Nominal Abstractions

- Nominal abstractions need to satisfy:
  - $\pi{\bullet}([a].x) = [\pi{\bullet}a].(\pi{\bullet}x)$
  - $[a].x \; = \; [b].y$ iff

    $$(a = b \wedge x = y) \vee$$
    $$(a \neq b \wedge x = (a\,b){\bullet}y \wedge a \# y)$$

- These properties imply that for finitely supported x:

$$\mathsf{supp}([a].x) = \mathsf{supp}(x) - \{a\}$$

- Are there any structures that satisfy these properties?

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  - $[a].x = [b].y$ iff

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    $(a \neq b \wedge x = (a\,b) \bullet y \wedge a \,\#\, y)$

- These properties imply that for finitely
  sup

From some perspective, this is a completely uninteresting question, because the user will only see this "interface".

- Are there any structures that satisfy these properties?

Yes...

- α-equivalence classes (sets of syntax trees), e.g. $[\lambda a.(a\ c)]_\alpha = [\lambda b.(b\ c)]_\alpha$
- terms with de-Bruijn indices and named free variables, like $\lambda(1\ c)$.
  (you need a function abs which "abstracts" a variable:
  $abs(x, t) \mapsto \lambda(\ldots)$)
- a weak HOAS encoding (lambdas as functions — the function for $\lambda a.(a\ c)$ will be the same as the one for $\lambda b.(b\ c)$)

Yes...

- α-equivalence classes (sets of syntax trees), e.g. $[\lambda a.(a\ c)]_\alpha = [\lambda b.(b\ c)]_\alpha$

- I could now stop here and probably go for α-equivalence classes (Norrish is doing this now with the help of a package by Hohmeier for HOL4).

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Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

$$[a].t \stackrel{\mathsf{def}}{=} (\lambda b.\,\mathsf{if}\ a = b$$
$$\mathsf{then\ Some}(t)$$
$$\mathsf{else\ if}\ b \,\#\, t\ \mathsf{then\ Some}((b\,a)\bullet t)\ \mathsf{else\ None})$$

$\boxed{\text{type: } \alpha \longrightarrow \iota\ \mathsf{option}}$

Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

$$
\begin{aligned}
[a].t \stackrel{\text{def}}{=} (\lambda b.\ & \text{if } a = b \\
& \text{then Some}(t) \\
& \text{else if } b \# t \text{ then Some}((b\,a) \bullet t) \text{ else None})
\end{aligned}
$$

This is supposed to stand for the α-equivalence class of $\lambda a.t$.

Function $[a].t$ ‘$=$’ $[\lambda a.t]_\alpha$

$$[a].\mathsf{App}(a,c) \stackrel{\text{def}}{=}$$
$$(\lambda b.\ \text{if } a = b$$
$$\text{then Some}(\mathsf{App}(a,c))$$
$$\text{else if } b \;\#\; \mathsf{App}(a,c)$$
$$\text{then Some}((b\,a) \bullet \mathsf{App}(a,c)) \text{ else None})$$

Let’s check this for $[a].\mathsf{App}(a,c)$:

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Let's check this for $[a].\mathsf{App}(a,c)$:

$[a].\mathsf{App}(a,c)$ 'applied to' a 'gvs' $\mathsf{Some}(\mathsf{App}(a,c))$

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$[a].\mathsf{App}(a,c)$ 'applied to' d 'gvs' $\mathsf{Some}(\mathsf{App}(d,c))$

$\vdots$

Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

$$[a].\mathsf{App}(a,c) \stackrel{\mathsf{def}}{=} (\lambda b.\ \mathsf{if}\ a = b\ \mathsf{then}\ \mathsf{Some}(\mathsf{App}(a,c))\ \mathsf{else\ if}\ b \# \mathsf{App}(a,c)\ \mathsf{then}\ \mathsf{Some}((b\,a)\bullet\mathsf{App}(a,c))\ \mathsf{else\ None})$$

Let's check this for $[a].\mathsf{App}(a,c)$:

$[a].\mathsf{App}(a,c)$ 'applied to' a 'gvs' $\mathsf{Some}(\mathsf{App}(a,c))$ '$\lambda a.(a\,c)$'

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$[a].\mathsf{App}(a,c)$ 'applied to' c 'gvs' None

$[a].\mathsf{App}(a,c)$ 'applied to' d 'gvs' $\mathsf{Some}(\mathsf{App}(d,c))$ '$\lambda d.(d\,c)$'

$\vdots$

Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$[a].\mathsf{App}(a,c) \stackrel{\mathsf{def}}{=}$
$(\lambda b.$ if $a = b$
then Some(App(a,c))
else if $b \# \mathsf{App}(a,c)$
then Some($(b\,a) \bullet \mathsf{App}(a,c)$) else None)

Let's check this for $[a].\mathsf{App}(a,c)$:

$[\lambda a.(a\,c)]_\alpha$:

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$[a].\mathsf{App}(a,c)$ 'applied to' c 'gvs' None

$[a].\mathsf{App}(a,c)$ 'applied to' d 'gvs' Some(App(d,c)) '$\lambda d.(d\,c)$'

$\vdots$ $\vdots$

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\end{array}
$$

Weak HOAS Encoding

I define **inductively** the α-equivalence classes of lambda-terms—but they still have **names**.

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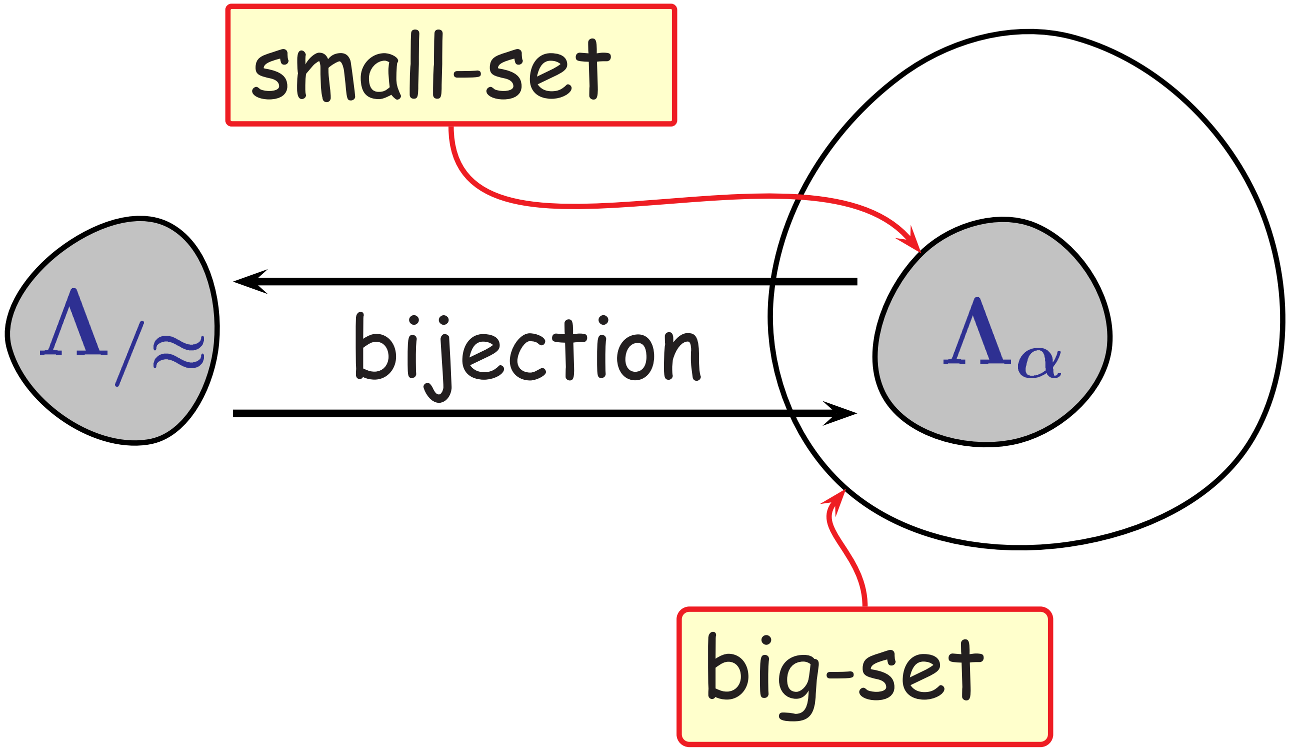

Big-Set / $Ftrms$

One step back: naïve attempt for big-set

$$
\begin{array}{rcll}
Ftrm ::= & \mathsf{Var} & : & Atom \\
\mid & \mathsf{App} & : & Ftrm \times Ftrm \\
\mid & \mathsf{Lam} & : & Ftrm\ Set
\end{array}
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$$

we try to encode the α-equivalence class as the set of lambda-terms

$$[t]_\alpha \stackrel{\mathsf{def}}{=} \{t' \mid t \approx t'\}$$

Big-Set / $Ftrms$

Better attempt for big-set

$$
\begin{array}{lll}
Ftrm ::= & \mathsf{Var} & : Atom \\
\quad\mid & \mathsf{App} & : Ftrm \times Ftrm \\
\quad\mid & \mathsf{Lam} & : Atom \rightarrow Ftrm\ \mathsf{option}
\end{array}
$$

same idea, but encoding with (partial) functions, along the lines:

"if $t' \in [\lambda a.t]_\alpha$ then 'yes' else None"

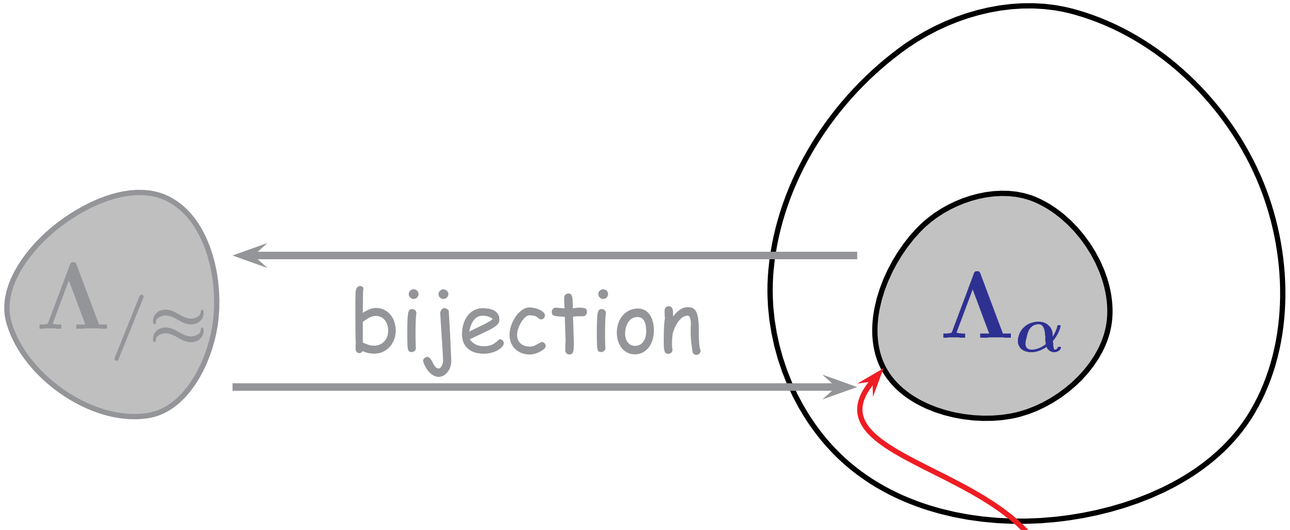
Permutations for Big-Set

Given the permutation action for atoms, we can permute all "**free**" atoms in $Ftrms$:

$$
\begin{array}{lcl}
\pi \bullet \mathsf{Var}(a) & \stackrel{\mathrm{def}}{=} & \mathsf{Var}(\pi \bullet a) \\
\pi \bullet \mathsf{App}(t_1, t_2) & \stackrel{\mathrm{def}}{=} & \mathsf{App}(\pi \bullet t_1, \pi \bullet t_2) \\
\pi \bullet \mathsf{Lam}(fn) & \stackrel{\mathrm{def}}{=} & \mathsf{Lam}(\lambda a. \pi \bullet (fn(\pi^{-1} \bullet a)))
\end{array}
$$

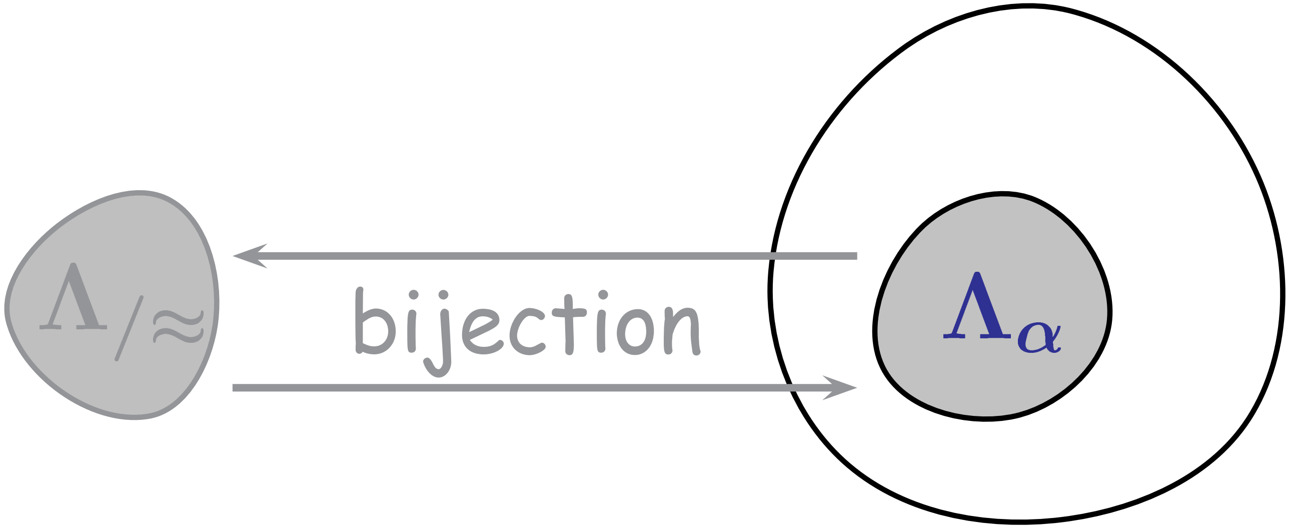
The $pt_{\alpha, ftrm}$-properties are satisfied.

Definition of Small-Set



$$
\begin{array}{rcl}
t ::= & & \mathsf{Var}(a) \\
 & | & \mathsf{App}(t_1, t_2) \\
 & | & \mathsf{Lam}\,[a].t
\end{array}
$$

Definition of Small-Set



$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

Definition of Small-Set

Which also means that we have a familiar induction principle in place for Λ_α (in a moment). And all terms in Λ_α have finite support.

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Definition of Small-Set

Which also means that we have a familiar induction principle in place for Λ_α (in a moment). And all terms in Λ_α have finite support.

$$
\begin{aligned}
\mathsf{supp}(\mathsf{Var}(a)) &= \{a\} \\
\mathsf{supp}(\mathsf{App}(t_1, t_2)) &= \mathsf{supp}(t1, t2) \\
\mathsf{supp}(\mathsf{Lam}\,[a].t) &= \mathsf{supp}([a].t) = \mathsf{supp}(t) - \{a\}
\end{aligned}
$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

Bijection

In order to show that $\Lambda_{/\approx}$ and Λ_α are bijective we define a function q from Λ to Λ_α:

$$
\begin{aligned}
q(a) &\stackrel{\text{def}}{=} \mathsf{Var}(a)\\
q(t_1\,t_2) &\stackrel{\text{def}}{=} \mathsf{App}(q(t_1), q(t_2))\\
q(\lambda a.t) &\stackrel{\text{def}}{=} \mathsf{Lam}\,[a].q(t)
\end{aligned}
$$

with the property

$$t_1 \approx t_2 \;\Leftrightarrow\; q(t_1) = q(t_2)$$

Struct. Induction on Λ_α

$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{\mathsf{Lam}\,[a].t \in \Lambda_\alpha}$$

Structural Induction Principle:

$$\frac{\begin{array}{l}(\forall a)\ P\,(\mathsf{Var}(a)) \\ (\forall t_1, t_2)\ P\,t_1 \wedge P\,t_2 \Rightarrow P\,(\mathsf{App}(t_1, t_2)) \\ (\forall a, t)\ P\,t \Rightarrow P\,(\mathsf{Lam}\,[a].t)\end{array}}{(\forall t)\ P\,t}$$

Substitution Lemma: If $x \not\equiv y$ and $x \notin FV(L)$, then

$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]].$$

Proof: By induction on the structure of M.

- **Case 1:** M is a variable.

  Case 1.1. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.

  Case 1.2. $M \equiv y$. Then both sides equal L, for $x \notin FV(L)$ implies $L[x := \ldots] \equiv L$.

  Case 1.3. $M \equiv z \not\equiv x, y$. Then both sides equal z.

- **Case 2:** $M \equiv \lambda z.M_1$. By the variable convention we may assume that $z \not\equiv x, y$ and z is not free in N, L. Then by induction hypothesis

$$\begin{aligned} & (\lambda z.M_1)[x := N][y := L] \\ \equiv\ & \lambda z.(M_1[x := N][y := L]) \\ \equiv\ & \lambda z.(M_1[y := L][x := N[y := L]]) \\ \equiv\ & (\lambda z.M_1)[y := L][x := N[y := L]]. \end{aligned}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. "$\square$"

Can We Do Better?

We consider induction hypotheses of the form $P\ t\ x$ with x **finitely supported**. In the case of the substitution lemma:

$$P = \lambda M.\ \lambda(x, y, N, L).$$
$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]]$$
$$x = (x, y, N, L)$$

$$\frac{\begin{array}{l}\forall a.\ P\,(\mathsf{Var}(a))\,x \\ \forall t_1, t_2.\ (\forall z.\, P\,t_1\,z) \wedge (\forall z.\, P\,t_2\,z) \Rightarrow P\,(\mathsf{App}(t_1, t_2))\,x \\ \forall a.\ a \# x \Rightarrow (\forall t.\ (\forall z.\, P\,t\,z) \Rightarrow P\,(\mathsf{Lam}\,[a].t)\,x)\end{array}}{P\,t\,x}$$

Proof

By an induction over Λ_α. We need to strengthen the induction hypothesis to

$$\forall t, \pi.\ P\,(\pi \bullet t)\,x$$

In the lambda-case we need to show:

$$P\,(\pi \bullet \mathsf{Lam}\,[a].t)\,x$$

We have:

$$\forall t, \pi.\ P\,(\pi \bullet t)\,x$$

$$\forall a.\, a \# x \Rightarrow (\forall t.\, (\forall z.\, P\,t\,z) \Rightarrow P\,(\mathsf{Lam}\,[a].t)\,x)$$

Proof (II)

We have:

$$\forall t, \pi.\ P\ (\pi \bullet t)\ x$$

$$\forall a.\ a \# x \Rightarrow (\forall t.\ (\forall z.\ P\ t\ z) \Rightarrow P\ (\mathsf{Lam}\ [a].t)\ x)$$

We choose a c so that $c\ \#\ (x, \pi \bullet a, \pi \bullet t)$.
Hence $\forall t.\ (\forall z.\ P\ t\ z)\ \Rightarrow\ P\ (\mathsf{Lam}\ [c].t)\ x$.
Moreover $P\ ((c\ \ \pi \bullet a) :: \pi \bullet t)\ x$. Therefore
$P\ (\mathsf{Lam}\ [c].((c\ \ \pi \bullet a) :: \pi \bullet t))\ x$. Now

$$\mathsf{Lam}\ [c].((c\ \pi \bullet a) :: \pi \bullet t) = \mathsf{Lam}\ [\pi \bullet a].(\pi \bullet t).$$

So we have $P\ (\pi \bullet \mathsf{Lam}\ [a].t)\ x$. Done.

Substitution

We do not know yet how to define functions, but assume we have the following **total** definition for substitution

$$
\begin{aligned}
\mathsf{Var}(a)[b := t'] &\stackrel{\text{def}}{=} \mathtt{if}\ a = b\ \mathtt{then}\ t'\ \mathtt{else}\ \mathsf{Var}(a)\\
\mathsf{App}(t_1, t_2)[b := t'] &\stackrel{\text{def}}{=} \mathsf{App}(t_1[b := t'], t_2[b := t'])\\
\mathsf{Lam}\,[a].t[b := t'] &\stackrel{\text{def}}{=} \mathsf{Lam}\,[a].(t[b := t'])\\
&\qquad\qquad \text{provided } a \# (b, t')
\end{aligned}
$$

Total, "because":

$$\forall\, t, b, t'.\ \exists\, r.\ \mathsf{subst\_rel}(t, b, t', r)\ .$$

“Forget” Lemma

This lemma is part of the substitution lemma.

$$b \,\#\, t \longrightarrow t[b := t'] = t$$

Induction on t.

$$P = \lambda t.\ \lambda(b, t').\ \ a \,\#\, t \longrightarrow t[b := t'] = t$$
$$x = (b, t')$$

We want to prove

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We want to prove

$$P\, t\, (b, t')\ .$$

Proof (I)

We have to show that:

$$\forall\, a.\ b \# \mathsf{Var}(a) \longrightarrow \mathsf{Var}(a)[b := t'] = \mathsf{Var}(a)$$

$$\frac{\begin{array}{l}\forall a.\ P\,(\mathsf{Var}(a))\,x \\ \forall t_1, t_2.\ (\forall z.\, P\,t_1\,z) \wedge (\forall z.\, P\,t_2\,z) \Rightarrow P\,(\mathsf{App}(t_1, t_2))\,x \\ \forall a.\ a \# x \Rightarrow (\forall t.\ (\forall z.\, P\,t\,z) \Rightarrow P\,(\mathsf{Lam}\,[a].t)\,x)\end{array}}{(\forall t)\ P\,t\,x}$$

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We have to show that:

$$\forall\, a.\; b \# \mathsf{Var}(a) \longrightarrow \mathsf{Var}(a)[b{:=}t'] = \mathsf{Var}(a)$$

This amounts to showing:

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Proof (II)

We have:

$$\forall (b, t'). \; b \# t_1 \longrightarrow t_1[b := t'] = t_1$$
$$\forall (b, t'). \; b \# t_2 \longrightarrow t_2[b := t'] = t_2$$

We have to show:

$$b \# \mathsf{App}(t_1, t_2) \longrightarrow$$
$$\mathsf{App}(t_1, t_2)[b := t'] = \mathsf{App}(t_1, t_2)$$

$$\frac{\begin{array}{l}\forall a. \; P \, (\mathsf{Var}(a)) \, x \\ \forall t_1, t_2. \; (\forall z. \, P \, t_1 \, z) \wedge (\forall z. \, P \, t_2 \, z) \Rightarrow P \, (\mathsf{App}(t_1, t_2)) \, x \\ \forall a. \; a \# x \Rightarrow (\forall t. \; (\forall z. \, P \, t \, z) \Rightarrow P \, (\mathsf{Lam} \, [a].t) \, x)\end{array}}{(\forall t) \;\; P \, t \, x}$$

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Proof (III)

We have:

$$a \;\#\; (b, t')$$
$$\forall (b, t').\; b \;\#\; t \longrightarrow t[b := t'] = t$$

We have to show:

$$b \;\#\; \mathsf{Lam}\,[a].t \longrightarrow$$
$$(\mathsf{Lam}\,[a].t)[b := t'] = \mathsf{Lam}\,[a].t$$

$$\frac{\begin{array}{l}\forall a.\; P\,(\mathsf{Var}(a))\;x \\ \forall t_1, t_2.\; (\forall z.\, P\,t_1\,z) \wedge (\forall z.\, P\,t_2\,z) \Rightarrow P\,(\mathsf{App}(t_1, t_2))\;x \\ \forall a.\; a \;\#\; x \Rightarrow (\forall t.\; (\forall z.\, P\,t\,z) \Rightarrow P\,(\mathsf{Lam}\,[a].t)\;x)\end{array}}{(\forall t)\;\; P\,t\,x}$$

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We have to show:

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$$\frac{t[b := t'] = t}{(\mathsf{Lam}\,[a].t)[b := t'] = \mathsf{Lam}\,[a].t}$$

$$\begin{aligned}
\mathsf{Var}(a)[b := t'] &\stackrel{\text{def}}{=} \texttt{if } a = b \texttt{ then } t' \texttt{ else } \mathsf{Var}(a) \\
\mathsf{App}(t_1, t_2)[b := t'] &\stackrel{\text{def}}{=} \mathsf{App}(t_1[b := t'], t_2[b := t']) \\
\mathsf{Lam}\,[a].t[b := t'] &\stackrel{\text{def}}{=} \mathsf{Lam}\,[a].(t[b := t']) \\
&\qquad\qquad \text{provided } a \# (b, t')
\end{aligned}$$

Proof (III)

We have:

$$a \# (b, t') \;\Rightarrow a \# b \wedge a \# t'$$

$$\forall (b, t').\; b \# t \longrightarrow t[b := t'] = t$$

We have to show:

$$\frac{t[b := t'] = t}{\mathsf{Lam}\,[a].(t[b := t']) = \mathsf{Lam}\,[a].t}$$

Proof (III)

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$$a \# (b, t') \;\Rightarrow a \# b \wedge a \# t'$$

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We have to show:

$$\frac{t[b := t'] = t}{\mathsf{Lam}\,[a].t = \mathsf{Lam}\,[a].t}$$

```
lemma forget:
shows "b # t ⟶ t[b := t'] = t"
proof (nominal_induct t rule: lam_induct)
  case (Var b t' a) (* case 1: variables *)
  show "b # Var a ⟶ (Var a)[b:=t'] = Var a" by (simp add: fresh_atm)
next
  case (App b t' t₁ t₂) (* case 3: applications *)
  thus "b # App t₁ t₂ ⟶ (App t₁ t₂)[b:=t'] = App t₁ t₂" by simp
next
  case (Lam b t' a t) (* case 2: lambdas *)
  have ih: "∀ b t'. b # t ⟶ t[b := t'] = t" by fact
  have f1: "a#(b, t')" by fact
  hence f2: "a ≠ b" by (simp add: fresh_prod fresh_atm)
  show "b # Lam [a].t ⟶ (Lam [a].t)[b:=t'] = Lam [a].t"
  proof
    assume "b # Lam [a].t"
    hence "b # t" using f2 by (simp add: abs_fresh)
    hence "t[b := t'] = t" using ih by simp
    thus "(Lam [a].t)[b := t'] = Lam [a].t" using f1 by simp
  qed
qed
```

Authoritative Version

lemma forget:
shows "$b \# t \longrightarrow t[b := t'] = t$"
apply (nominal_induct t rule: lam_induct)
apply (auto simp add: abs_fresh fresh_prod fresh_atm)
done

What is Coming Next

- a bit more about inductions
- how can functions be defined over α-equivalence classes (this is a sore point in the current version of the nominal datatype package)
- $\ldots$