Nominal Logic and Reasoning about α-Equivalent Syntax

Christian Urban

every Tuesday from
14:15 to 16:00 in Room E27

(except 26. April, 17. May and 5. July)

Do You Remember the Quiz?

Assuming that a and b are distinct variables, is it possible to find λ-terms M_1 to M_7 that make the following pairs α-equivalent?

- $\lambda a.\lambda b.(M_1\ b)$ and $\lambda b.\lambda a.(a\ M_1)$
- $\lambda a.\lambda b.(M_2\ b)$ and $\lambda b.\lambda a.(a\ M_3)$
- $\lambda a.\lambda b.(b\ M_4)$ and $\lambda b.\lambda a.(a\ M_5)$
- $\lambda a.\lambda b.(b\ M_6)$ and $\lambda a.\lambda a.(a\ M_7)$

If there is one solution for a pair, can you describe all its solutions?

First-Order Unification

- $\mathtt{f}(t_1,\ldots,t_n) \approx ?\ \mathtt{f}(s_1,\ldots,s_n) \cup P$
  $\Longrightarrow t_1 \approx ?\ s_1 \cup \ldots \cup t_n \approx ?\ s_n \cup P$
- $X \approx ?\ t \cup P$
  $\stackrel{[X:=t]}{\Longrightarrow} P[X := t]$ $[X := t]$ part of the answer
- ...
- if σ is an answer for a problem P, then σ is the most general solution

First-Order Unification

- $\mathtt{f}(t_1, \ldots, t_n) \approx ? \; \mathtt{f}(s_1, \ldots, s_n) \cup P$
  $\Longrightarrow t_1 \approx ? \; s_1 \cup \ldots \cup t_n \approx ? \; s_n \cup P$
- $X \approx ? \; t \cup P$
  $\xRightarrow{[X:=t]}$ ... part of the answer

In this lecture I shall give a **slightly** more complicated algorithm for unification modulo α.

- ...
- if σ is an answer for a problem P, then σ is the most general solution

Simple Problem

Given the inference rule:

$$\frac{\mathrm{app}(\mathtt{fn}\, \boldsymbol{a}.\boldsymbol{Y}, \boldsymbol{X}) \Downarrow \boldsymbol{V}}{\mathtt{let}\, \boldsymbol{a} = \boldsymbol{X}\, \mathtt{in}\, \boldsymbol{Y} \Downarrow \boldsymbol{V}}$$

Simple Problem

Given the inference rule:

$$\frac{\mathrm{app}(\mathtt{fn}\,\boldsymbol{a}.\boldsymbol{Y},\boldsymbol{X})\Downarrow\boldsymbol{V}}{\mathtt{let}\;\boldsymbol{a}=\boldsymbol{X}\;\mathtt{in}\;\boldsymbol{Y}\Downarrow\boldsymbol{V}}$$

$\mathtt{fn}\,\boldsymbol{a}.(-)$ and $\mathtt{let}\;\boldsymbol{a}=\boldsymbol{X}\;\mathtt{in}\;(-)$
bind all occurrences of $\boldsymbol{a}$ in $(-)$

Simple Problem

Given the inference rule:

$$\frac{\mathrm{app}(\mathtt{fn}\, a.Y, X) \Downarrow V}{\mathtt{let}\; a = X \;\mathtt{in}\; Y \Downarrow V}$$

- $\mathtt{let}\; a = 1 \;\mathtt{in}\; a \Downarrow 1$

Simple Problem

Given the inference rule:

$$\frac{\mathrm{app}(\mathtt{fn}\, a.\, a, 1) \Downarrow 1}{\mathtt{let}\ a = 1\ \mathtt{in}\ a \Downarrow 1}$$

- $\mathtt{let}\ a = 1\ \mathtt{in}\ a \Downarrow 1\ \ [Y := a; X, V := 1]$

Simple Problem

Given the inference rule:

$$\frac{\mathrm{app}(\mathtt{fn}\, a.\, b,\ 1\ ) \Downarrow 1}{\mathtt{let}\ b\ =\ 1\ \mathtt{in}\ b \Downarrow 1}$$

error!

- $\mathtt{let}\ a = 1\ \mathtt{in}\ a \Downarrow 1 \quad [Y := a; X, V := 1]$
- $\mathtt{let}\ b\ = 1\ \mathtt{in}\ b \Downarrow 1 \quad [Y := b; X, V := 1]$

Simple Problem

Given the inference rule:

$$\frac{\texttt{app}(\texttt{fn}\,\boldsymbol{\lambda a.Fa})\,\boldsymbol{X} \;\Downarrow\; \boldsymbol{V}}{\texttt{let}\,\boldsymbol{X}(\boldsymbol{\lambda a.Fa}) \;\Downarrow\; \boldsymbol{V}}$$

Simple Problem

Given the inference rule:

$$\frac{\mathtt{app}(\mathtt{fn}\ \boldsymbol{F})\ \boldsymbol{X}\ \Downarrow\ \boldsymbol{V}}{\mathtt{let}\ \boldsymbol{X}\ \boldsymbol{F}\ \Downarrow\ \boldsymbol{V}}$$

- $\mathtt{let}\ 1\ \boldsymbol{\lambda a.a} \Downarrow 1$ or $\mathtt{let}\ 1\ \boldsymbol{\lambda b.b} \Downarrow 1$

Simple Problem

Given the inference rule:

$$\frac{\mathtt{app}(\mathtt{fn}\,\boldsymbol{F})\;\boldsymbol{X} \;\Downarrow\; \boldsymbol{V}}{\mathtt{let}\;\boldsymbol{X}\;\boldsymbol{F} \;\Downarrow\; \boldsymbol{V}}$$

- $\mathtt{let}\;1\;\boldsymbol{\lambda a.a} \Downarrow 1$ or $\mathtt{let}\;1\;\boldsymbol{\lambda b.b} \Downarrow 1$

  Does it have to be so complicated? No!

Recap α-Equivalence

$$\frac{}{a \approx a}\ \approx\text{-atm} \qquad \frac{t_1 \approx s_1 \quad t_2 \approx s_2}{t_1\, t_2 \approx s_1\, s_2}\ \approx\text{-app}$$

$$\frac{t \approx s}{\lambda a.t \approx \lambda a.s}\ \approx\text{-lam}_1 \qquad \frac{t \approx (a\, b)\bullet s \quad a \# s}{\lambda a.t \approx \lambda b.s}\ \approx\text{-lam}_2$$

$$\frac{}{a \# b}\ \#\text{-atm} \qquad \frac{a \# t_1 \quad a \# t_2}{a \# t_1\, t_2}\ \#\text{-app}$$

$$\frac{}{a \# \lambda a.t}\ \#\text{-lam}_1 \qquad \frac{a \# t}{a \# \lambda b.t}\ \#\text{-lam}_2$$

assuming $a \neq b$

Permutations on λ-Terms

$$
\begin{aligned}
[]\bullet a &\stackrel{\text{def}}{=} a\\
((a_1\,a_2) :: \pi)\bullet a &\stackrel{\text{def}}{=} \begin{cases} a_1 & \text{if } \pi\bullet a = a_2\\ a_2 & \text{if } \pi\bullet a = a_1\\ \pi\bullet a & \text{otherwise}\end{cases}\\
\pi\bullet(t_1\,t_2) &\stackrel{\text{def}}{=} (\pi\bullet t_1)(\pi\bullet t_2)\\
\pi\bullet(\lambda a.t) &\stackrel{\text{def}}{=} \lambda(\pi\bullet a).(\pi\bullet t)
\end{aligned}
$$

Problems from the Quiz

- $\lambda a.\lambda b.(M_1\ b) \approx ?\ \lambda b.\lambda a.(a\ M_1)$
- $\lambda a.\lambda b.(M_2\ b) \approx ?\ \lambda b.\lambda a.(a\ M_3)$
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What to do with permutations on 'holes' ?

$$\pi \bullet \square$$

Problems from the Quiz

- $\lambda a.\lambda b.(\Box_1\ b) \approx ?\ \lambda b.\lambda a.(a\ \Box_1)$
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What to do with permutations on 'holes' ?

$$\pi \bullet \Box$$

Answer: Nothing! Just keep them.

Terms

- $\langle\rangle$ Units
- $\langle t, t'\rangle$ Pairs
- $F\, t$ Funct.

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- $F\,t$ Funct.
- a Atoms
- $a.t$ Abstractions

bindable names (of object-level variables etc.)

generic binder:

$\ulcorner \lambda a.a \urcorner \mapsto \mathtt{fn}\ a.a$

constructions like $\mathtt{fn}\ X.X$ are not allowed

Terms

- $\langle\rangle$ Units
- $\langle t, t' \rangle$ Pairs
- $F\,t$ Funct.
- a Atoms
- $a.t$ Abstractions
- $\pi \cdot X$ Suspensions

Terms

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- $\pi{\cdot}X$ Suspensions

π is an explicit permutation, which is a list of swappings $(a_1\,b_1)\ldots(a_n\,b_n)$, waiting to be applied to the term that is substituted for X

X is a variable standing for an unknown term

Permutations

a permutation applied to a term:

- $$[\,]\bullet a \stackrel{\mathsf{def}}{=} a$$
- $$(b\,c)::\pi \bullet a \stackrel{\mathsf{def}}{=} \begin{cases} c & \text{if } \pi\bullet a = b \\ b & \text{if } \pi\bullet a = c \\ \pi\bullet a & \text{otherwise} \end{cases}$$

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- $\pi\bullet a.t \stackrel{\text{def}}{=} \pi\bullet a.\pi\bullet t$

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- $$\pi\bullet a.t \stackrel{\mathsf{def}}{=} \pi\bullet a.\pi\bullet t$$
- $$\pi\bullet\pi'\cdot X \stackrel{\mathsf{def}}{=} (\pi @ \pi')\cdot X$$

Permutations

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- $$\pi\bullet a.t \stackrel{\text{def}}{=} \pi\bullet a.\pi\bullet t$$
- $$\pi\bullet\pi'\cdot X \stackrel{\text{def}}{=} (\pi @ \pi')\cdot X$$

inverse permutation given by reversing the list of swappings

Permutations on atoms are bijections!

$$\pi\bullet a = b \quad \text{iff} \quad a = (\pi^{-1})\bullet b$$

Freshness Relation

We will identify

$$\texttt{fn}\, a.X \;\approx\; \texttt{fn}\, b.(a\, b)\cdot X$$

provided that 'b is fresh for X — ($b \# X$)', i.e., does not occur freely in any ground term that might be substituted for X.

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If we know more about X, e.g., if we knew that $a \# X$ and $b \# X$, then we can replace $(a\,b)\cdot X$ by X.

Freshness Assumptions

Our equality is **not** just

$$t \approx t' \qquad \alpha\text{-equivalence}$$

Freshness Assumptions

but judgements

$$\nabla \vdash t \approx t' \qquad \alpha\text{-equivalence}$$

where

$$\nabla = \{a_1 \# X_1, \ldots, a_n \# X_n\}$$

is a finite set of freshness assumptions.

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is a finite set of freshness assumptions.

$$\{a \# X, b \# X\} \vdash \mathtt{fn}\, a.X \approx \mathtt{fn}\, b.X$$

Freshness Assumptions

but judgements

$\boxed{\nabla \vdash t \approx t'}$ α-equivalence

$\boxed{\nabla \vdash a \# t}$ freshness

where

$$\nabla = \{a_1 \# X_1, \ldots, a_n \# X_n\}$$

is a finite set of freshness assumptions.

$$\begin{array}{rcl} \{b \# X\} & \vdash & b \# a.X \\ \{\} & \vdash & a \# a.X \end{array}$$

Rules for Equivalence

Excerpt
(i.e. only the interesting rules)

Rules for Equivalence

$$\frac{\nabla \vdash t \approx t'}{\nabla \vdash a.t \approx a.t'}$$

$$\frac{a \neq b \qquad \nabla \vdash t \approx (a\,b)\bullet t' \qquad \nabla \vdash a \# t'}{\nabla \vdash a.t \approx b.t'}$$

Rules for Equivalence

$$\frac{(a \mathrel{\#} X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi \cdot X \approx \pi' \cdot X}$$

Rules for Equivalence

$$\frac{(a \# X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi \cdot X \approx \pi' \cdot X}$$

for example

$$\{a \# X, b \# X\} \vdash X \approx (a\, b) \cdot X$$

Rules for Equivalence

$$\frac{(a \;\#\; X) \in \nabla \quad \text{for all } a \text{ with } \pi \bullet a \neq \pi' \bullet a}{\nabla \vdash \pi \cdot X \approx \pi' \cdot X}$$

for example

$$\{a \# X, c \# X\} \vdash (a\,c)(a\,b)\cdot X \approx (b\,c)\cdot X$$

because $(a\,c)(a\,b)$: $a \mapsto b$, $b \mapsto c$, $c \mapsto a$ $\quad (b\,c)$: $a \mapsto a$, $b \mapsto c$, $c \mapsto b$

disagree at a and c.

Rules for Freshness

Excerpt
(again only the interesting rules)

Rules for Freshness

$$\frac{a \neq b}{\nabla \vdash a \# b}$$

$$\frac{}{\nabla \vdash a \# a.t} \qquad \frac{a \neq b \quad \nabla \vdash a \# t}{\nabla \vdash a \# b.t}$$

$$\frac{(\pi^{-1} \bullet a \# X) \in \nabla}{\nabla \vdash a \# \pi \cdot X}$$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

(Reflexivity)	$\nabla \vdash t \approx t$
(Symmetry)	if $\nabla \vdash t_1 \approx t_2$ then $\nabla \vdash t_2 \approx t_1$
(Transitivity)	if $\nabla \vdash t_1 \approx t_2$ and $\nabla \vdash t_2 \approx t_3$ then $\nabla \vdash t_1 \approx t_3$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

because $\approx$ has very good properties:

- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi \bullet t \approx \pi \bullet t'$
- $\nabla \vdash a \# t$ then $\nabla \vdash \pi \bullet a \# \pi \bullet t$

$\approx$ is an Equivalence

Theorem: $\approx$ is an equivalence relation.

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- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi{\bullet}t \approx \pi{\bullet}t'$
- $\nabla \vdash a \ \# \ t$ then $\nabla \vdash \pi{\bullet}a \ \# \ \pi{\bullet}t$
- $\nabla \vdash t \approx \pi{\bullet}t'$ then $\nabla \vdash (\pi^{-1}){\bullet}t \approx t'$
- $\nabla \vdash a \ \# \ \pi{\bullet}t$ then $\nabla \vdash (\pi^{-1}){\bullet}a \ \# \ t$

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Theorem: $\approx$ is an equivalence relation.

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- $\nabla \vdash t \approx t'$ then $\nabla \vdash \pi{\bullet}t \approx \pi{\bullet}t'$
- $\nabla \vdash a \# t$ then $\nabla \vdash \pi{\bullet}a \# \pi{\bullet}t$
- $\nabla \vdash t \approx \pi{\bullet}t'$ then $\nabla \vdash (\pi^{-1}){\bullet}t \approx t'$
- $\nabla \vdash a \# \pi{\bullet}t$ then $\nabla \vdash (\pi^{-1}){\bullet}a \# t$
- $\nabla \vdash a \# t$ and $\nabla \vdash t \approx t'$ then $\nabla \vdash a \# t'$

Comparison with $=_\alpha$

Traditionally $=_\alpha$ is defined as

> least congruence which identifies $a.t$ with $b.[a := b]t$ provided b is not free in t

where $[a := b]t$ replaces all free occurrences of a by b in t.

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For ground terms:

> Theorem: $t =_\alpha t'$ iff $\varnothing \vdash t \approx t'$
>
> $a \notin FA(t)$ iff $\varnothing \vdash a \# t$

Comparison with $=_\alpha$

Traditionally $=_\alpha$ is defined as

least congruence which identifies $a.t$ with $b.[a := b]t$ provided b is not free in t

where $[a := b]t$ replaces all free occurrences of a by b in t.

In general $=_\alpha$ and $\approx$ are distinct!

$a.X =_\alpha b.X$ but not

$\varnothing \vdash a.X \approx b.X$ $(a \neq b)$

Comparison with $=_\alpha$

That is a crucial point: if we had

$$\varnothing \vdash a.X \approx b.X\,,$$

then applying $[X := a]$, $[X := b]$, . . .
give two terms that are **not** α-equivalent.

The freshness constraints $a \# X$ and $b \# X$ rule out the problematic substitutions. Therefore

$$\{a \# X, b \# X\} \vdash a.X \approx b.X$$

does hold.

f

Substitutions

- $\sigma(a.t) \stackrel{\text{def}}{=} a.\sigma(t)$
- $\sigma(\pi{\cdot}X) \stackrel{\text{def}}{=} \begin{cases} \pi \bullet \sigma(X) & \text{if } \sigma(X) \neq X \\ \pi{\cdot}X & \text{o'wise do nothing} \end{cases}$

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for example

$$a.(a\,b){\cdot}X \;\; [X := \langle b, Y \rangle]$$

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for example

$$\underline{a.(a\,b)\cdot X\ \ [X := \langle b, Y\rangle]}$$

$$\Rightarrow \quad a.\underline{(a\,b)\cdot X[X := \langle b, Y\rangle]}$$

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for example

$$a.(a\,b)\cdot X \;\; [X := \langle b, Y\rangle]$$

$$\Rightarrow \quad a.\underline{(a\,b)\cdot X[X := \langle b, Y\rangle]}$$

$$\Rightarrow \quad a.(a\,b)\bullet\underline{\langle b, Y\rangle}$$

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for example

$$a.(a\,b){\cdot}X\ \ [X := \langle b, Y\rangle]$$

$$\Rightarrow\quad a.(a\,b){\cdot}X[X := \langle b, Y\rangle]$$

$$\Rightarrow\quad a.\underline{(a\,b)}\bullet\langle b, Y\rangle$$

$$\Rightarrow\quad a.\langle a, (a\,b){\cdot}Y\rangle$$

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- if $\nabla \vdash t \approx t'$ and $\nabla' \vdash \sigma(\nabla)$

  then $\nabla' \vdash \sigma(t) \approx \sigma(t')$

Substitutions

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- if $\nabla \vdash t \approx t'$ and $\nabla' \vdash \sigma(\nabla)$
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this means
$\nabla' \vdash a \# \sigma(X)$
holds for all
$(a \# X) \in \nabla$

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- if $\nabla \vdash t \approx t'$ and $\nabla' \vdash \sigma(\nabla)$

  then $\nabla' \vdash \sigma(t) \approx \sigma(t')$
- $\sigma(\pi \bullet t) = \pi \bullet \sigma(t)$

Equational Problems

An equational problem

$$t \approx ? \ t'$$

is solved by

- a substitution σ (terms for variables)
- and a set of freshness assumptions ∇

so that $\nabla \vdash \sigma(t) \approx \sigma(t')$.

Unifying equations may entail solving freshness problems.

E.g. assuming that $a \neq a'$, then

$$a.t \approx ?\; a'.t'$$

can only be solved if

$$t \approx ?\; (a\, a')\bullet t' \quad \text{and} \quad a \,\#?\; t'$$

can be solved.

Freshness Problems

A freshness problem

$$a \mathrel{\#?} t$$

is solved by

- a substitution σ
- and a set of freshness assumptions ∇

so that $\nabla \vdash a \mathrel{\#} \sigma(t)$.

Most General Unifiers

Definition: for a unification problem P, a solution (σ_1, ∇_1) is more general than another solution (σ_2, ∇_2), iff there exists a substitution σ with

- $\nabla_2 \vdash \sigma(\nabla_1)$
- $\nabla_2 \vdash \sigma_2 \approx \sigma \circ \sigma_1$

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- $\nabla_2 \vdash \sigma(\nabla_1)$

  $\nabla_2 \vdash a \# \sigma(X)$
  holds for all
  $(a \# X) \in \nabla_1$

- $\nabla_2 \vdash \sigma_2 \approx \sigma \circ \sigma_1$

Most General Unifiers

<u>Definition:</u> for a unification problem P, a solution (σ_1, ∇_1) is more general than another solution (σ_2, ∇_2), iff the

$\nabla_2 \vdash \sigma_2(X) \approx \sigma(\sigma_1(X))$
holds for all
$X \in \mathrm{dom}(\sigma_2) \cup \mathrm{dom}(\sigma \circ \sigma_1)$

- $\nabla_2 \vdash \sigma(\nabla$
- $\nabla_2 \vdash \sigma_2 \approx \sigma \circ \sigma_1$

Existence of MGUs

<u>Theorem</u>: there is an algorithm which, given a nominal unification problem P, decides whether or not it has a solution (σ, ∇), and returns a most general one if it does.

Existence of MGUs

Theorem: there is an algorithm which, given a nominal unification problem P, decides whether or not it has a solution (σ, ∇), and returns a most general one if it does.

straightforward definition:
"iff there exists a τ such that ..."

Existence of MGUs

Theorem: there is an algorithm which, given a nominal unification problem P, decides whether or not it has a solution (σ, ∇), and returns a most general one if it does.

Proof: one can reduce all the equations to 'solved form' first (creating a substitution), and then solve the freshness problems (easy).

Reductions

A set of $(t \approx ?\ t')$ and $(a \# ?\ t)$ problems can be reduced by

$$\stackrel{\sigma}{\Longrightarrow} \quad \text{or} \quad \stackrel{\nabla}{\Longrightarrow}$$

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \xRightarrow{\varepsilon}$ if $a \neq b$
  $\{t \approx ?\ (a\,b) \bullet t', a \#?\ t'\} \cup P$

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \overset{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx ?\ (a\, b)\bullet t', a \,\#?\ t'\} \cup P$
- $\{a.t \approx ?\ a.t'\} \uplus P \overset{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$

  $\{t \approx ?\ (a\, b) \bullet t', a \#?\ t'\} \cup P$

- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$

- $\{\pi \bullet X \approx ?\ \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$

  $\{a \#?\ X | a \in ds(\pi, \pi')\} \cup P$

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$

  $\{t \approx ?\ (a\,b)\bullet t', a \#?\ t'\} \cup P$

- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$

- $\{\pi \bullet X \approx ?\ \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$

  $\{a \#?\ X | a \in ds(\pi, \pi')\} \cup P$

- $\{\pi \bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$

  if X does not occur in t

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$

  $\{t \approx ?\ (a\,b) \bullet t', a \# ?\ t'\} \cup P$

- $\{a.t \approx ?\ a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ?\ t'\} \cup P$

- $\{\pi \bullet X \approx ?\ \pi' \bullet X\}$

  $\sigma = [X :=$
  $\pi^{-1} \bullet t]$

  $\{a \# ?\ X \mid a \in ds(\pi, \pi')\} \cup P$

- $\{\pi \bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$

  if X does not occur in t

Reductions

- $\{a.t \approx ?\ b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx ?\ (a\,b)\bullet t', a \;\#?\; t'\} \cup P$
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- $\{\pi \bullet X \approx ?\ \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$
  $\{a \;\#?\; X | a \in ds(\pi, \pi')\} \cup P$
- $\{\pi \bullet X \approx ?\ t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t
- $\{a \;\#?\; b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \;\#?\; t\} \cup P$ if $a \neq b$

Reductions

- $\{a.t \approx ? \; b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$
  $\{t \approx ? \; (a\,b)\bullet t', a \;\#?\; t'\} \cup P$
- $\{a.t \approx ? \; a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx ? \; t'\} \cup P$
- $\{\pi \bullet X \approx ? \; \pi' \bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$
  $\{a \;\#?\; X | a \in ds(\pi, \pi')\} \cup P$
- $\{\pi \bullet X \approx ? \; t\} \uplus P \stackrel{\sigma}{\Longrightarrow} \sigma P$
  if X does not occur in t
- $\{a \;\#?\; b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \;\#?\; t\} \cup P$ if $a \neq b$
- $\{a \;\#?\; \pi \bullet X\} \uplus P \stackrel{\nabla}{\Longrightarrow} P$

Reductions

- $\{a.t \approx? b.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$ if $a \neq b$

  $\{t \approx? (a\,b)\bullet t', a \#? t'\} \cup P$

- $\{a.t \approx? a.t'\} \uplus P \stackrel{\varepsilon}{\Longrightarrow} \{t \approx? t'\} \cup P$

- $\{\pi\bullet X \approx? \pi'\bullet X\} \uplus P \stackrel{\varepsilon}{\Longrightarrow}$

  $\{a \#? X | a \in ds(\pi, \pi')\} \cup P$

- $\{\pi\bullet X \approx? t\} \uplus P \stackrel{\sigma}{\Longrightarrow}$

  if X does not occur in t

  $\nabla = \{\pi^{-1}\bullet a \# X\}$

- $\{a \#? b.t\} \uplus P \stackrel{\varnothing}{\Longrightarrow} \{a \#? t\} \cup P$ if $a \neq b$

- $\{a \#? \pi\bullet X\} \uplus P \stackrel{\nabla}{\Longrightarrow} P$

Reductions

A set of $(t \approx ? \ t')$ and $(a \ \#? \ t)$ problems can be reduced by

$$\stackrel{\sigma}{\Longrightarrow} \quad \text{or} \quad \stackrel{\nabla}{\Longrightarrow}$$

Reductions

A set of $(t \approx ?\ t')$ and $(a\ \#?\ t)$ problems can be reduced by

$$\stackrel{\sigma}{\Longrightarrow} \quad \text{or} \quad \stackrel{\nabla}{\Longrightarrow}$$

If there is a reduction

$$P \stackrel{\sigma_1}{\Longrightarrow} \ldots \stackrel{\sigma_n}{\Longrightarrow} P' \stackrel{\nabla_1}{\Longrightarrow} \ldots \stackrel{\nabla_m}{\Longrightarrow} \varnothing$$

then

$$(\sigma_n \circ \ldots \circ \sigma_1, \nabla_1 \cup \ldots \cup \nabla_m)$$

is a most general unifier for P.

Remember the Quiz?

Assuming that a and b are distinct variables, is it possible to find λ-terms M_1 to M_7 that make the following pairs α-equivalent?

- $\lambda a.\lambda b.(M_1\ b)$ and $\lambda b.\lambda a.(a\ M_1)$
- $\lambda a.\lambda b.(M_2\ b)$ and $\lambda b.\lambda a.(a\ M_3)$
- $\lambda a.\lambda b.(b\ M_4)$ and $\lambda b.\lambda a.(a\ M_5)$
- $\lambda a.\lambda b.(b\ M_6)$ and $\lambda a.\lambda a.(a\ M_7)$

If there is one solution for a pair, can you describe all its solutions?

Answers to the Quiz

$\lambda a.\lambda b.(M_1\, b)$ and $\lambda b.\lambda a.(a\, M_1)$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\; \approx? \;\; b.a.\langle a, M_1\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? (a\,b)\bullet a.\langle a, M_1\rangle \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle\ ,\ a \#? a.\langle a, M_1\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx? \langle b, (a\,b)\cdot M_1\rangle\ ,\ a \#? a.\langle a, M_1\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\xRightarrow{\varepsilon} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

$$\xRightarrow{\varepsilon} \langle M_1, b\rangle \approx? \langle b, (a\,b)\cdot M_1\rangle \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

$$\xRightarrow{\varepsilon} M_1 \approx? b \;,\; b \approx? (a\,b)\cdot M_1 \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \ \approx? \ b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? \ b.\langle b, (a\,b)\cdot M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle M_1, b\rangle \approx? \ \langle b, (a\,b)\cdot M_1\rangle \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} M_1 \approx? \ b \ , \ b \approx? \ (a\,b)\cdot M_1 \ , \ a \ \#? \ a.\langle a, M_1\rangle$$

$$\stackrel{[M_1:=b]}{\Longrightarrow} b \approx? \ (a\,b)\bullet b \ , \ a \ \#? \ a.\langle a, b\rangle$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \#? a.\langle a, M_1\rangle$$

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Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

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$$\overset{[M_1:=b]}{\Longrightarrow} b \approx? a \;,\; a \;\#?\; a.\langle a, b\rangle$$

$$\Longrightarrow FAIL$$

Answers to the Quiz

$$a.b.\langle M_1, b\rangle \;\approx?\; b.a.\langle a, M_1\rangle$$

$$\xRightarrow{\varepsilon} b.\langle M_1, b\rangle \approx? b.\langle b, (a\,b)\cdot M_1\rangle \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

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$$\xRightarrow{\varepsilon} M_1 \approx? b \;,\; b \approx? (a\,b)\cdot M_1 \;,\; a \;\#?\; a.\langle a, M_1\rangle$$

$$\xRightarrow{[M_1:=b]} b \approx? a \;,\; a \;\#?\; a.\langle a, b\rangle$$

$$\Longrightarrow FAIL$$

$\boxed{\lambda a.\lambda b.(M_1\, b) =_\alpha \lambda b.\lambda a.(a\, M_1) \text{ has no solution}}$

Answers to the Quiz

$\lambda a.\lambda b.(b\, M_6)$ and $\lambda a.\lambda a.(a\, M_7)$

Answers to the Quiz

$a.b.\langle b, M_6 \rangle \;\; \approx ? \;\; a.a.\langle a, M_7 \rangle$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \;\; \approx? \;\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? \; a.\langle a, M_7\rangle$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \approx? \; a.a.\langle a, M_7\rangle$$

$$\xRightarrow{\varepsilon} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\xRightarrow{\varepsilon} \langle b, M_6\rangle \approx? \langle b, (b\,a)\cdot M_7\rangle \;,\; b \,\#?\, \langle a, M_7\rangle$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \;\; \approx? \;\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? \; a.\langle a, M_7\rangle$$

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$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

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$$\stackrel{\varepsilon}{\Longrightarrow} M_6 \approx? (b\,a)\cdot M_7 \;,\; b \,\#?\, \langle a, M_7\rangle$$

$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \,\#?\, \langle a, M_7\rangle$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\overset{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx? \langle b, (b\,a)\cdot M_7\rangle \;,\; b \#? \langle a, M_7\rangle$$

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$$\overset{\varepsilon}{\Longrightarrow} M_6 \approx? (b\,a)\cdot M_7 \;,\; b \#? \langle a, M_7\rangle$$

$$\overset{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \#? \langle a, M_7\rangle$$

$$\overset{\varnothing}{\Longrightarrow} b \#? a \;,\; b \#? M_7$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\stackrel{\varepsilon}{\Longrightarrow} \langle b, M_6\rangle \approx? \langle b, (b\,a)\cdot M_7\rangle \;,\; b \;\#?\; \langle a, M_7\rangle$$

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$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \;\#?\; \langle a, M_7\rangle$$

$$\stackrel{\varnothing}{\Longrightarrow} b \;\#?\; a \;,\; b \;\#?\; M_7$$

$$\stackrel{\varnothing}{\Longrightarrow} b \;\#?\; M_7$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \;\approx?\; a.a.\langle a, M_7\rangle$$

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$$\stackrel{[M_6:=(b\,a)\cdot M_7]}{\Longrightarrow} b \#? \langle a, M_7\rangle$$

$$\stackrel{\varnothing}{\Longrightarrow} b \#? a \;,\; b \#? M_7$$

$$\stackrel{\varnothing}{\Longrightarrow} b \#? M_7$$

$$\stackrel{\{b\#M_7\}}{\Longrightarrow} \varnothing$$

Answers to the Quiz

$$a.b.\langle b, M_6\rangle \ \approx? \ a.a.\langle a, M_7\rangle$$

$$\xRightarrow{\varepsilon} b.\langle b, M_6\rangle \approx? a.\langle a, M_7\rangle$$

$$\xRightarrow{\varepsilon} \langle b, M_6\rangle \approx$$

$$\xRightarrow{\varepsilon} b \approx? b \ , \ M$$

$$\xRightarrow{\varepsilon} M_6 \approx? (b$$

$$\xRightarrow{[M_6:=(b\,a)\cdot M_7]} b \#?$$

$$\xRightarrow{\varnothing} b \#? a \ , \ b \#? M_7$$

$$\xRightarrow{\varnothing} b \#? M_7$$

$$\xRightarrow{\{b\#M_7\}} \varnothing$$

$$\lambda a.\lambda b.(b\, M_6) \ =_\alpha \ \lambda a.\lambda a.(a\, M_7)$$

we can take M_7 to be any λ-term that does not contain free occurrences of b, so long as we take M_6 to be the result of swapping all occurrences of b and a throughout M_7

Correct/Incorrect?

Does the following Prolog program produce for every lambda-term the correct type?

$$\frac{x : A \in \Gamma}{\Gamma \rhd x : A} \qquad \frac{\Gamma \rhd M : A \supset B \quad \Gamma \rhd N : A}{\Gamma \rhd M\,N : B} \qquad \frac{x : A, \Gamma \rhd M : B}{\Gamma \rhd \lambda x.M : A \supset B}$$

```
type Gamma (var X) A :- member (pair X A) Gamma.

type Gamma (app M N) B :- type Gamma M (arrow A B),
                          type Gamma N A.

type Gamma (lam X M) (arrow A B) :-
                          type (pair X A)::Gamma M B.

member A A::Tail.

member A B::Tail :- member A Tail.
```

Correct/Incorrect?

NO! The term $\lambda x.\lambda x.(x \;\; x)$ should not be typable. But

```
type [] (lam X (lam X (app X X))) T
```

returns two answers

- $\tau \supset (\tau \supset \sigma) \supset \sigma$
- $(\tau \supset \sigma) \supset \tau \supset \sigma$

ev-

```
type Gamma (lam X M) (arrow A B) :-
                              type (pair X A)::Gamma M B.

member A A::Tail.

member A B::Tail :- member A Tail.
```

Is it useful?

applications to logic programming (with J. Cheney)

$$\frac{x : A \in \Gamma}{\Gamma \triangleright x : A} \qquad \frac{\Gamma \triangleright M : A \supset B \quad \Gamma \triangleright N : A}{\Gamma \triangleright M\, N : B} \qquad \frac{x : A, \Gamma \triangleright M : B}{\Gamma \triangleright \lambda x.M : A \supset B}$$

```
type Gamma (var X) A :- member (pair X A) Gamma.

type Gamma (app M N) B :- type Gamma M (arrow A B),
                          type Gamma N A.

type Gamma (lam x.M) (arrow A B) / x#Gamma :-
                          type (pair x A)::Gamma M B.

member A A::Tail.

member A B::Tail :- member A Tail.
```

Conclusion

- computed with freshness assumptions. This allowed us to define $\approx$ so that substitution respects α-equivalence. Furthermore have concrete names for binders (nominal unification) and not de-Bruijn indices.
- it is a completely first-order language
- verified everything in Isabelle
- as powerful as Miller's pattern-unification
- algorithm is quadratic (linear???)

Possible Extensions

- Atoms are constants: otherwise

$$a.M \approx ?\ X.Y.\mathtt{f}(a)$$

  - 1st solution: $[M := X.\mathtt{f}(X), Y := a]$

$$a.X.\mathtt{f}(X) \approx X.a.\mathtt{f}(a)$$

  - 2nd solution: $[M := Y.\mathtt{f}(a), X := a]$

$$a.Y.\mathtt{f}(a) \approx a.Y.\mathtt{f}(a)$$