Freshness Lemma

A criterion when a construction that "picks a fresh atom" is independent of which fresh atom is chosen:

Given $pt_{\alpha,\iota}$, at_α and a function $h : \alpha \longrightarrow \iota$:
if $\mathsf{finite}(\mathsf{supp}_\alpha\ h)$ and

$$\exists a.\ a \# h \wedge a \# (h\ a)$$

then

$$\exists! fresh.\ \forall a.\ a \# h \ \Rightarrow\ (h\ a) = fresh(h)$$

Structural Recursion for Λ

Given

- $g_{var} : name \longrightarrow x$
- $g_{app} : x \times x \longrightarrow x$
- $g_{lam} : name \times x \longrightarrow x$

there exists a function $\overline{g} : \Lambda \longrightarrow x$ with

- $\overline{g}(var\ a) = g_{var}(a)$
- $\overline{g}(app\ t_1\ t_2) = g_{app}(\overline{g}(t_1), \overline{g}(t_2))$
- $\overline{g}(lam\ a\ t) = g_{lam}(a, \overline{g}(t))$

Structural Recursion for Λ_α

Given (all finitely supported by A)

- $f_{VAR} : name \longrightarrow x$
- $f_{APP} : x \times x \longrightarrow x$
- $f_{LAM} : name \times x \longrightarrow x$ with $\exists a \notin A.\, \forall x.a \# f_{LAM}(a, x)$

there exists a function $\overline{f} : \Lambda_\alpha \longrightarrow x$ with

- $\overline{f}(VAR\ a) = f_{VAR}(a)$
- $\overline{f}(APP\ t_1\ t_2) = f_{APP}(\overline{f}(t_1), \overline{f}(t_2))$
- $a \notin A \Rightarrow \overline{f}(LAM\ a\ t) = f_{LAM}(a, \overline{f}(t))$

Use Recursion for Λ

Instantiate x with $\pi \longrightarrow x$

- $g_{var} : name \longrightarrow (\pi \longrightarrow x)$
- $g_{app} : (\pi \longrightarrow x) \times (\pi \longrightarrow x) \longrightarrow (\pi \longrightarrow x)$
- $g_{lam} : name \times (\pi \longrightarrow x) \longrightarrow (\pi \longrightarrow x)$

and set...

Use Recursion for Λ

Instantiate x with $\pi \longrightarrow x$

- $g_{var}(a)\ \pi = f_{VAR}(\pi \bullet x)$
- $g_{app}(s_1, s_2)\ \pi := f_{APP}(s_1\ \pi, s_2\ \pi)$
- $g_{lam}(a, s)\ \pi =$

$$fresh(\underbrace{\lambda a'.\ f_{LAM}(a', s\ (\pi \bullet (a\, a')))}_{h})$$

We know that there exists a $\overline{g}(s)\ \pi$ with...

Use Recursion for Λ

We know that there exists a $\overline{g}(s)\ \pi$ with...

- $\overline{g}(var\ a)\ \pi = g_{var}(a)\ \pi$
- $\overline{g}(app\ t_1\ t_2)\ \pi =$
  $g_{app}(\overline{g}(t_1)\ \pi, \overline{g}(t_2)\ \pi)$
- $\overline{g}(lam\ a\ t)\ \pi = g_{lam}(a, \overline{g}(t)\ \pi)$

We set $\overline{f} : \Lambda_\alpha \longrightarrow x$ to $\overline{g}(t)\ []$ (t is any representative Λ-term).

Properties

Now we have to check the properties:

- $\overline{f}(VAR\ a) = f_{VAR}(a)$
- $\overline{f}(APP\ t_1\ t_2) = f_{APP}(\overline{f}(t_1), \overline{f}(t_2))$
- $a \not\in A \Rightarrow \overline{f}(LAM\ a\ t) = f_{LAM}(a, \overline{f}(t))$

Interesting because of the way we used $fresh$.
(Assumptions A supports "everything";
$\exists a \not\in A.\ \forall x.\ a \# f_{LAM}(a, x)$.)

Freshness Lemma

A criterion when a construction that "picks a fresh atom" is independent of which fresh atom is chosen:

Given $pt_{\alpha,\iota}$, at_α and a function $h : \alpha \longrightarrow \iota$:
if $\mathsf{finite}(\mathsf{supp}_\alpha\ h)$ and

$$\exists a.\ a \# h \wedge a \# (h\ a)$$

then

$$\exists! fresh.\ \forall a.\ a \# h \ \Rightarrow\ (h\ a) = fresh(h)$$

BUT

I do not like it :o(