Nominal Logic and Reasoning about α-Equivalent Syntax

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every Tuesday from 14:15 to 16:00 in Room E27

(except 26. April, 17. May and 5. July)

Formalising Nominal Sets in HOL

- In the first two lectures we worked with concrete objects – syntax trees; we defined permutation operation over them and found they have some nice properties.
- Now, we will work abstractly. We have different kinds of atoms (in the lambda-calculus you have only one, because only one kind of binder; Pi-calculus or System F have more than one kind of binder),
- and separate 'our world' into atoms and the 'rest'.

Some Polymorphic Functions

- α will stand for type of atoms; ι will be rest
- type-constructor for permutations
  - $\alpha\ prm : (\alpha \times \alpha)\ list$
- functions
  - $\_ \bullet \_ : \alpha\ prm \Rightarrow \iota \Rightarrow \iota$
  - $swap : \alpha \times \alpha \Rightarrow \alpha \Rightarrow \alpha$
  - $[\_].\_ : \alpha \Rightarrow \iota \Rightarrow \iota$

(i) is applying a permutation, (ii) is swapping an atom, (ii) is abstracting an atom

Permutations on Atoms

Permutations on atoms need to satisfy:

- $[]\bullet a = a$
- $((a\ b) :: \pi)\bullet c = swap\ (a\, b)\ (\pi\bullet c)$
- $swap\ (a\ b)\ c =$
  $$\texttt{if}\ a = c\ \texttt{then}\ b\ \texttt{else}$$
  $$\texttt{if}\ b = c\ \texttt{then}\ a\ \texttt{else}\ c$$
- each set of all atoms of type α is infinite

We shall collectively refer to these properties as at_α (this makes use of Locales in Isabelle).

Atoms

In Isabelle we can create such atoms by declaring new types α_k each with an axiom that assumes infinity and two primitive recursive functions:

- $swap_k\ (a\ b)\ c = \mathtt{if} \ldots$
- $[]\bullet_k c = c$
- $((a\ b) :: \pi)\bullet_k c = swap_k\ (a\ b)\ (\pi \bullet c)$

Trivial task to show that at_{α_k} are satisfied.

Properties

Assuming at_α, we have

- $(\pi_1 @ \pi_2) \bullet c = \pi_1 \bullet (\pi_2 \bullet c)$

  induction on π_1
- $\pi^{-1} \bullet \pi \bullet c = c$

  induction on π and swapping being infolutive
- $\pi \bullet a = \pi \bullet b$ iff $a = b$

  by a calculation using the above
- ...

Equality on Permutations

This list-representation of permutations is very nice to work with in Isabelle, but it is not a unique representation (very laborious to achieve such a representation in a theorem-prover, I think).

$$\pi_1 \sim \pi_2 \stackrel{\text{def}}{=} \{a : \alpha \mid \pi_1 \bullet a \neq \pi_2 \bullet a\} = \varnothing$$

- $[(a\ a)] \sim []$
- $[(a\ b)] \sim [(b\ a)]$

- $[(a\ c), (b\ c), (a\ c)] \sim [(a\ b)]$
  provided a, b and c are distinct

General Permutations

So far we have only considered permutations acting on atoms. We are now going to overload $\_ \bullet \_ : \alpha\ prm \Rightarrow \iota \Rightarrow \iota$ to act on the 'rest of the world'. Permutations have to satisfy:

- $[] \bullet x = x$
- $(\pi_1 @ \pi_2) \bullet x = \pi_1 \bullet (\pi_2 \bullet x)$
- $\pi_1 \sim \pi_2$ implies $\pi_1 \bullet x = \pi_2 \bullet x$

We refer to these properties as $pt_{\alpha,\iota}$.

From these Properties

Given $pt_{\alpha,\iota}$ and at_α:

- $\pi^{-1} \bullet \pi \bullet x = x$

  using $(\pi^{-1} @ \pi) \sim []$

- $\pi \bullet x = \pi \bullet y$ iff $x = y$

  using the above and some calculations

- $\forall a, b.\ (a\,b) \bullet x = x$ iff $\forall \pi.\ \pi \bullet x = x$

  in one direction induction on π

Overloading of $\bullet$

- $\pi \bullet a \stackrel{\mathsf{def}}{=} a \qquad \pi : \alpha\ prm, a : \alpha', \alpha \neq \alpha'$
  we have $pt_{\alpha,\alpha'}$ (and at_α implies $pt_{\alpha,\alpha}$)
- $\pi \bullet X \stackrel{\mathsf{def}}{=} \{\pi \bullet x \mid x \in X\}$
  $pt_{\alpha,\iota}$ implies $pt_{\alpha,\iota\ set}$
- $\pi \bullet (x_1, x_2) \stackrel{\mathsf{def}}{=} (\pi \bullet x_1, \pi \bullet x_2)$
  pt_{α,ι_1} and pt_{α,ι_2} implies $pt_{\alpha,\iota_1 \times \iota_2}$
- $\pi \bullet [] \stackrel{\mathsf{def}}{=} []$
  $\pi \bullet (x :: xs) \stackrel{\mathsf{def}}{=} (\pi \bullet x) :: (\pi \bullet xs)$
  $pt_{\alpha,\iota}$ implies $pt_{\alpha,\iota\ list}$

Permutations on Functions

- $\pi \bullet f \stackrel{\text{def}}{=} \lambda x.\pi \bullet (f(\pi^{-1} \bullet x))$
  $$\pi : \alpha\ prm, f : \iota_1 \Rightarrow \iota_2$$
- given at_α, pt_{α,ι_1} and pt_{α,ι_2} then $pt_{\alpha,\iota_1 \Rightarrow \iota_2}$
  (we have to know $\pi_1 \sim \pi_2$ implies $\pi_1^{-1} \sim \pi_2^{-1}$)
- assuming at_α and pt_{α,ι_1}, then
  $$\pi \bullet (f\ x) = (\pi \bullet f)(\pi \bullet x)$$
  where $\pi : \alpha\ prm$ and $f : \iota_1 \Rightarrow \iota_2$

Support and Freshness

- $supp_\alpha\ x \stackrel{\text{def}}{=} \{a : \alpha \mid \text{infinite}\{b \mid (a\ b)\bullet x \neq x\}\}$ $\quad x : \iota$
- $a \# x \stackrel{\text{def}}{=} a \notin supp_\alpha(x)$ $\quad a : \alpha, x : \iota$
- $X\ supports\ x \stackrel{\text{def}}{=} \forall a, b \notin X.\ \ (a\ b)\bullet x = x$ $\quad X : \alpha\ set, x : \iota$

I will drop the α in $supp_\alpha$.
The notion $supports$ will play a role in the next lecture (it s'times helps for calculating the $supp$).

Simple Properties

- $supp\ (x_1, x_2) = (supp\ x_1) \cup (supp\ x_2)$

  $a \ \# \ (x_1, x_2)$ iff $a \ \# \ x_1 \wedge a \ \# \ x_2$
- $supp\ (a : \alpha) = \{a\}$ provided at_α

  $a \ \# \ b$ iff $a \neq b$
- $supp\ A = \varnothing$ A set of all atoms α

  since $\forall a, b.\ (a\ b) \bullet A = A$
- $supp\ F = \{a_1, \ldots, a_n\}$ assuming F is a finite set of atoms $a_1, \ldots, a_n$

  not every set of atoms has finite support: e.g. $A/2$

Existence of a Fresh Atom

Q: Why do we assume that there are infinitely many atoms?

A: For any finitely supported x:

$$\exists c.\ \ c \# x$$

If something is finitely supported, then we can always choose a fresh atom (also for functions).

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$\pi \bullet supp(x) = supp(\pi \bullet x)$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$\pi \bullet supp(x)$$

$$\stackrel{\text{def}}{=} \pi \bullet \{a \mid \inf\{b \mid (a\,b) \bullet x \neq x\}\}$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$\pi \bullet supp(x)$$

$$\stackrel{\text{def}}{=} \pi \bullet \{a \mid \inf\{b \mid (a\,b) \bullet x \neq x\}\}$$

$$\stackrel{\text{def}}{=} \{\pi \bullet a \mid \inf\{b \mid (a\,b) \bullet x \neq x\}\}$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$
\begin{aligned}
& \pi\bullet supp(x) \\
\stackrel{\text{def}}{=}\ & \pi\bullet\{a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\} \\
\stackrel{\text{def}}{=}\ & \{\pi\bullet a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\} \\
=\ & \{\pi\bullet a \mid \inf\{\pi\bullet b \mid (a\,b)\bullet x \neq x\}\}
\end{aligned}
$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$
\begin{aligned}
& \pi \bullet supp(x) \\
\stackrel{\text{def}}{=}\ & \pi \bullet \{a \mid \inf\{b \mid (a\,b) \bullet x \neq x\}\} \\
\stackrel{\text{def}}{=}\ & \{\pi \bullet a \mid \inf\{b \mid (a\,b) \bullet x \neq x\}\} \\
=\ & \{\pi \bullet a \mid \inf\{\pi \bullet b \mid (a\,b) \bullet x \neq x\}\} \\
=\ & \{a \mid \inf\{b \mid (\pi^{-1} \bullet a\ \ \pi^{-1} \bullet b) \bullet x \neq x\}\}
\end{aligned}
$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$
\begin{aligned}
&\pi\bullet supp(x)\\
\stackrel{\text{def}}{=}\;& \pi\bullet\{a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\}\\
\stackrel{\text{def}}{=}\;& \{\pi\bullet a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\}\\
=\;& \{\pi\bullet a \mid \inf\{\pi\bullet b \mid (a\,b)\bullet x \neq x\}\}\\
=\;& \{a \mid \inf\{b \mid (\pi^{-1}\bullet a\;\; \pi^{-1}\bullet b)\bullet x \neq x\}\}\\
=\;& \{a \mid \inf\{b \mid \pi\bullet(\pi^{-1}\bullet a\;\; \pi^{-1}\bullet b)\bullet x \neq \pi\bullet x\}\}
\end{aligned}
$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$
\begin{aligned}
& \pi \bullet supp(x) \\
\stackrel{\text{def}}{=}\; & \pi \bullet \{a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\} \\
\stackrel{\text{def}}{=}\; & \{\pi \bullet a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\} \\
=\; & \{\pi \bullet a \mid \inf\{\pi \bullet b \mid (a\,b)\bullet x \neq x\}\} \\
=\; & \{a \mid \inf\{b \mid (\pi^{-1}\bullet a\ \ \pi^{-1}\bullet b)\bullet x \neq x\}\} \\
=\; & \{a \mid \inf\{b \mid \pi\bullet(\pi^{-1}\bullet a\ \ \pi^{-1}\bullet b)\bullet x \neq \pi\bullet x\}\} \\
=\; & \{a \mid \inf\{b \mid (a\,b)\bullet\pi\bullet x \neq \pi\bullet x\}\}
\end{aligned}
$$

More Interesting Prop's

Assuming at_α, $pt_{\alpha,\iota}$:

$$
\begin{aligned}
&\pi \bullet supp(x)\\
\overset{\text{def}}{=}\ &\pi \bullet \{a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\}\\
\overset{\text{def}}{=}\ &\{\pi \bullet a \mid \inf\{b \mid (a\,b)\bullet x \neq x\}\}\\
=\ &\{\pi \bullet a \mid \inf\{\pi \bullet b \mid (a\,b)\bullet x \neq x\}\}\\
=\ &\{a \mid \inf\{b \mid (\pi^{-1}\bullet a\ \ \pi^{-1}\bullet b)\bullet x \neq x\}\}\\
=\ &\{a \mid \inf\{b \mid \pi \bullet (\pi^{-1}\bullet a\ \ \pi^{-1}\bullet b)\bullet x \neq \pi \bullet x\}\}\\
=\ &\{a \mid \inf\{b \mid (a\,b)\bullet \pi \bullet x \neq \pi \bullet x\}\}\\
\overset{\text{def}}{=}\ &supp(\pi \bullet x)
\end{aligned}
$$

It is as Simple as That. . .

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

It is as Simple as That…

$$at_\alpha, pt_{\alpha,\iota} : a \mathrel{\#} x \wedge b \mathrel{\#} x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a = b$ clear

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

$$a \# x \stackrel{\text{def}}{=} a \notin \mathsf{supp}(x)$$

$$\mathsf{supp}(x) \stackrel{\text{def}}{=} \{a \mid \mathsf{inf}\{c \mid (a\,c)\bullet x \neq x\}\}$$

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\mathrm{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathrm{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2) $\mathrm{fin}(\{c \mid (a\,c)\bullet x \neq x\} \cup \{c \mid (b\,c)\bullet x \neq x\})$ f. (1)

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \,\#\, x \wedge b \,\#\, x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2') $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2') $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3) $\mathsf{inf}\{c \mid \neg((a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x)\}$ f. (2')

Given a finite set of atoms,
its 'co-set' must be infinite.

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

<u>Proof:</u> case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\mathsf{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2') $\mathsf{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3') $\mathsf{inf}\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$ f. (2')

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) fin$\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$
fin$\{c \mid (b\,c)\bullet x \neq x\}$

(2′) fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3′) inf$\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$ f. (2′)

(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ for a $c \in$ (3′)

If a set is infinite, it must contain a few elements. Let's pick c so that $c \neq a, b$.

It is as Simple as That...

$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\, b) \bullet x = x$

Proof: case $a \neq b$:

(1) fin$\{c \mid (a\, c) \bullet x \neq x\}$ from Ass. +Def. of $\#$
fin$\{c \mid (b\, c) \bullet x \neq x\}$

(2′) fin$\{c \mid (a\, c) \bullet x \neq x \vee (b\, c) \bullet x \neq x\}$ f. (1)

(3′) inf$\{c \mid (a\, c) \bullet x = x \wedge (b\, c) \bullet x = x)\}$ f. (2′)

(4) (i) $(a\, c) \bullet x = x$ (ii) $(b\, c) \bullet x = x$ for a $c \in$ (3′)

(5) $(a\, c) \bullet x = x$ by (4i)

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \mathrel{\#} x \wedge b \mathrel{\#} x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) fin$\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$
fin$\{c \mid (b\,c)\bullet x \neq x\}$

(2') fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3') inf$\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$ f. (2')

(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ for a $c \in$ (3')

(5) $(a\,c)\bullet x = x$ by (4i)

(6) $(b\,c)\bullet(a\,c)\bullet x = (b\,c)\bullet x$ by bij.

$\boxed{\text{bij.: } x = y \text{ iff } \pi\bullet x = \pi\bullet y}$

It is as Simple as That. . .

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b){\bullet}x = x$$

Proof: case $a \neq b$:

(1)	fin$\{c \mid (a\,c){\bullet}x \neq x\}$	from Ass. +Def. of $\#$
	fin$\{c \mid (b\,c){\bullet}x \neq x\}$	
(2')	fin$\{c \mid (a\,c){\bullet}x \neq x \vee (b\,c){\bullet}x \neq x\}$	f. (1)
(3')	inf$\{c \mid (a\,c){\bullet}x = x \wedge (b\,c){\bullet}x = x)\}$	f. (2')
(4)	(i) $(a\,c){\bullet}x = x$ (ii) $(b\,c){\bullet}x = x$	for a $c \in$ (3')
(5)	$(a\,c){\bullet}x = x$	by (4i)
(6')	$(b\,c){\bullet}(a\,c){\bullet}x = x$	by bij.,(4ii)

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\, b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\mathsf{fin}\{c \mid (a\, c)\bullet x \neq x\}$ from Ass. +Def. of $\#$
$\mathsf{fin}\{c \mid (b\, c)\bullet x \neq x\}$

(2′) $\mathsf{fin}\{c \mid (a\, c)\bullet x \neq x \vee (b\, c)\bullet x \neq x\}$ f. (1)

(3′) $\mathsf{inf}\{c \mid (a\, c)\bullet x = x \wedge (b\, c)\bullet x = x)\}$ f. (2′)

(4) (i) $(a\, c)\bullet x = x$ (ii) $(b\, c)\bullet x = x$ for a $c \in$ (3′)

(5) $(a\, c)\bullet x = x$ by (4i)

(6′) $(b\, c)\bullet(a\, c)\bullet x = x$ by bij.,(4ii)

(7) $(a\, c)\bullet(b\, c)\bullet(a\, c)\bullet x = (a\, c)\bullet x$ by bij.

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b) \bullet x = x$$

Proof: case $a \neq b$:

(1) fin$\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

fin$\{c \mid (b\,c)\bullet x \neq x\}$

(2′) fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3′) inf$\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$ f. (2′)

(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ for a $c \in$ (3′)

(5) $(a\,c)\bullet x = x$ by (4i)

(6′) $(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4ii)

(7′) $(a\,c)\bullet(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4i)

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1) $\text{fin}\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$

$\quad\ \text{fin}\{c \mid (b\,c)\bullet x \neq x\}$

(2') $\text{fin}\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3') $\text{inf}\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$ f. (2')

(4) (i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$ for a $c \in$ (3')

(5) $(a\,c)\bullet x = x$ by (4i)

(6') $(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4ii)

(7') $(a\,c)\bullet(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4i)

$$[(a\,c),(b\,c),(a\,c)] \sim [(a\,b)]$$

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

<u>Proof:</u> case $a \neq b$:

(1) fin$\{c \mid (a\,c)\bullet x \neq x\}$ from Ass. +Def. of $\#$
fin$\{c \mid (b\,c)\bullet x \neq x\}$

(2′) fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$ f. (1)

(3′) inf$\{c \mid (a$ f. (2′)

(4) (i) $(a\,c)\bullet$ a $c \in$ (3′)

3rd property of $pt_{\alpha,\iota}$:
$\pi_1 \sim \pi_2 \Rightarrow \pi_1\bullet x = \pi_2\bullet x$

(5) $(a\,c)\bullet x = x$ by (4i)

(6′) $(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4ii)

(7′) $(a\,c)\bullet(b\,c)\bullet(a\,c)\bullet x = x$ by bij.,(4i)

(8) $(a\,b)\bullet x = x$ by 3rd. prop.

It is as Simple as That...

$$at_\alpha, pt_{\alpha,\iota} : a \# x \wedge b \# x \Rightarrow (a\,b)\bullet x = x$$

Proof: case $a \neq b$:

(1)	fin$\{c \mid (a\,c)\bullet x \neq x\}$	from Ass. +Def. of $\#$
	fin$\{c \mid (b\,c)\bullet x \neq x\}$	
(2′)	fin$\{c \mid (a\,c)\bullet x \neq x \vee (b\,c)\bullet x \neq x\}$	f. (1)
(3′)	inf$\{c \mid (a\,c)\bullet x = x \wedge (b\,c)\bullet x = x)\}$	f. (2′)
(4)	(i) $(a\,c)\bullet x = x$ (ii) $(b\,c)\bullet x = x$	for a $c \in$ (3′)
(5)	$(a\,c)\bullet x = x$	by (4i)
(6′)	$(b\,c)\bullet(a\,c)\bullet x = x$	by bij.,(4ii)
(7′)	$(a\,c)\bullet(b\,c)\bullet(a\,c)\bullet x = x$	by bij.,(4i)
(8)	$(a\,b)\bullet x = x$	by 3rd. prop.

Done.

Support on Functions

Assuming at_α, pt_{α,ι_1}, pt_{α,ι_2}:

If

- $\mathrm{fin}(supp\ x)$,
- $\mathrm{fin}(supp\ f)$,
- $\mathrm{fin}(supp\ (f\ x))$

then $supp\ (f\ x) \subseteq (supp\ f) \cup (supp\ x)$

Support on Functions (II)

Assuming the contrary, that is there exists an a with

$$a \# x, a \# f \text{ and } \neg a \# (f\ x)$$

Because of the finite support of x, f and $(f\ x)$, there exists a b with $b \# x, f, (f\ x)$. So we have

$$(a\,b)\bullet(f\ x) \neq f\ x$$

$$(a\,b)\bullet(f\ x) = ((a\,b)\bullet f)((a\,b)\bullet x) = (f\ x)$$

Empty Support

Assuming at_α, pt_{α,ι_1}, pt_{α,ι_2}:

$$(supp\ f) = \varnothing$$

iff $\forall \pi, x.\ \ \pi \bullet (f\ x) = f(\pi \bullet x)$

We shall call either side as a function being **equivariant**. For example, support is an equivariant function:

$$\pi \bullet supp(x) = supp(\pi \bullet x)$$

Empty Support (II)

($\Rightarrow$) Assuming $(supp\ f) = \varnothing$; to prove

$$\forall \pi. \pi \bullet (f\ x) = f(\pi \bullet x).$$

$$\forall a, b.\ a \# f \wedge b \# f \Rightarrow (a\,b) \bullet f = f$$

Since support of f is empty, that is

$$\forall a, b.\ (a\,b) \bullet f = f$$

which implies

$$\forall \pi. \pi \bullet f = f$$

and then

$$\forall \pi. \pi \bullet (f\ x) = (\pi \bullet f)(\pi \bullet x) = f(\pi \bullet x)$$

The other direction is easy.

Non-Equivariant Functions

Assuming natural numbers were atoms (they are infinite and we can define swapping and permutation over them). Then

$$f\ x \stackrel{\text{def}}{=} x'$$

is not an equivariant function. Assume it is. Then

$$(x\ x')\bullet(f\ x) = x$$

$$(x\ x')\bullet(f\ x) = f((x\ x')\bullet x) = f\ x' = x''$$

However, all 'interesting' ('non-interesting'?) functions over syntax will be equivariant.

Abstractions

We are now going to specify what abstraction 'abstractly' means: it is an operation $[\_].(\_) : \alpha \Rightarrow \iota \Rightarrow \iota$ which has to satisfy:

- $\pi \bullet ([a].x) = [\pi \bullet a].(\pi \bullet x)$
- $[a].x = [b].y$ iff

  $$(a = b \wedge x = y) \vee$$
  $$(a \neq b \wedge x = (a\,b) \bullet y \wedge a \# y)$$

We refer to these properties as $abst_{\alpha,\iota}$. We assume that only one atom at a time is abstracted; more interesting abstractions are possible.

Freshness and Abstractions

Given $abst_{\alpha,\iota}$, $pt_{\alpha,\iota}$ and at_α:
If $\mathrm{fin}(supp\ x)$ and $a \neq b$ then

$$a \# x \text{ iff } a \# [b].x$$

Proof. There exists a c with $c \# a, b, x, [b].x$.

($\Leftarrow$) From $a \# [b].x$ and $c \# [b].x$

$$[b].x = (a\,b)\bullet([b].x) = [b].(a\,c)\bullet x$$

By $abst_{\alpha,\iota}$(ii) hence $x = (a\,c)\bullet x$. Now from $c \# x$:

$$c \# x \Leftrightarrow (a\,c)\bullet c \# (a\,c)\bullet x \Leftrightarrow a \# x$$

Freshness and Abstractions

Given $abst_{\alpha,\iota}$, $pt_{\alpha,\iota}$ and at_α:
If $\mathrm{fin}(supp\ x)$ and $a \neq b$ then

$$a \;\#\; x \text{ iff } a \;\#\; [b].x$$

Proof. There exists a c with $c \;\#\; a, b, x, [b].x$.

($\Rightarrow$) From $c \;\#\; [b].x$ we also have

$$(a\,c)\bullet c \;\#\; (a\,c)\bullet[b].x$$

and

$$a \;\#\; [b].(a\,c)\bullet x$$

Because $a \;\#\; x$ and $c \;\#\; x$, $(a\,c)\bullet x = x$.

Freshness and Abstractions

We also have

$$a \# [a].x$$

Again from $c \# a, x, [a].x$ we can infer

$$\begin{array}{rcl} c \# [a].x & \Leftrightarrow & (a\,c)\bullet c \# (a\,c)\bullet [a].x \\ & \Leftrightarrow & a \# [c].(a\,c)\bullet x. \end{array}$$

However:

$$[c].(a\,c)\bullet x = [a].x$$

(since $c \neq a$, by $abst_{\alpha,\iota}$(ii) $[c].(a\,c)\bullet x = [a].x$
iff $(a\,c)\bullet x = (a\,c)\bullet x \wedge c \# x$)

Conclusion

You might call all this 'abstract non-sense', and very, very strictly speaking none of this is really necessary for reasoning about syntax, **BUT**

- it is some very simple and beautiful mathematics
- next week we show how this can all made concrete for actual syntax
- it is a good 'organising' framework for building a general datatype-package to reason about **any** kind of syntax involving binders, I claim.