $O(\log n)$ per query, using $O(n \log n)$ space. Restricted versions (e.g., an offline version and a problem of finding all rich substrings of a string) require $O(n)$ space.

Ingredients: eertree + a persistent lazy version of segment tree (a data structure for computing symmetric functions on arrays)

Factorization with Bit Compression

Bit Compression (Four Russians' trick):

- $\log n$ bits are assumed to fit into a machine word
- packing a bit array into machine words, we can process subarrays of $\log n$ bits in $O(1)$ time using
  - standard operations
  - custom functions given by precomputed tables of size $o(n)$

$\log n$ speed-up of an algorithm can (sometimes) be obtained

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Theorem (KRS, SOFSEM 2015)

There is an online algorithm solving the k-factorization problem in $O(kn)$ time and $O(n)$ space.

Theorem (BKRS, CPM 2017)

There is an online algorithm solving the palindromic length problem in $O(n)$ time and space.

Palindromic Series Example

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

Palindromic Series Example

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

period = 1:
a

Palindromic Series Example

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

period = 1:
a

period = 2:
aba

Palindromic Series Example

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

period = 1:
a

period = 2:
aba

period = 3:
abaaba
abaabaaba

Palindromic Series Example

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

period = 1:
a

period = 2:
aba

period = 3:
abaaba
abaabaaba

period = 10:
abaabaabaaabaabaaba
abaabaabaaabaabaabaaabaabaaba
abaabaabaaabaabaabaaabaabaabaaabaabaaba

Sketch for $O(n \log n)$

- Let $ans[n]$ be the palindromic length of the processed string
- Let u_1, ..., u_t be all its suffix palindromes
- $ans[n] = 1 + \min_{i\in[1..t]} ans[n-|u_i|]$
- Let U_1, ..., U_k be all its series
- $ans[n] = 1 + \min_{i\in[1..k]} \min_{u\in U_i} ans[n-|u|]$
- After appending a symbol
  - Each internal minimum can be computed in $O(1)$ time, $O(k)$ in total
  - The list of series can be recalculated in $O(k)$ time

Good and Bad Iterations

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba

<u>append(a)</u>

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba**a**

<u>append(b)</u>

aaabaabaabaaabaabaabaaabaabaabaaabaabaaba**b**

Streaming

In the streaming model, the input string arrives online symbol by symbol and cannot be stored: the available memory is sublinear

- Still, some pattern matching and search of regularities can be performed (wait for Tanya's lecture for a comprehensive account...)

Features of streaming model:

- Many problems can be solved only approximately, or w.h.p., or both
- Many trade-offs (e.g., memory vs approximation ratio)
- Real-time algorithms are utterly important

Longest Palindrome in a Stream

Finding the longest palindrome in a stream “requires both”:

- provably, the problem can be solved only by a Monte Carlo algorithm, reaching the required approximation ratio w.h.p.

Tool to detect palindromes: Karp-Rabin hash

- $\alpha > 0$, $p \in [n^{3+\alpha}, n^{4+\alpha}] \cap PRIMES$
- r is a fixed integer randomly chosen from $\{1, \ldots, p-1\}$

For S, its forward and reversed hash are defined as

$$\phi^F(S) = \left(\sum_{i=1}^{n} S[i] \cdot r^i\right) \bmod p; \ \phi^R(S) = \left(\sum_{i=1}^{n} S[i] \cdot r^{n-i+1}\right) \bmod p$$

- condition $\phi^F(u) = \phi^R(u)$ defines a palindrome modulo the (improbable) collisions of hashes
- $O(1)$ computation of $\phi(S[1..i+1])$ from $\phi(S[1..i])$; of $\phi(S[i..j])$ from $\phi(S[1..i])$ and $\phi(S[1..j])$

Longest Palindrome in a Stream (2)

Theorem (Gawrichowski, Uznanski, MS, CPM2016)

1. Longest palindrome in a stream cannot be found approximately within $o(M \log \min\{\sigma, M\})$ bits of memory, where $M = n/E$ for approximating the answer with additive error E and $M = \frac{\log n}{\log(1+\varepsilon)}$ for approximating with multiplicative error ε.
2. For both additive and multiplicative error, there exist real-time algorithms finding a longest palindrome with a given error within $O(M)$ words of memory.

★ If a string is close to random, it contains palindromes of length $O(\log n)$ only; a sliding-window real-time modification of Manacher's algorithm can found a longest palindrome exactly in $O(\log n)$ space!