# Recap from Last Time

- we discussed a general theory of nominal sets using certain properties
- Permutations on atoms need to satisfy:
  - $[] \bullet a = a$
  - $((a\ b) :: \pi) \bullet c = swap\ (a\ b)\ (\pi \bullet c)$
  - $swap\ (a\ b)\ c =$
    $\texttt{if}\ a = c\ \texttt{then}\ b\ \texttt{else}$
    $\texttt{if}\ b = c\ \texttt{then}\ a\ \texttt{else}\ c$
  - every set of all atoms of type $\alpha$ is infinite

# General Permutations

Permutations have to satisfy:

- $[] \bullet x = x$
- $(\pi_1 @ \pi_2) \bullet x = \pi_1 \bullet (\pi_2 \bullet x)$
- $\pi_1 \sim \pi_2$ implies $\pi_1 \bullet x = \pi_2 \bullet x$

Two important definitions:

- $supp_\alpha\ x \stackrel{\text{def}}{=} \{a : \alpha \mid \text{infinite}\{b \mid (a\ b) \bullet x \neq x\}\}$ $\qquad x : \iota$
- $a \# x \stackrel{\text{def}}{=} a \notin supp_\alpha(x) \qquad a : \alpha, x : \iota$

# Abstractions

- Nominal abstractions need to satisfy $abst_{\alpha,\iota}$.
  - $\pi \bullet ([a].x) = [\pi \bullet a].(\pi \bullet x)$
  - $[a].x = [b].y$ iff

    $(a = b \wedge x = y) \vee$
    $(a \neq b \wedge x = (a\,b) \bullet y \wedge a \# y)$
- Are there any structures that satisfy these properties?

# Yes...

- $\alpha$-equivalence classes (sets of syntax trees)

$$[\lambda a.(a\ c)]_\alpha = [\lambda b.(b\ c)]_\alpha$$

- terms with de-Bruijn indices and named free variables

$$\lambda(1\ c)$$

  (you need a function $abs$ which "abstracts" a variable: $abs(x, t) \mapsto \lambda(\ldots)$)

- a weak HOAS encoding (lambdas as functions — the function for $\lambda a.(a\ c)$ will be the same as the one for $\lambda b.(b\ c)$)

# Weak HOAS Encoding

I define **inductively** the $\alpha$-equivalence classes of lambda-terms—but they still have **names**.

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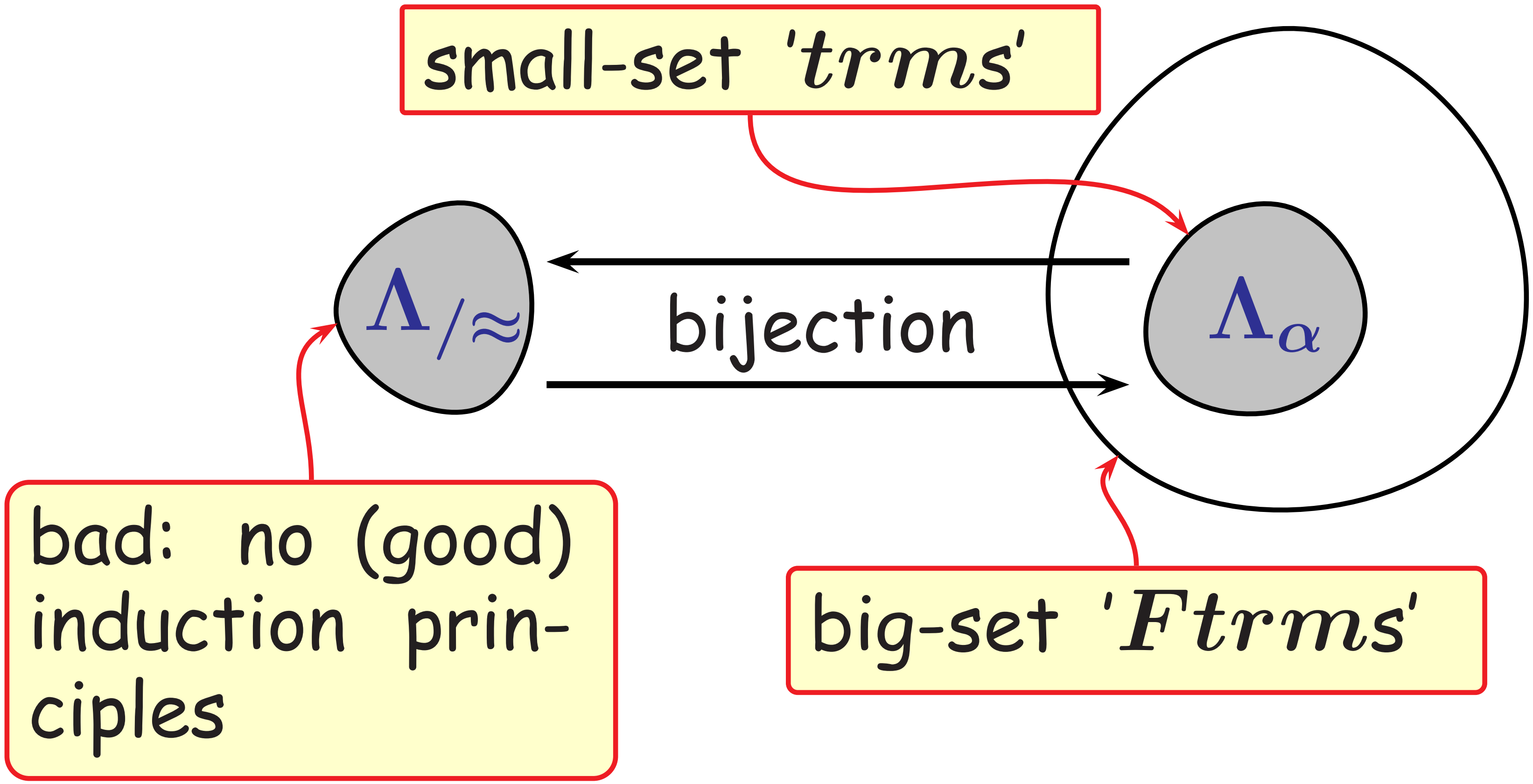

# Weak HOAS Encoding

I define **inductively** the $\alpha$-equivalence classes of lambda-terms—but they still have **names**.

The small set $\Lambda_\alpha$ will consist of $trm$s having the form

$$t ::= \mathsf{Var}(a)\ ^{*} \quad | \quad \mathsf{App}\ t\ t \quad | \quad [a].t$$

* $atoms$ come from an infinite set, e.g. the natural numbers

be
in
ciples

# Big-Set / $Ftrms$

One step back: naïve attempt for big-set

$$
\begin{array}{rlll}
Ftrm ::= & & & \\
 & \text{Var} & : Atom & \text{'atoms'} \\
 \mid & \text{App} & : Ftrm \times Ftrm & \text{'pairs'} \\
 \mid & \text{Se} & : Ftrm\ Set & \text{'}\alpha\text{-eq-cl'}
\end{array}
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$$

we try to encode the $\alpha$-equivalence class as the set of lambda-terms

$$[t]_\alpha \stackrel{\text{def}}{=} \{t' \mid t \approx t'\}$$

# Big-Set / $Ftrms$

Better attempt for big-set

$$
\begin{array}{lllll}
Ftrm ::= & \text{er} & & & \text{'error'} \\
\mid & \text{Var} & : & Atom & \text{'atoms'} \\
\mid & \text{App} & : & Ftrm \times Ftrm & \text{'pairs'} \\
\mid & \text{Se} & : & Atom \longrightarrow Ftrm & \text{'}\alpha\text{-eq-cl'}
\end{array}
$$

same idea, but encoding with (partial) functions, along the lines:

"if $t' \in [\lambda a.t]_\alpha$ then 'yes' else er"

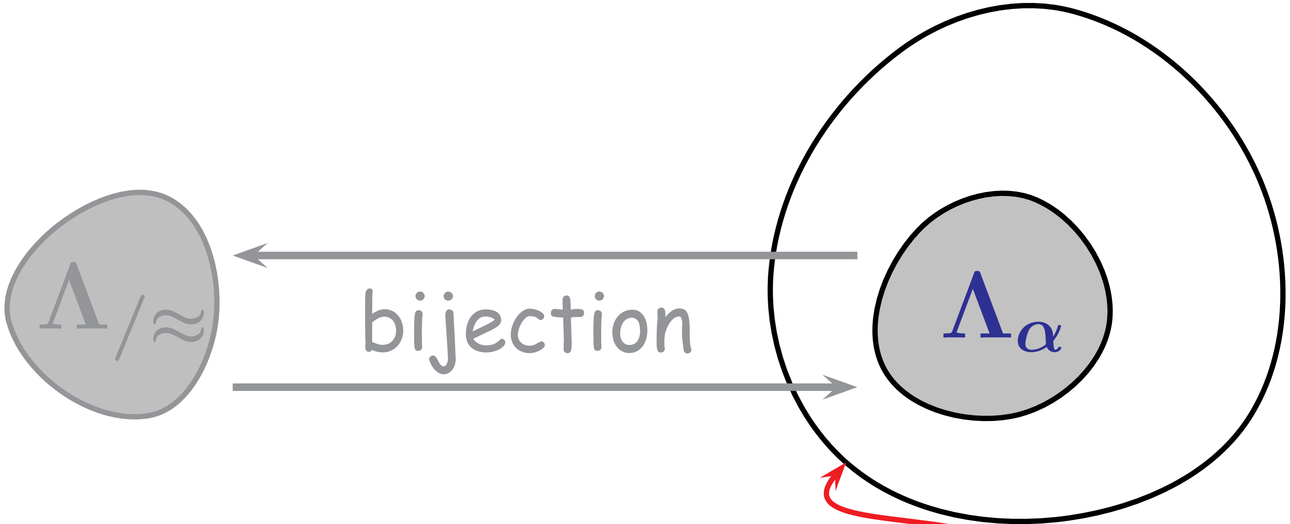
# Permutations for Big-Set

Given the permutation action for atoms, we can permute all "**free**" atoms in $Ftrm$s:

$$
\begin{aligned}
\pi \bullet \mathsf{er} &\stackrel{\text{def}}{=} \mathsf{er} \\
\pi \bullet \mathsf{Var}(a) &\stackrel{\text{def}}{=} \mathsf{Var}(\pi \bullet a) \\
\pi \bullet \mathsf{App}(t_1, t_2) &\stackrel{\text{def}}{=} \mathsf{App}(\pi \bullet t_1, \pi \bullet t_2) \\
\pi \bullet \mathsf{Se}(fn) &\stackrel{\text{def}}{=} \mathsf{Se}(\lambda a. \pi \bullet (fn(\pi^{-1} \bullet a)))
\end{aligned}
$$

The $pt_{\alpha, ftrm}$-properties are satisfied.

# What About Small-Set?




$$
\begin{array}{rcl}
t & ::= & \mathsf{er} \\
 & | & \mathsf{Var}(a) \\
 & | & \mathsf{App}(t_1, t_2) \\
 & | & \mathsf{Se}(fn)
\end{array}
$$

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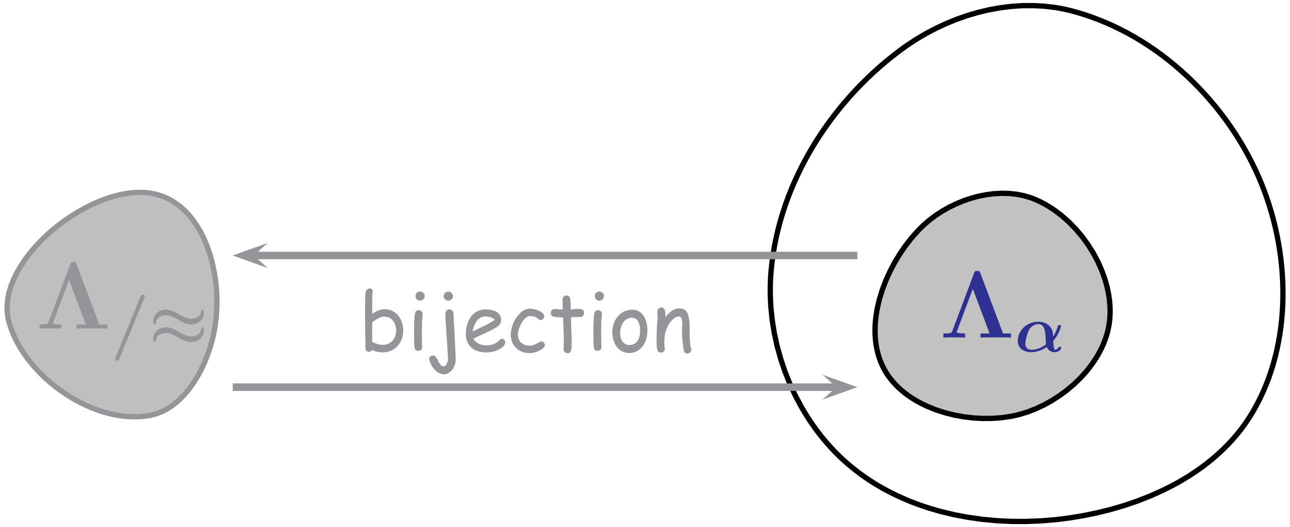


For $\Lambda_\alpha$, we are only interested in some very specific functions, namely

$$
\begin{aligned}
[a].t \stackrel{\text{def}}{=} \text{Se}\ (\lambda b.\ & \text{if } a = b \\
& \text{then } t \\
& \text{else if } b \# t \text{ then } (b\,a) \bullet t \text{ else er})
\end{aligned}
$$

# Function $[a].t$ '=' $[\lambda a.t]_\alpha$

$$[a].t \stackrel{\text{def}}{=} \text{Se}\,(\lambda b.\ \begin{array}{l}\text{if } a = b\\ \text{then } t\\ \text{else if } b \# t \text{ then } (b\,a)\bullet t \text{ else er})\end{array}$$

# Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

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This is supposed to stand for the $\alpha$-equivalence class of $\lambda a.t$.

# Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

$$
\begin{array}{l}
[a].\mathsf{App}(a,c) \stackrel{\mathsf{def}}{=} \\
\qquad\qquad \mathsf{Se}\ (\lambda b.\ \mathsf{if}\ a = b \\
\qquad\qquad\qquad\qquad \mathsf{then}\ \mathsf{App}(a,c) \\
\qquad\qquad\qquad\qquad \mathsf{else\ if}\ b \# \mathsf{App}(a,c) \\
\qquad\qquad\qquad\qquad\qquad \mathsf{then}\ (b\,a) \bullet \mathsf{App}(a,c)\ \mathsf{else\ er})
\end{array}
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Let's check this for $[a].\mathsf{App}(a,c)$:

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$\vdots$

# Function $[a].t$ '$=$' $[\lambda a.t]_\alpha$

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Let's check this for $[a].\mathsf{App}(a,c)$:

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$\vdots$

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$\quad\quad\quad$ then $(b\,a) \bullet \mathsf{App}(a,c)$ else er$)$

Let's check this for $[a].\mathsf{App}(a,c)$:

| | | | | | $[\lambda a.(a\,c)]_\alpha$: |
|---|---|---|---|---|---|
| $[a].\mathsf{App}(a,c)$ | 'applied to' | $a$ | 'gives' | $\mathsf{App}(a,c)$ | '$\lambda a.(a\,c)$' |
| $[a].\mathsf{App}(a,c)$ | 'applied to' | $b$ | 'gives' | $\mathsf{App}(b,c)$ | '$\lambda b.(b\,c)$' |
| $[a].\mathsf{App}(a,c)$ | 'applied to' | $c$ | 'gives' | er | |
| $[a].\mathsf{App}(a,c)$ | 'applied to' | $d$ | 'gives' | $\mathsf{App}(d,c)$ | '$\lambda d.(d\,c)$' |
| | ⋮ | | | | ⋮ |

# Properties of $[a].t$

Nominal Abstractions need to satisfy $abst_{\alpha,\iota}$:

- $\pi \bullet ([a].t) = [\pi \bullet a].(\pi \bullet t)$
- $[a].x = [b].y$ iff

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The last axiom says that $[a].t$ behaves like an abstraction.

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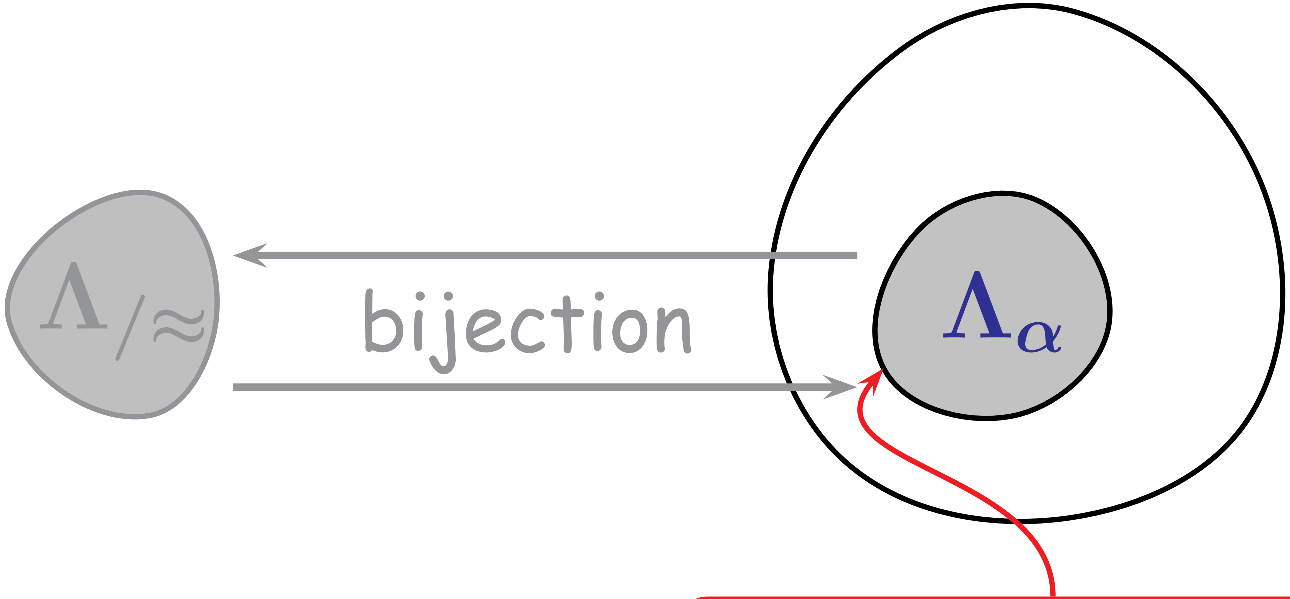
The last axiom says that $[a].t$ behaves like an abstraction.

Remember the definition of $\alpha$-equivalence from the detour:

$$\frac{t_1 \approx t_2}{\lambda a.t_1 \approx \lambda a.t_2} \qquad \frac{t_1 \approx (a\, b) \bullet t_2 \quad a \notin FV(t_2)}{\lambda a.t_1 \approx \lambda b.t_2}$$

# Definition of Small-Set

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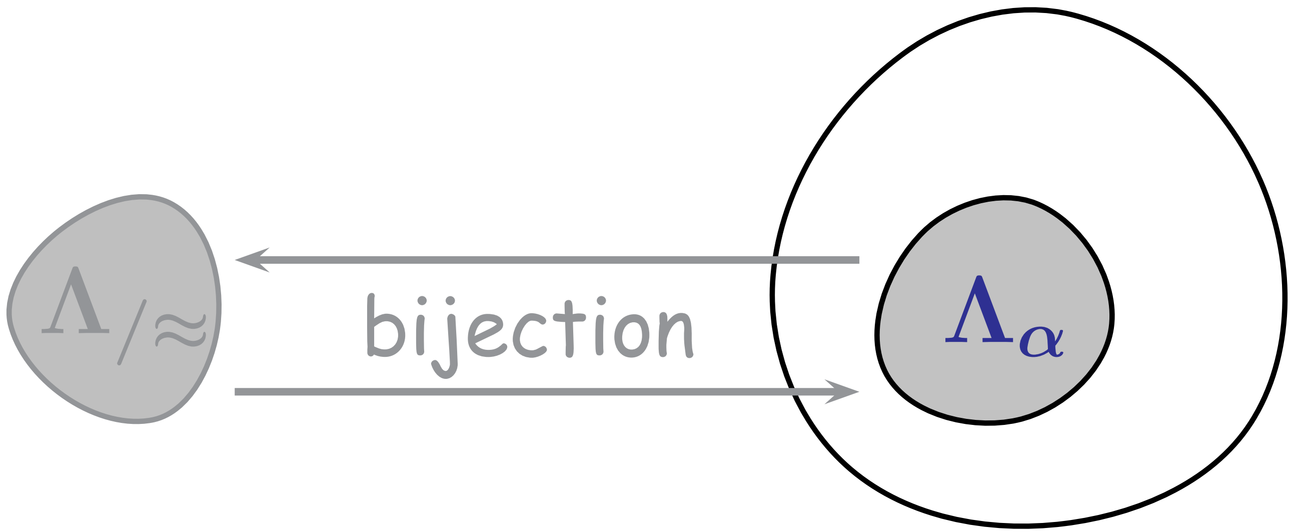


$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{[a].t \in \Lambda_\alpha}$$

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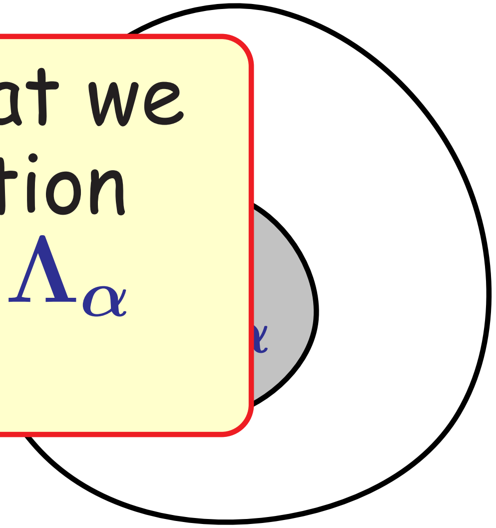

Which also means that we have a familiar induction principle in place for $\Lambda_\alpha$ (in a moment).

$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{[a].t \in \Lambda_\alpha}$$

# Bijection

In order to show that $\Lambda_{/\approx}$ and $\Lambda_\alpha$ are bijective we define a function $q$ from $\Lambda$ to $\Lambda_\alpha$:

$$
\begin{array}{lcl}
q(a) & \stackrel{\text{def}}{=} & \mathsf{Var}(a) \\
q(t_1\, t_2) & \stackrel{\text{def}}{=} & \mathsf{App}(q(t_1), q(t_2)) \\
q(\lambda a.t) & \stackrel{\text{def}}{=} & [a].q(t)
\end{array}
$$

with the property

$$t_1 \approx t_2 \;\Leftrightarrow\; q(t_1) = q(t_2)$$

# Struct. Induction on $\Lambda_\alpha$

$$\frac{}{\mathsf{Var}(a) \in \Lambda_\alpha} \qquad \frac{t_1 \in \Lambda_\alpha \quad t_2 \in \Lambda_\alpha}{\mathsf{App}(t_1, t_2) \in \Lambda_\alpha}$$

$$\frac{t \in \Lambda_\alpha}{[a].t \in \Lambda_\alpha}$$

Structural Induction Principle:

$$\frac{\begin{array}{l}(\forall a)\ P\,(\mathsf{Var}(a)) \\ (\forall t_1, t_2)\ P\,t_1 \wedge P\,t_2 \Rightarrow P\,(\mathsf{App}(t_1, t_2)) \\ (\forall a, t)\ P\,t \Rightarrow P\,([a].t)\end{array}}{(\forall t)\ P\,t}$$

**Substitution Lemma:** If $x \not\equiv y$ and $x \notin FV(L)$, then

$$M[x := N][y := L] \equiv M[y := L][x := N[y := L]].$$

**Proof:** By induction on the structure of $M$.

- **Case 1:** $M$ is a variable.

  Case 1.1. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.

  Case 1.2. $M \equiv y$. Then both sides equal $L$, for $x \notin FV(L)$ implies $L[x := \ldots] \equiv L$.

  Case 1.3. $M \equiv z \not\equiv x, y$. Then both sides equal $z$.

- **Case 2:** $M \equiv \lambda z.M_1$. By the variable convention we may assume that $z \not\equiv x, y$ and $z$ is not free in $N, L$. Then by induction hypothesis

$$\begin{aligned} &(\lambda z.M_1)[x := N][y := L] \equiv \lambda z.(M_1[x := N][y := L]) \\ \equiv\ & \lambda z.(M_1[y := L][x := N[y := L]]) \\ \equiv\ & (\lambda z.M_1)[y := L][x := N[y := L]]. \end{aligned}$$

- **Case 3:** $M \equiv M_1 M_2$. The statement follows again from the induction hypothesis. "$\square$"

# Can We Do Better?

We consider induction hypotheses of the form $P\ t\ x$ with $x$ finitely supported. In the case of the substitution lemma:

$$P = \lambda M.\ \lambda(x, y, N, L).\ M[x := N][y := L]$$
$$\equiv M[y := L][x := N[y := L]]$$

$$x = (x, y, N, L)$$

$$\frac{\begin{array}{l}(\forall a)\ P\,(\mathsf{Var}(a))\,x \\ (\forall t_1, t_2)\ P\,t_1\,x \wedge P\,t_2\,x \Rightarrow P\,(\mathsf{App}(t_1, t_2))\,x \\ (\forall a)\ a \,\#\, x \Rightarrow ((\forall t)\ P\,t\,x \Rightarrow P\,([a].t)\,x)\end{array}}{(\forall t)\ P\,t\,x}$$

# Proof

By an induction over $\Lambda_\alpha$. We need to strengthen the induction hypothesis to

$$(\forall t, \pi)\ P\,(\pi \bullet t)\,x$$

In the lambda-case we need to show:

$$P\,(\pi \bullet [a].t)\,x$$

We have:

$$(\forall t, \pi)\ P\,(\pi \bullet t)\,x$$

$$(\forall a)\ a \# x \Rightarrow ((\forall t)\ P\,t\,x \Rightarrow P\,([a].t)\,x)$$

```
lemma subs_lemma: assumes a1: "x ≠ y" and a2: "x#L"
shows "M[x := N][y := L] = M[y := L][x := N[y := L]]"
proof (nominal_induct M)
  case (Var z) (* case 1: variables *)
  have "z = x ∨ (z ≠ x ∧ z = y) ∨ (z ≠ x ∧ z ≠ y)" by force
  thus "(Var z)[x:=N][y:=L] = (Var z)[y:=L][x:=N[y:=L]]"
    using a1 a2 forget by force
next
  case ([z].M1) (* case 2: lambdas *)
  assume ih: "M1[x:=N][y:=L] = M1[y:=L][x:=N[y:=L]]"
  assume f1: "z#(L, N, x, y)"
  from f1 fresh_fact1 have f2: "z#N[y := L]" by simp
  show "([z].M1)[x:=N][y:=L] = ([z].M1)[y:=L][x:=N[y:=L]]"
(is "?LS = ?RS")
  proof -
    have "?LS = [z].(M1[x := N][y := L])" using f1 by simp
    also have "... = [z].(M1[y:=L][x:=N[y:=L]])" using ih by simp
    also have "... =  ?RS" using f1 f2 by simp
    finally show "?LS = ?RS" by simp
  qed
next
  case (App M1 M2) (* case 3: applications *)
  thus "?case" by simp
qed
```